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Investigation of C_m-Supermagic Properties in the Union of C_n and C_m Graph Structures

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Abstract— Graph labeling, the assignment of numbers to the vertices or edges of a graph, finds applications in diverse fields such as network addressing, channel allocation, data mining, image processing, cryptography, and logistics. A C_m-supermagic labeling involves assigning integers to a graph's edges and vertices such that the labels' sum for all C_m cycles equals a constant value. This paper explores the C_m-supermagic properties of the $Cl_{n,m}$ graph, formed by the union of a C_n graph and n C_m graphs. It comprehensively analyzes the conditions under which $Cl_{n,m}$ exhibits C_m-supermagic properties and derive explicit labeling constructions. These results contribute to understanding C_m-supermagic graphs and their potential applications in theoretical and applied domains.

Keywords — *Graph labeling, C_m-supermagic labeling, circular graph, magic sum*

Mathematics Subject Classification (2020) 05C78, 05C30

1. Introduction

Graph labeling, an important area of research in graph theory, refers to assigning numbers to the vertices or edges of a graph. There are various types of labeling in the literature. This paper focuses on C_m-supermagic labeling, a special case of H-supermagic labeling primarily studied on graphs containing circular structures. C_m-supermagic labeling is a type of labeling where integers are assigned to the edges and vertices of a graph according to specific rules, ensuring that the sum of the labels of each cycle C_m is equal to a constant value. Some of the recent studies in this area are as follows: Roswitha et al. [1] investigated C_m-supermagic labeling for some classes of graphs, such as Jahangir graph, wheel graph for even n, and complete bipartite graph $K_{m,n}$ for $m = 2$. Rizvi et al. [2] proved that if a graph has a cycle-(super)magic labeling, then its uniform subdivided graph also has a cycle-(super)magic labeling. They also discussed some cycle-super magic labelings for nonuniform subdivided fans and triangular ladders. Rizvi et al. [3] formulated cycle-supermagic labelings for the disjoint union of isomorphic copies of different families of graphs and proved that the disjoint union of non isomorphic copies of fans and ladders are cycle-supermagic. In [4], Numana et al. studied the cycle-supermagic labeling of a pumpkin graph and two classes of planar maps. Azeem [5] investigated cycle-super magic labeling of zig-zag, linear chains, and the disjoint union of non-isomorphic copies of both chains. [6–8] investigated

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C_m-supermagic labeling methods for the specific graph class, such as book-snake, polygonal snake, and friendship graphs. Pradipta et al. [9] provided some cycle-supermagic labelings of calendula graphs, denoted by $Cl_{n,m}$, using the m-balanced multisets.

This paper focuses on the C_m-supermagic properties of calendula graphs, where the labeling function is derived using a different method. The rest of the paper is organized as follows: Section 2 investigates the conditions for the C_m-supermagic properties of the graph $Cl_{n,m}$. Section 3 provides explicit constructions and C_m-supermagic labelings for various circulant graphs $Cl_{n,m}$, illustrating the proposed theorems.

2. Supermagic Properties of the Graph $Cl_{n,m}$

A graph $G(V, E)$ admits an H-covering if every edge belongs to a subgraph of G isomorphic to a simple graph H. A bijection $\lambda : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ is called an H-magic labeling of G if there exists a constant $m(\lambda)$, known as the magic sum, such that for all subgraph $H' = (V', E')$ of G, which is isomorphic to H, the following equality hold:

$$\sum_{v \in V'} \lambda(v) + \sum_{e \in E'} \lambda(e) = m(\lambda)$$

A graph is H-magic if it satisfies this property. Furthermore, an H-magic labeling λ is considered H-supermagic if the labels assigned to the vertices of G cover the set $\{1, 2, \ldots, |V|\}$ exactly [10]. Let $n, m \geq 3$. The graph $Cl_{n,m}$ is obtained by amalgamating an edge of a copy G_i of C_m, for $i \in \{1, 2, \ldots, n\}$, with an edge e_i of the graph C_n [9], where the central cycle is C_m and thus the parameters have opposite meanings. This results in a graph with nm edges and $n(m-1)$ vertices (see Figure 1).

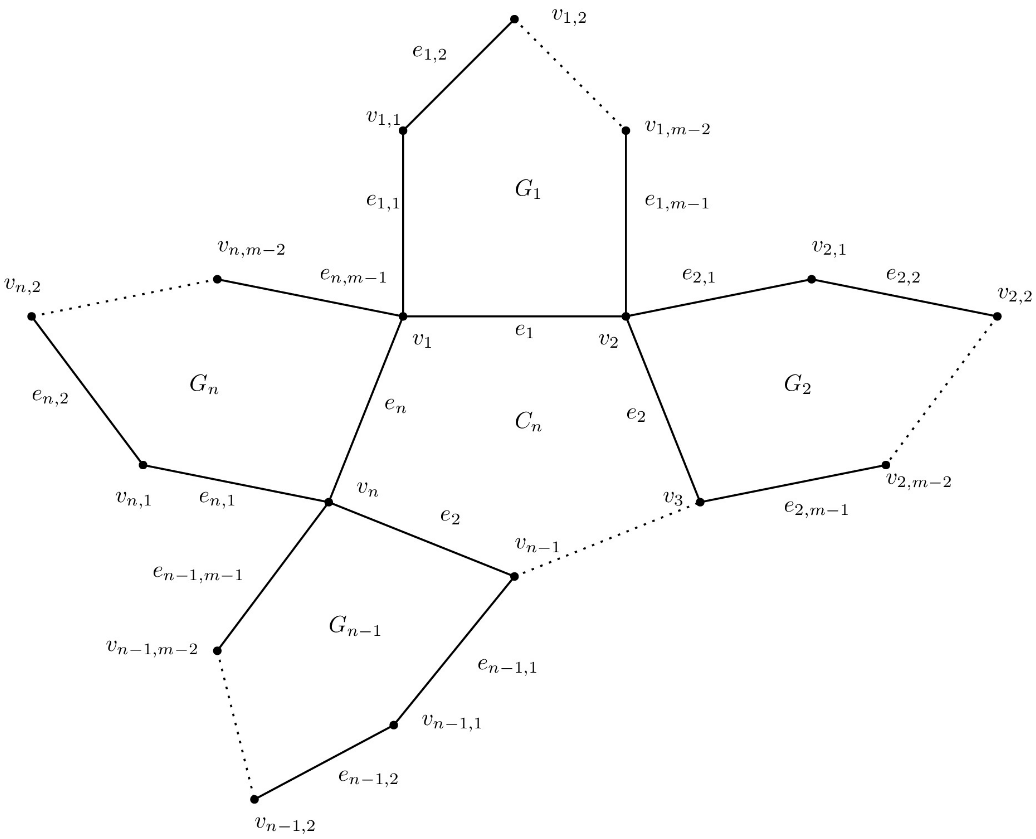


Figure 1. $Cl_{n,m}$

Denote these vertices and edges as follows:

$$V = \{v_i : i \in \{1, \dots, n\}\} \cup \{v_{i,j} : i \in \{1, \dots, n\} \text{ and } j \in \{1, \dots, m-2\}\}$$

and

$$\begin{aligned} E = &\{e_i : i \in \{1, \dots, n\}\} \cup \{e_{i,1} : e_{i,1} = v_i v_{i,1},\ i \in \{1, \dots, n\}\} \\ &\cup \{e_{i,j} : e_{i,j} = v_{i,j-1} v_{i,j},\ i \in \{1, \dots, n\} \text{ and } j \in \{2, \dots, m-2\}\} \\ &\cup \{e_{i,m-1} : e_{i,m-1} = v_{i,m-2} v_{i+1},\ i \in \{1, \dots, n\}\} \end{aligned}$$

If $n \neq m$, then there exist n subgraphs in $Cl_{n,m}$ that are isomorphic to C_m. If $n = m$, then the number of these subgraphs is $n+1$. In both cases, these subgraphs represent a C_m-covering for $Cl_{n,m}$. Therefore, whether $Cl_{n,m}$ has a C_m-supermagic labeling can be questioned. In the following, we investigate the C_m-supermagic labeling of these graphs for different values of n and m:

Case I: If $n = m$, then the C_n-supermagic labeling of $Cl_{n,n}$ graph has been analyzed by separately considering the odd and even cases for the value of n.

Theorem 2.1. For even n, $Cl_{n,n}$ has a C_n-supermagic labeling.

Proof. Consider the following labeling function:

$$\lambda(v_i) = \left(\frac{n}{2} - 1\right) n + n + 2 - i + \left\lfloor \frac{1}{i} \right\rfloor (-n), \quad i \in \{1, 2, \dots, n\}$$

and

$$\lambda(v_{i,j}) = \begin{cases} (j-1)n + i, & j \in \{1, 2, \dots, \frac{n}{2} - 1\} \text{ and } i \in \{1, 2, \dots, n\} \\ \frac{n}{2} n + \left\lfloor \frac{1}{i} \right\rfloor n + i - 1, & j = \frac{n}{2} \text{ and } i \in \{1, 2, \dots, n\} \\ jn + n + 1 - i, & j \in \{\frac{n}{2} + 1, \frac{n}{2} + 2, , \dots, n-2\} \text{ and } i \in \{1, 2, \dots, n\} \end{cases}$$

Handle the edge labels in two cases:

Case 1: If $n \equiv 0 \pmod 4$, then

$$\lambda(e_i) = \begin{cases} n^2 + (\frac{n}{2} - 2) n + i, & i \equiv 0, 1 \pmod 4 \\ n^2 + (\frac{n}{2} - 1) n + n + 1 - i, & i \equiv 2, 3 \pmod 4 \end{cases}$$

and

$$\lambda(e_{i,j}) = \begin{cases} n^2 - n + (j-1)n + i, & j \in \{1, 2, \dots, \frac{n}{2} - 1\} \text{ and } i \in \{1, 2, \dots, n\} \\ n^2 + (\frac{n}{2} - 2) n + i, & j = \frac{n}{2} \text{ and } i \equiv 2, 3 \pmod 4 \\ n^2 + (\frac{n}{2} - 1) n + n + 1 - i, & j = \frac{n}{2} \text{ and } i \equiv 0, 1 \pmod 4 \\ n^2 + (j-1)n + n + 1 - i, & j \in \{\frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n-1\} \text{ and } i \in \{1, 2, \dots, n\} \end{cases}$$

Case 2: If $n \equiv 2 \pmod 4$, then

$$\lambda(e_i) = \begin{cases} n^2 + (\frac{n}{2} - 2) n + i, & i \equiv 0, 1 \pmod 4 \text{ and } i \in \{1, 2, , \dots, n-4\} \\ n^2 + (\frac{n}{2} - 1) n + n + 1 - i, & i \equiv 2, 3 \pmod 4 \text{ and } i \in \{1, 2, , \dots, n-4\} \\ n^2 + (\frac{n}{2} - 2) n + i, & i = n - 3 \\ n^2 + (\frac{n}{2} - 1) n + n + 1 - i, & i \in \{n, n-1, n-2\} \end{cases}$$

and

$$\lambda\left(e_{i,j}\right)=\begin{cases} n^2-n+(j-1)\,n+i, & j\in\{1,2,\ldots,\frac{n}{2}-1\} \text{ and } i\in\{1,2,\ldots,n\} \\ n^2+\left(\frac{n}{2}-1\right)n+n+1-i, & j=\frac{n}{2}, i\equiv 0,1 \pmod 4 \text{ and } i\in\{1,2,,\ldots,n-4\} \\ n^2+\left(\frac{n}{2}-2\right)n+i, & j=\frac{n}{2}, i\equiv 2,3 \pmod 4 \text{ and } i\in\{1,2,,\ldots,n-4\} \\ n^2+\left(\frac{n}{2}-1\right)n+n+1-i, & j=\frac{n}{2} \text{ and } i=n-3 \\ n^2+\left(\frac{n}{2}-2\right)n+i, & j=\frac{n}{2} \text{ and } i\in\{n,n-1,n-2\} \\ n^2+(j-1)\,n+n+1-i, & j\in\{\frac{n}{2}+1,\frac{n}{2}+2,,\ldots,n-1\} \text{ and } i\in\{1,2,\ldots,n\} \end{cases}$$

Vertex Sum: For all G_i, if $i\in\{1,2,\ldots,n-1\}$, then

$$\begin{aligned}
\sum_{v\in G_i}\lambda(v) &= \sum_{j=1}^{n-2}\lambda(v_{i,j})+\lambda(v_i)+\lambda(v_{i+1}) \\
&= \sum_{j=1}^{\frac{n}{2}-1}\lambda(v_{i,j})+\lambda(v_{i,\frac{n}{2}})+\sum_{j=\frac{n}{2}+1}^{n-2}\lambda(v_{i,j})+\lambda(v_i)+\lambda(v_{i+1}) \\
&= \sum_{j=1}^{\frac{n}{2}-1}((j-1)\,n+i)+\left(\frac{n}{2}n+\left\lfloor\frac{1}{i}\right\rfloor n+i-1\right)+\sum_{j=\frac{n}{2}+1}^{n-2}(jn+n+1-i) \\
&\quad +\left(\left(\frac{n}{2}-1\right)n+n+2-i+\left\lfloor\frac{1}{i}\right\rfloor(-n)\right)+\left(\left(\frac{n}{2}-1\right)n+n+2-(i+1)+\left\lfloor\frac{1}{i+1}\right\rfloor(-n)\right) \\
&= \left(\frac{1}{2}n+2i+\frac{1}{8}n\left(n^2-6n+4i\right)+n\left\lfloor\frac{1}{i}\right\rfloor-\frac{1}{2}ni-\frac{3}{4}n^2+\frac{3}{8}n^3-3\right) \\
&\quad +2n-2i-n\left\lfloor\frac{1}{i+1}\right\rfloor+2n\left(\frac{1}{2}n-1\right)-n\left\lfloor\frac{1}{i}\right\rfloor+3 \\
&= \frac{5}{2}n-n\left\lfloor\frac{1}{i+1}\right\rfloor+\frac{1}{8}n\left(n^2-6n+4i\right)+2n\left(\frac{1}{2}n-1\right)-\frac{1}{2}ni-\frac{3}{4}n^2+\frac{3}{8}n^3 \\
&= 2n-2x-n\left\lfloor\frac{1}{i+1}\right\rfloor+2n\left(\frac{1}{2}n-1\right)-n\left\lfloor\frac{1}{i}\right\rfloor+3 \\
&= \frac{1}{2}n\left(n^2-n+1\right)
\end{aligned}$$

If $i=n$, then

$$\begin{aligned}
\sum_{v\in G_n}\lambda(v) &= \sum_{j=1}^{n-2}\lambda(v_{n,j})+\lambda(v_n)+\lambda(v_1) \\
&= \sum_{j=1}^{\frac{n}{2}-1}\lambda(v_{i,j})+\lambda(v_{i,\frac{n}{2}})+\sum_{j=\frac{n}{2}+1}^{n-2}\lambda(v_{i,j})+\lambda(v_n)+\lambda(v_1) \\
&= \sum_{j=1}^{\frac{n}{2}-1}((j-1)\,n+n)+\left(\frac{n}{2}n+\left\lfloor\frac{1}{n}\right\rfloor n+n-1\right)+\sum_{j=\frac{n}{2}+1}^{n-2}(jn+n+1-n) \\
&\quad +\left(\left(\frac{n}{2}-1\right)n+n+2-n+\left\lfloor\frac{1}{n}\right\rfloor(-n)\right)+\left(\left(\frac{n}{2}-1\right)n+n+2-1+\left\lfloor\frac{1}{1}\right\rfloor(-n)\right) \\
&= \left(\frac{5}{2}n+\frac{1}{8}n^2(n-2)-\frac{5}{4}n^2+\frac{3}{8}n^3-3\right)+\left(2n\left(\frac{1}{2}n-1\right)+3\right) \\
&= \frac{5}{2}n+2n\left(\frac{1}{2}n-1\right)+\frac{1}{8}n^2(n-2)-\frac{5}{4}n^2+\frac{3}{8}n^3 \\
&= \frac{1}{2}n\left(n^2-n+1\right)
\end{aligned}$$

and for C_n,

$$\begin{aligned}\sum_{v\in C_n}\lambda(v) &= \sum_{i=1}^{n}\lambda(v_i)\\ &= \sum_{i=1}^{n}\left(\left(\frac{n}{2}-1\right)n+n+2-i+\left\lfloor\frac{1}{i}\right\rfloor(-n)\right)\\ &= \left(\frac{n}{2}-1\right)n^2+n^2+2n-\frac{n(n+1)}{2}-n\\ &= n+n^2\left(\frac{1}{2}n-1\right)-\frac{1}{2}n(n+1)+n^2\\ &= \frac{1}{2}n\left(n^2-n+1\right)\end{aligned}$$

The final formula for the sum of vertex labels for graphs G_i and C_n is

$$\sum\lambda(v)=\frac{1}{2}n\left(n^2-n+1\right)$$

Edge Sum: For G_i graphs, the following cases apply:

Case 1.1: If $n\equiv 0 \pmod 4$ and $i\equiv 0,1 \pmod 4$, then

$$\begin{aligned}\sum_{e\in G_i}\lambda(e) &= \sum_{j=1}^{n-1}\lambda(e_{i,j})+\lambda(e_i)\\ &= \sum_{j=1}^{\frac{n}{2}-1}\lambda(e_{i,j})+\lambda(e_{i,\frac{n}{2}})+\sum_{j=\frac{n}{2}+1}^{n-1}\lambda(e_{i,j})+\lambda(e_i)\\ &= \sum_{j=1}^{\frac{n}{2}-1}\left(n^2-n+(j-1)n+i\right)+\left(n^2+\left(\tfrac{n}{2}-1\right)n+n+1-i\right)\\ &\quad+\sum_{j=\frac{n}{2}+1}^{n-1}\left(n^2+(j-1)n+n+1-i\right)+\left(n^2+\left(\tfrac{n}{2}-2\right)n+i\right)\\ &= \tfrac{7}{2}n+n\left(\tfrac{1}{2}n-1\right)+n\left(\tfrac{1}{2}n-2\right)-2n^2+\tfrac{3}{2}n^3\\ &= \tfrac{1}{2}n\left(3n^2-2n+1\right)\end{aligned}$$

Case 1.2: If $n\equiv 0 \pmod 4$ and $i\equiv 2,3 \pmod 4$, then

$$\begin{aligned}\sum_{e\in G_i}\lambda(e) &= \sum_{j=1}^{n-1}\lambda(e_{i,j})+\lambda(e_i)\\ &= \sum_{j=1}^{\frac{n}{2}-1}\lambda(e_{i,j})+\lambda(e_{i,\frac{n}{2}})+\sum_{j=\frac{n}{2}+1}^{n-1}\lambda(e_{i,j})+\lambda(e_i)\\ &= \sum_{j=1}^{\frac{n}{2}-1}\left(n^2-n+(j-1)n+i\right)+\left(n^2+\left(\tfrac{n}{2}-2\right)n+i\right)\\ &\quad+\sum_{j=\frac{n}{2}+1}^{n-1}\left(n^2+(j-1)n+n+1-i\right)+\left(n^2+\left(\tfrac{n}{2}-1\right)n+n+1-i\right)\\ &= \tfrac{7}{2}n+n\left(\tfrac{1}{2}n-1\right)+n\left(\tfrac{1}{2}n-2\right)-2n^2+\tfrac{3}{2}n^3\\ &= \tfrac{1}{2}n\left(3n^2-2n+1\right)\end{aligned}$$

Case 2.1: If $n \equiv 2 \pmod 4$ and $i \equiv 0,1 \pmod 4$, $i = n-3$, and $i \notin \{n-1, n-2\}$, then

$$\begin{aligned}
\sum_{e\in G_i} \lambda(e) &= \sum_{j=1}^{n-1} \lambda(e_{i,j}) + \lambda(e_i) \\
&= \sum_{j=1}^{\frac{n}{2}-1} \lambda(e_{i,j}) + \lambda(e_{i,\frac{n}{2}}) + \sum_{j=\frac{n}{2}+1}^{n-1} \lambda(e_{i,j}) + \lambda(e_i) \\
&= \sum_{j=1}^{\frac{n}{2}-1} \left(n^2 - n + (j-1)n + i\right) + \left(n^2 + \left(\tfrac{n}{2}-1\right)n + n + 1 - i\right) \\
&\quad + \sum_{j=\frac{n}{2}+1}^{n-1} \left(n^2 + (j-1)n + n + 1 - i\right) + \left(n^2 + \left(\tfrac{n}{2}-2\right)n + i\right) \\
&= \tfrac{7}{2}n + n\left(\tfrac{1}{2}n - 1\right) + n\left(\tfrac{1}{2}n - 2\right) - 2n^2 + \tfrac{3}{2}n^3 \\
&= \tfrac{1}{2}n\left(3n^2 - 2n + 1\right)
\end{aligned}$$

Case 2.2: If $n \equiv 2 \pmod 4$ and $i \equiv 2,3 \pmod 4$, $i \in \{n-1, n-2\}$, and $i \neq n-3$, then

$$\begin{aligned}
\sum_{e\in G_i} \lambda(e) &= \sum_{j=1}^{n-1} \lambda(e_{i,j}) + \lambda(e_i) \\
&= \sum_{j=1}^{\frac{n}{2}-1} \lambda(e_{i,j}) + \lambda(e_{i,\frac{n}{2}}) + \sum_{j=\frac{n}{2}+1}^{n-1} \lambda(e_{i,j}) + \lambda(e_i) \\
&= \sum_{j=1}^{\frac{n}{2}-1} \left(n^2 - n + (j-1)n + i\right) + \left(n^2 + \left(\tfrac{n}{2}-2\right)n + i\right) \\
&\quad + \sum_{j=\frac{n}{2}+1}^{n-1} \left(n^2 + (j-1)n + n + 1 - i\right) + \left(n^2 + \left(\tfrac{n}{2}-1\right)n + n + 1 - i\right) \\
&= \tfrac{7}{2}n + n\left(\tfrac{1}{2}n - 1\right) + n\left(\tfrac{1}{2}n - 2\right) - 2n^2 + \tfrac{3}{2}n^3 \\
&= \tfrac{1}{2}n\left(3n^2 - 2n + 1\right)
\end{aligned}$$

For C_n,

Case 1: For $n \equiv 0 \pmod 4$:

$$\begin{aligned}
\sum_{e\in C_n} \lambda(e) &= \sum_{i=1}^{n} \lambda(e_i) \\
&= \lambda(e_1) + \lambda(e_2) + \cdots + \lambda(e_n) \\
&= \tfrac{n}{2}\lambda(e_1) + \tfrac{n}{2}\lambda(e_2) \\
&= \tfrac{1}{2}n\left(n\left(\tfrac{1}{2}n - 2\right) + n^2 + i\right) + \tfrac{1}{2}n\left(n + n\left(\tfrac{1}{2}n - 1\right) + n^2 + 1 - i\right) \\
&= \tfrac{n}{2}\left(3n^2 - 2n + 1\right)
\end{aligned}$$

Case 2: For $n \equiv 2 \pmod 4$: First, we calculate the last six terms separately:

$$\lambda(e_{n-5}) = n^2 + \left(\frac{n}{2} - 2\right)n + n - 5$$

$$\lambda(e_{n-4}) = n^2 + \left(\frac{n}{2} - 1\right)n + n + 1 - (n-4) = n^2 + \left(\frac{n}{2} - 1\right)n + 5$$

$$\lambda(e_{n-3}) = n^2 + \left(\frac{n}{2} - 2\right)n + n - 3$$

$$\lambda(e_{n-2}) = n^2 + \left(\frac{n}{2} - 1\right) n + n + 1 - (n-2) = n^2 + \left(\frac{n}{2} - 1\right) n + 3$$

$$\lambda(e_{n-1}) = n^2 + \left(\frac{n}{2} - 1\right) n + n + 1 - (n-1) = n^2 + \left(\frac{n}{2} - 1\right) n + 2$$

and

$$\lambda(e_n) = n^2 + \left(\frac{n}{2} - 1\right) n + n + 1 - n = n^2 + \left(\frac{n}{2} - 1\right) n + 1$$

Therefore,

$$\begin{aligned}
\sum_{e\in C_n} \lambda(e) &= \sum_{i=1}^{n} \lambda(e_i) \\
&= \lambda(e_1) + \lambda(e_2) + \cdots + \lambda(e_n) \\
&= \tfrac{n-6}{2}\lambda(e_1) + \tfrac{n-6}{2}\lambda(e_2) + \lambda(e_{n-5}) + \lambda(e_{n-4}) + \lambda(e_{n-3}) + \lambda(e_{n-2}) + \lambda(e_{n-1}) + \lambda(e_n) \\
&= \tfrac{n-6}{2}\left(n^2 + \left(\tfrac{n}{2} - 2\right) n + i\right) + \tfrac{n-6}{2}\left(n^2 + \left(\tfrac{n}{2} - 1\right) n + n + 1 - i\right) \\
&\quad + \left(n^2 + \left(\tfrac{n}{2} - 2\right) n + n - 5\right) + \left(n^2 + \left(\tfrac{n}{2} - 1\right) n + n + 1 - (n-4)\right) \\
&\quad + \left(n^2 + \left(\tfrac{n}{2} - 2\right) n + n - 3\right) + \left(n^2 + \left(\tfrac{n}{2} - 1\right) n + 3\right) \\
&\quad + \left(n^2 + \left(\tfrac{n}{2} - 1\right) n + 2\right) + \left(n^2 + \left(\tfrac{n}{2} - 1\right) n + 1\right) \\
&= \tfrac{1}{2} n \left(3n^2 - 2n + 1\right)
\end{aligned}$$

Similarly, the final formula for the sum of edge labels for both all G_i and C_n graphs is

$$\sum \lambda(e) = \frac{1}{2} n \left(3n^2 - 2n + 1\right)$$

Magic Sum:

$$\begin{aligned}
\mu(\lambda) &= \sum \lambda(v) + \sum \lambda(e) \\
&= \tfrac{1}{2} n \left(n^2 - n + 1\right) + \tfrac{1}{2} n \left(3n^2 - 2n + 1\right) \\
&= \tfrac{1}{2} n \left(4n^2 - 3n + 2\right)
\end{aligned}$$

Thus, this labeling function satisfies the properties required for a C_n-supermagic labeling, showing that such labeling exists for $Cl_{n,n}$ when n is even. □

Theorem 2.2. For odd n, $Cl_{n,n}$ has a C_n-supermagic labeling.

PROOF. Consider the following labeling function:

$$\lambda(v_i) = 2n + 1 - i, \quad i \in \{1, 2, \ldots, n\}$$

$$\lambda(v_{i,j}) = \begin{cases} i, & j = 1 \text{ and } i \in \{1, 2, \ldots, n\} \\ jn + i, & j \in \{2, 4, \ldots, n-3\} \text{ and } i \in \{1, 2, \ldots, n\} \\ (j+1)\, n + 1 - i, & j \in \{3, 5, \ldots, n-2\} \text{ and } i \in \{1, 2, \ldots, n\} \end{cases}$$

$$\lambda(e_i) = \begin{cases} n^2 - n + (n-3)\, n + 1 + i, & i \in \{1, 2, \ldots, n-1\} \\ n^2 - n + (n-3)\, n + 2 + 2n - 1, & i = n \end{cases}$$

and

$$\lambda(e_{i,j}) = \begin{cases} n^2 - n + (j-1)n + i, & j \in \{1,3,\ldots,n-4\} \text{ and } i \in \{1,2,\ldots,n\} \\ n^2 - n + jn + 1 - i, & j \in \{2,4,\ldots,n-3\} \text{ and } i \in \{1,2,\ldots,n\} \\ n^2 - n + (n-3)n + n + i, & j = n-2 \text{ and } i \in \{1,2,\ldots,n-1\} \\ n^2 - n + (n-3)n + 1, & j = n-2 \text{ and } i = n \\ 2n^2 - n + 1 - i, & j = n-1 \text{ and } i \in \{1,2,\ldots,n-1\} \\ n^2 - n + (n-3)n + 2n, & j = n-1 \text{ and } i = n \end{cases}$$

Vertex Sum: For all G_i, if $i \in \{1,2,\ldots,n-1\}$, then

$$\begin{aligned} \sum_{v \in G_i} \lambda(v) &= \sum_{j=1}^{n-2} \lambda(v_{i,j}) + \lambda(v_i) + \lambda(v_{i+1}) \\ &= \lambda(v_{i,1}) + \sum_{j=2,4,\cdots}^{n-3} \lambda(v_{i,j}) + \sum_{j=3,5,\cdots}^{n-2} \lambda(v_{i,j}) + \lambda(v_i) + \lambda(v_{i+1}) \\ &= i + \sum_{j=2,4,\cdots}^{n-3} (jn+i) + \sum_{j=3,5,\cdots}^{n-2} ((j+1)n + 1 - i) + (2n+1-i) + (2n+1-(i+1)) \\ &= \sum_{j=1}^{\frac{n-3}{2}} (2jn+i) + \sum_{j=1}^{\frac{n-3}{2}} (((2j+1)+1)n + 1 - i) + 4n + 1 - i \\ &= \tfrac{1}{2}n^3 - n^2 + 3n - \tfrac{1}{2} - i \end{aligned}$$

If $i = n$, then

$$\begin{aligned} \sum_{v \in G_n} \lambda(v) &= \sum_{j=1}^{n-2} \lambda(v_{n,j}) + \lambda(v_n) + \lambda(v_1) \\ &= \lambda(v_{n,1}) + \sum_{j=2,4,\cdots}^{n-3} \lambda(v_{n,j}) + \sum_{j=3,5,\cdots}^{n-2} \lambda(v_{n,j}) + \lambda(v_n) + \lambda(v_1) \\ &= \sum_{j=1}^{\frac{n-3}{2}} (2jn+i) + \sum_{j=1}^{\frac{n-3}{2}} (((2j+1)+1)n + 1 - i) + 4n + 1 \\ &= \tfrac{1}{2}n^3 - n^2 + 3n - \tfrac{1}{2} \end{aligned}$$

For C_n,

$$\begin{aligned} \sum_{v \in C_n} \lambda(v) &= \sum_{i=1}^{n} \lambda(v_i) \\ &= \sum_{i=1}^{n} (2n+1-i) \\ &= \tfrac{3}{2}n^2 + \tfrac{1}{2}n \end{aligned}$$

Edge Sum: For all G_i, if $i \in \{1,2,\ldots,n-1\}$, then

$$\begin{aligned} \sum_{e \in G_i} \lambda(e) &= \sum_{j=1}^{n-1} \lambda(e_{i,j}) + \lambda(e_i) \\ &= \sum_{j=1,3,\cdots}^{n-4} \lambda(e_{i,j}) + \sum_{j=2,4,\cdots}^{n-3} \lambda(e_{i,j}) + \lambda(e_{i,n-2}) + \lambda(e_{i,n-1}) + \lambda(e_i) \\ &= \sum_{j=1,3,\cdots}^{n-4} (n^2 - n + (j-1)n + i) + \sum_{j=2,4,\cdots}^{n-3} (n^2 - n + jn + 1 - i) \\ &\quad + (n^2 - n + (n-3)n + n + i) + (2n^2 - n + 1 - i) + (n^2 - n + (n-3)n + 1 + i) \end{aligned}$$

$$= \sum_{j=1}^{\frac{n-3}{2}} \left(n^2 - n + ((2j-1)-1)\,n + i\right) + \sum_{j=1}^{\frac{n-3}{2}} \left(n^2 - n + 2jn + 1 - i\right) + 6n^2 - 8n + 2 + i$$
$$= \tfrac{3}{2}n^3 - n^2 + \tfrac{1}{2} + i$$

If $i = n$, then

$$\sum_{e\in G_n} \lambda(e) = \sum_{j=1}^{n-1} \lambda(e_{n,j}) + \lambda(e_n)$$
$$= \sum_{j=1,3,\cdots}^{n-4} \lambda(e_{n,j}) + \sum_{j=1,3,\cdots}^{n-3} \lambda(e_{n,j}) + \lambda(e_{n,n-2}) + \lambda(e_{n,n-1}) + \lambda(e_n)$$
$$= \sum_{j=1,3,\cdots}^{n-4} \left(n^2 - n + (j-1)\,n + i\right) + \sum_{j=2,4,\cdots}^{n-3} \left(n^2 - n + jn + 1 - i\right) + \left(n^2 - n + (n-3)\,n + 1\right)$$
$$+ \left(n^2 - n + (n-3)\,n + 2n\right) + \left(n^2 - n + (n-3)\,n + 2 + 2n - 1\right)$$
$$= \sum_{j=1}^{\frac{n-3}{2}} \left(n^2 - n + ((2j-1)-1)\,n + i\right) + \sum_{j=1}^{\frac{n-3}{2}} \left(n^2 - n + 2jn + 1 - i\right) + 6n^2 - 8n + 2$$
$$= \tfrac{3}{2}n^3 - n^2 + \tfrac{1}{2}$$

For C_n,

$$\sum_{e\in C_n} \lambda(e) = \sum_{i=1}^{n} \lambda(e_i)$$
$$= \sum_{i=1}^{n-1} \lambda(e_i) + \lambda(e_n)$$
$$= \sum_{i=1}^{n-1} \left(n^2 - n + (n-3)\,n + 1 + i\right) + \left(n^2 - n + (n-3)\,n + 2 + 2n - 1\right)$$
$$= \tfrac{1}{2}n\left(4n^2 - 7n + 5\right)$$

Magic Sum: For all G_i, if $i \in \{1, 2, \ldots, n-1\}$, then

$$\mu(\lambda) = \sum_{v\in G_i} \lambda(v) + \sum_{e\in G_i} \lambda(e)$$
$$= \left(\tfrac{1}{2}n^3 - n^2 + 3n - \tfrac{1}{2} - i\right) + \left(\tfrac{3}{2}n^3 - n^2 + \tfrac{1}{2} + i\right)$$
$$= 2n^3 - 2n^2 + 3n$$

If $i = n$, then

$$\mu(\lambda) = \sum_{v\in G_n} \lambda(v) + \sum_{e\in G_n} \lambda(e)$$
$$= \left(\tfrac{1}{2}n^3 - n^2 + 3n - \tfrac{1}{2}\right) + \left(\tfrac{3}{2}n^3 - n^2 + \tfrac{1}{2}\right)$$
$$= 2n^3 - 2n^2 + 3n$$

For C_n,

$$\mu(\lambda) = \sum_{v\in C_n} \lambda(v) + \sum_{e\in C_n} \lambda(e)$$
$$= \left(\tfrac{3}{2}n^2 + \tfrac{1}{2}n\right) + \left(\tfrac{1}{2}n\left(4n^2 - 7n + 5\right)\right)$$
$$= 2n^3 - 2n^2 + 3n$$

Therefore, the proposed labeling function satisfies all the necessary conditions for C_n-supermagic labeling. This establishes the existence of a C_n-supermagic labeling for $Cl_{n,n}$ when n is odd. □

Corollary 2.3. $Cl_{n,n}$ has a C_n-supermagic labeling.

Case II: If $n \neq m$, then the C_m-supermagic labeling of $Cl_{n,m}$ graph has been examined separately for the cases $m = 3$, $m > 3$ and m odd, and $m > 3$ and m even.

Theorem 2.4. If $n \neq m = 3$, then $Cl_{n,3}$ has a C_3-supermagic labeling.

Proof. Consider the following labeling function:

$$\lambda(v_i) = i, \quad i \in \{1, 2, \dots, n\}$$

$$\lambda(v_{i,1}) = 2n + 1 - i, \quad i \in \{1, 2, \dots, n\}$$

$$\lambda(e_i) = \begin{cases} 3n - i, & i \in \{1, 2, \dots, n-1\} \\ 3n, & i = n \end{cases}$$

and

$$\lambda(e_{i,j}) = \begin{cases} 3n + i, & j = 1 \text{ and } i \in \{1, 2, \dots, n\} \\ 5n + 1 - i, & j = 2 \text{ and } i \in \{1, 2, \dots, n\} \end{cases}$$

Vertex Sum: For all G_i, if $i \in \{1, 2, \dots, n-1\}$, then

$$\begin{aligned} \sum_{v \in G_i} \lambda(v) &= \lambda(v_i) + \lambda(v_{i+1}) + \lambda(v_{i,1}) \\ &= i + (i+1) + (2n + 1 - i) \\ &= 2n + 2 + i \end{aligned}$$

If $i = n$, then

$$\begin{aligned} \sum_{v \in G_n} \lambda(v) &= \lambda(v_n) + \lambda(v_1) + \lambda(v_{n,1}) \\ &= n + 1 + (n+1) \\ &= 2n + 2 \end{aligned}$$

Edge Sum: For all G_i, if $i \in \{1, 2, \dots, n-1\}$, then

$$\begin{aligned} \sum_{e \in G_i} \lambda(e) &= \lambda(e_{i,1}) + \lambda(e_{i,2}) + \lambda(e_i) \\ &= (3n + i) + (5n + 1 - i) + (3n - i) \\ &= 11n + 1 - i \end{aligned}$$

If $i = n$, then

$$\begin{aligned} \sum_{e \in G_n} \lambda(e) &= \lambda(e_{n,1}) + \lambda(e_{n,2}) + \lambda(e_n) \\ &= (4n) + (4n + 1) + (3n) \\ &= 11n + 1 \end{aligned}$$

Magic Sum: For all G_i, if $i \in \{1, 2, \dots, n-1\}$, then

$$\begin{aligned} \mu(\lambda) &= \sum_{v \in G_i} \lambda(v) + \sum_{e \in G_i} \lambda(e) \\ &= (2n + 2 + i) + (11n + 1 - i) \\ &= 13n + 3 \end{aligned}$$

If $i = n$, then

$$\begin{aligned}\mu(\lambda) &= \sum_{v\in G_n} \lambda(v) + \sum_{e\in G_n} \lambda(e)\\ &= (2n+2) + (11n+1)\\ &= 13n+3\end{aligned}$$

Thus, the labeling function satisfies the conditions for a C_3-supermagic labeling, proving the theorem. □

Theorem 2.5. If $n \neq m$, $m > 3$, and m is odd, then $Cl_{n,m}$ has a C_m-supermagic labeling.

Proof. Consider the following labeling function:

$$\lambda(v_i) = i, \quad i \in \{1,2,\ldots,n\}$$

$$\lambda(v_{i,1}) = 2n+1-i, \quad i \in \{1,2,\ldots,n\}$$

$$\lambda(v_{i,2}) = \begin{cases} 3n-i, & i \in \{1,2,\ldots,n-1\} \\ 3n, & i = n \end{cases}$$

$$\lambda(v_{i,j}) = \begin{cases} jn+i, & i \in \{1,2,\ldots,n\} \text{ and } j \in \{3,5,\ldots,m-2\} \\ (j+1)n+1-i, & i \in \{1,2,\ldots,n\} \text{ and } j \in \{4,6,\ldots,m-3\} \end{cases}$$

$$\lambda(e_i) = mn+1-i, \quad i \in \{1,2,\ldots,n\}$$

and

$$\lambda(e_{i,j}) = \begin{cases} (m+j-1)n+i, & i \in \{1,2,\ldots,n\} \text{ and } j \in \{1,3,5,\ldots,m-2\} \\ (m+j)n+1-i, & i \in \{1,2,\ldots,n\} \text{ and } j \in \{2,4,6,\ldots,m-1\} \end{cases}$$

Vertex Sum: For all G_i, if $i \in \{1,2,\ldots,n-1\}$, then

$$\begin{aligned}\sum_{v\in G_i} \lambda(v) &= \lambda(v_{i,1}) + \lambda(v_{i,2}) + \sum_{j=3,5,\cdots}^{m-2} \lambda(v_{i,j}) + \sum_{j=4,6,\cdots}^{m-3} \lambda(v_{i,j}) + \lambda(v_i) + \lambda(v_{i+1})\\ &= (2n+1-i) + (3n-i) + \sum_{j=3,5,\cdots}^{m-2} (jn+i) + \sum_{j=4,6,\cdots}^{m-3} ((j+1)n+1-i) + (i) + (i+1)\\ &= 5n+2 + \sum_{j=1}^{\frac{m-3}{2}} ((2j+1)n+i) + \sum_{j=1}^{\frac{m-5}{2}} ((2j+2)n+1-i)\\ &= \tfrac{1}{2}m + 3n + \tfrac{1}{2}m^2n - \tfrac{3}{2}mn - \tfrac{1}{2} + i\end{aligned}$$

If $i = n$, then

$$\begin{aligned}\sum_{v\in G_n} \lambda(v) &= \lambda(v_{n,1}) + \lambda(v_{n,2}) + \sum_{j=3,5,\cdots}^{m-2} \lambda(v_{n,j}) + \sum_{j=4,6,\cdots}^{m-3} \lambda(v_{n,j}) + \lambda(v_n) + \lambda(v_1)\\ &= (n+1) + (3n) + \sum_{j=3,5,\cdots}^{m-2} (jn+n) + \sum_{j=4,6,\cdots}^{m-3} ((j+1)n+1-n) + n+1\\ &= 5n+2 + \sum_{j=1}^{\frac{m-3}{2}} ((2j+1)n+n) + \sum_{j=1}^{\frac{m-5}{2}} ((2j+2)n+1-n)\\ &= \frac{1}{2}m + 4n + \frac{1}{2}m^2n - \frac{3}{2}mn - \frac{1}{2}\end{aligned}$$

Edge Sum: For all G_i, if $i \in \{1, 2, \ldots, n-1\}$, then

$$\begin{aligned}
\sum_{e \in G_i} \lambda(e) &= \sum_{j=1}^{m-1} \lambda(e_{i,j}) + \lambda(e_i) \\
&= \sum_{j=1,3,\cdots}^{m-2} \lambda(e_{i,j}) + \sum_{j=2,4,\cdots}^{m-1} \lambda(e_{i,j}) + \lambda(e_i) \\
&= \sum_{j=1,3,\cdots}^{m-2} ((m+j-1)n+i) + \sum_{j=2,4,\cdots}^{m-1} ((m+j)n+1-i) + mn + 1 - i \\
&= \sum_{j=1}^{\frac{m-1}{2}} ((m+(2j-1)-1)n+i) + \sum_{j=1}^{\frac{m-1}{2}} ((m+2j)n+1-i) + mn + 1 - i \\
&= \frac{1}{2}m + \frac{1}{2}n + \frac{3}{2}m^2 n - mn + \frac{1}{2} - i
\end{aligned}$$

If $i = n$, then

$$\begin{aligned}
\sum_{e \in G_n} \lambda(e) &= \sum_{j=1}^{m-1} \lambda(e_{n,j}) + \lambda(e_n) \\
&= \sum_{j=1,3,\cdots}^{m-2} \lambda(e_{n,j}) + \sum_{j=2,4,\cdots}^{m-1} \lambda(e_{n,j}) + \lambda(e_n) \\
&= \sum_{j=1,3,\cdots}^{m-2} ((m+j-1)n+n) + \sum_{j=2,4,\cdots}^{m-1} ((m+j)n+1-n) + mn + 1 - n \\
&= \sum_{j=1}^{\frac{m-1}{2}} ((m+(2j-1)-1)n+n) + \sum_{j=1}^{\frac{m-1}{2}} ((m+2j)n+1-n) + mn + 1 - n \\
&= \frac{1}{2}m - \frac{1}{2}n + \frac{3}{2}m^2 n - mn + \frac{1}{2}
\end{aligned}$$

Magic Sum: For all G_i, if $i \in \{1, 2, \ldots, n-1\}$, then

$$\begin{aligned}
\mu(\lambda) &= \sum_{v \in G_i} \lambda(v) + \sum_{e \in G_i} \lambda(e) \\
&= \left(\frac{1}{2}m + 3n + \frac{1}{2}m^2 n - \frac{3}{2}mn - \frac{1}{2} + i\right) + \left(\frac{1}{2}m + \frac{1}{2}n + \frac{3}{2}m^2 n - mn + \frac{1}{2} - i\right) \\
&= m + \frac{7}{2}n + 2m^2 n - \frac{5}{2}mn
\end{aligned}$$

If $i = n$, then

$$\begin{aligned}
\mu(\lambda) &= \sum_{v \in G_n} \lambda(v) + \sum_{e \in G_n} \lambda(e) \\
&= \left(\frac{1}{2}m + 4n + \frac{1}{2}m^2 n - \frac{3}{2}mn - \frac{1}{2}\right) + \left(\frac{1}{2}m - \frac{1}{2}n + \frac{3}{2}m^2 n - mn + \frac{1}{2}\right) \\
&= m + \frac{7}{2}n + 2m^2 n - \frac{5}{2}mn
\end{aligned}$$

Thus, the labeling satisfies the conditions of a C_m-supermagic labeling for $Cl_{n,m}$. □

Theorem 2.6. If $n \neq m$, $m > 3$, and m is even, then $Cl_{n,m}$ has a C_m-supermagic labeling.

Proof. Consider the following labeling function:

$$\lambda(v_i) = i, \quad i \in \{1, 2, \ldots, n\}$$

$$\lambda(v_{i,1}) = 2n + 1 - i, \quad i \in \{1, 2, \ldots, n\}$$

$$\lambda\left(v_{i,2}\right)=\begin{cases}3n-i, & i\in\{1,2,\ldots,n-1\}\\ 3n, & i=n\end{cases}$$

$$\lambda\left(v_{i,j}\right)=\begin{cases}jn+i, & i\in\{1,2,\ldots,n\}\text{ and } j\in\{3,5,\ldots,m-3\}\\ (j+1)\,n+1-i, & i\in\{1,2,\ldots,n\}\text{ and } j\in\{4,6,\ldots,m-2\}\end{cases}$$

$$\lambda\left(e_i\right)=(m-1)\,n+i,\quad i\in\{1,2,\ldots,n\}$$

and

$$\lambda\left(e_{i,j}\right)=\begin{cases}(m+j)\,n+1-i, & i\in\{1,2,\ldots,n\}\text{ and } j\in\{1,3,5,\ldots,m-1\}\\ (m+j-1)\,n+i, & i\in\{1,2,\ldots,n\}\text{ and } j\in\{2,4,6,\ldots,m-2\}\end{cases}$$

Vertex Sum: For all G_i, if $i\in\{1,2,\ldots,n-1\}$, then

$$\begin{aligned}\sum_{v\in G_i}\lambda(v)&=\lambda\left(v_{i,1}\right)+\lambda\left(v_{i,2}\right)+\sum_{j=3,5,\cdots}^{m-3}\lambda\left(v_{i,j}\right)+\sum_{j=4,6,\cdots}^{m-2}\lambda\left(v_{i,j}\right)+\lambda\left(v_i\right)+\lambda\left(v_{i+1}\right)\\&=(2n+1-i)+(3n-i)+\sum_{j=3,5,\cdots}^{m-3}(jn+i)+\sum_{j=4,6,\cdots}^{m-2}((j+1)\,n+1-i)+(i)+(i+1)\\&=5n+2+\sum_{j=1}^{\frac{m-4}{2}}((2j+1)\,n+i)+\sum_{j=1}^{\frac{m-4}{2}}((2j+2)\,n+1-i)\\&=\frac{1}{2}m+3n+\frac{1}{2}m^2n-\frac{3}{2}mn\end{aligned}$$

If $i=n$, then

$$\begin{aligned}\sum_{v\in G_n}\lambda(v)&=\lambda\left(v_{n,1}\right)+\lambda\left(v_{n,2}\right)+\sum_{j=3,5,\cdots}^{m-3}\lambda\left(v_{n,j}\right)+\sum_{j=4,6,\cdots}^{m-2}\lambda\left(v_{n,j}\right)+\lambda\left(v_n\right)+\lambda\left(v_1\right)\\&=(n+1)+(3n)+\sum_{j=3,5,\cdots}^{m-3}(jn+n)+\sum_{j=4,6,\cdots}^{m-2}((j+1)\,n+1-n)+n+1\\&=5n+2+\sum_{j=1}^{\frac{m-4}{2}}((2j+1)\,n+n)+\sum_{j=1}^{\frac{m-4}{2}}((2j+2)\,n+1-n)\\&=\frac{1}{2}m+3n+\frac{1}{2}m^2n-\frac{3}{2}mn\end{aligned}$$

Edge Sum: For all G_i, if $i\in\{1,2,\ldots,n-1\}$, then

$$\begin{aligned}\sum_{e\in G_i}\lambda(e)&=\sum_{j=1}^{m-1}\lambda\left(e_{i,j}\right)+\lambda\left(e_i\right)\\&=\sum_{j=1,3,\cdots}^{m-1}\lambda\left(e_{i,j}\right)+\sum_{j=2,4,\cdots}^{m-2}\lambda\left(e_{i,j}\right)+\lambda\left(e_i\right)\\&=\sum_{j=1,3,\cdots}^{m-1}((m+j)\,n+1-i)+\sum_{j=2,4,\cdots}^{m-2}((m+j-1)\,n+i)+(m-1)\,n+i\\&=\sum_{j=1}^{\frac{m}{2}}((m+(2j-1))\,n+1-i)+\sum_{j=1}^{\frac{m-2}{2}}((m+2j-1)\,n+i)+(m-1)\,n+i\\&=\frac{1}{2}m+n+n\,(m-1)+\frac{3}{2}m^2n-2mn\end{aligned}$$

If $i = n$, then

$$\begin{aligned}\sum_{e\in G_i}\lambda(e) &= \sum_{j=1}^{m-1}\lambda(e_{n,j})+\lambda(e_n)\\ &= \sum_{j=1,3,\cdots}^{m-1}\lambda(e_{n,j})+\sum_{j=2,4,\cdots}^{m-2}\lambda(e_{n,j})+\lambda(e_n)\\ &= \sum_{j=1,3,\cdots}^{m-1}((m+j)n+1-n)+\sum_{j=2,4,\cdots}^{m-2}((m+j-1)n+n)+(m-1)n+n\\ &= \sum_{j=1}^{\frac{m}{2}}((m+(2j-1))n+1-n)+\sum_{j=1}^{\frac{m-2}{2}}((m+2j-1)n+n)+(m-1)n+n\\ &= \frac{1}{2}m+n+n(m-1)+\frac{3}{2}m^2n-2mn\end{aligned}$$

Magic Sum: For all G_i, if $i \in \{1, 2, \ldots, n-1\}$, then

$$\begin{aligned}\mu(\lambda) &= \sum_{v\in G_i}\lambda(v)+\sum_{e\in G_i}\lambda(e)\\ &= \left(\frac{1}{2}m+3n+\frac{1}{2}m^2n-\frac{3}{2}mn\right)+\left(\frac{1}{2}m+n+n(m-1)+\frac{3}{2}m^2n-2mn\right)\\ &= m+4n+n(m-1)+2m^2n-\frac{7}{2}mn\end{aligned}$$

If $i = n$, then

$$\begin{aligned}\mu(\lambda) &= \sum_{v\in G_n}\lambda(v)+\sum_{e\in G_n}\lambda(e)\\ &= \left(\frac{1}{2}m+3n+\frac{1}{2}m^2n-\frac{3}{2}mn\right)+\left(\frac{1}{2}m+n+n(m-1)+\frac{3}{2}m^2n-2mn\right)\\ &= m+4n+n(m-1)+2m^2n-\frac{7}{2}mn\end{aligned}$$

Thus, completing the proof, we have demonstrated that the labeling function satisfies the conditions for a C_m-supermagic labeling for $Cl_{n,m}$. □

Corollary 2.7. $Cl_{n,m}$ has a C_m-supermagic labeling.

3. Illustrative Examples

In this section, we illustrate the application of the proposed theorems by providing explicit constructions and C_m-supermagic labelings for various circulant graphs $Cl_{n,m}$. Each example demonstrates the labeling process, verifying the derived properties and values such as $\mu(\lambda)$.

Example 3.1. Consider the graph $Cl_{4,4}$. By Theorem 2.1, the magic constant $\mu(\lambda) = 108$, and the C_4-supermagic labeling of $Cl_{4,4}$ is illustrated in Figure 2 and detailed in Tables 1 and 2.

Table 1. Labeling for the vertices and edges of G_i in $Cl_{4,4}$

G_i	v_i	v_{i+1}	$v_{i,1}$	$v_{i,2}$	e_i	$e_{i,1}$	$e_{i,2}$	$e_{i,3}$
G_1	5	8	1	12	17	13	24	28
G_2	8	7	2	9	23	14	18	27
G_3	7	6	3	10	22	15	19	26
G_4	6	5	4	11	20	16	21	25

Table 2. Labeling for the vertices and edges of C_4 in $Cl_{4,4}$

	v_1	v_2	v_3	v_4	e_1	e_2	e_3	e_4
C_4	5	8	7	6	17	23	22	20

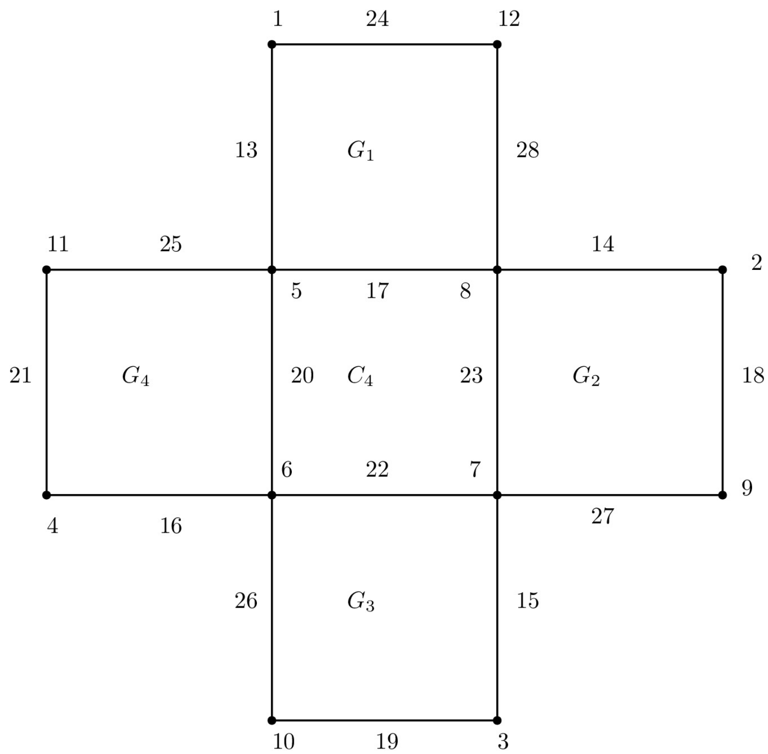


Figure 2. C_4-supermagic labeling of $Cl_{4,4}$

Example 3.2. Consider the graph $Cl_{6,6}$. By Theorem 2.1, the magic constant $\mu(\lambda) = 384$, and the C_6-supermagic labeling of $Cl_{6,6}$ is illustrated in Figure 6 and detailed in Tables 3 and 4.

Table 3. Labeling for the vertices and edges of G_i in $Cl_{6,6}$

	v_i	v_{i+1}	$v_{i,1}$	$v_{i,2}$	$v_{i,3}$	$v_{i,4}$	e_i	$e_{i,1}$	$e_{i,2}$	$e_{i,3}$	$e_{i,4}$	$e_{i,5}$
G_1	13	18	1	7	24	30	43	31	37	54	60	66
G_2	18	17	2	8	19	29	53	32	38	44	59	65
G_3	17	16	3	9	20	28	45	33	39	52	58	64
G_4	16	15	4	10	21	27	51	34	40	46	57	63
G_5	15	14	5	11	22	26	50	35	41	47	56	62
G_6	14	13	6	12	23	25	49	36	42	48	55	61

Table 4. Labeling for the vertices and edges of C_6 in $Cl_{6,6}$

	v_1	v_2	v_3	v_4	v_5	v_6	e_1	e_2	e_3	e_4	e_5	e_6
C_6	13	18	17	16	15	14	43	53	45	51	50	49

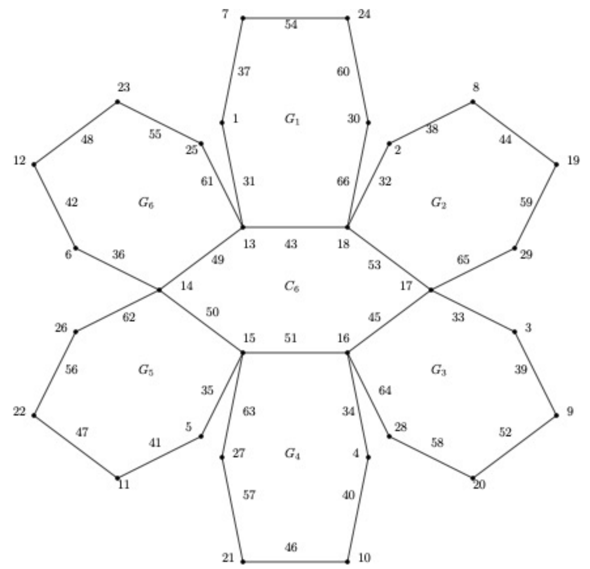


Figure 3. C_6-supermagic labeling of $Cl_{6,6}$

Example 3.3. Consider the graph $Cl_{5,5}$. By Theorem 2.2, the magic constant $\mu(\lambda) = 215$, and the C_5-supermagic labeling of $Cl_{5,5}$ is illustrated in Figure 4 and detailed in Tables 5 and 6.

Table 5. Labeling for the vertices and edges of G_i in $Cl_{5,5}$

	v_i	v_{i+1}	$v_{i,1}$	$v_{i,2}$	$v_{i,3}$	e_i	$e_{i,1}$	$e_{i,2}$	$e_{i,3}$	$e_{i,4}$
G_1	10	9	1	11	20	32	21	30	36	45
G_2	9	8	2	12	19	33	22	29	37	44
G_3	8	7	3	13	18	34	23	28	38	43
G_4	7	6	4	14	17	35	24	27	39	42
G_5	6	10	5	15	16	41	25	26	31	40

Table 6. Labeling for the vertices and edges of C_5 in $Cl_{5,5}$

	v_1	v_2	v_3	v_4	v_5	e_1	e_2	e_3	e_4	e_5
C_5	10	9	8	7	6	32	33	34	35	41

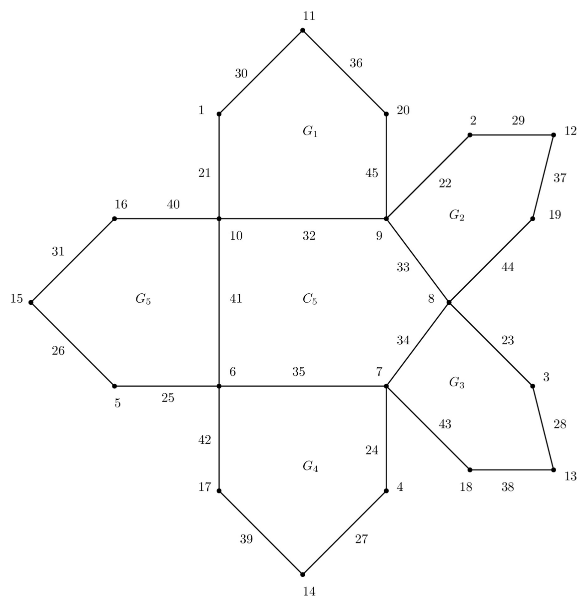


Figure 4. C_5-supermagic labeling of $Cl_{5,5}$

Example 3.4. Consider the graph $Cl_{5,3}$. By Theorem 2.4, the magic constant $\mu(\lambda) = 68$, and the C_3-supermagic labeling of $Cl_{5,3}$ is illustrated in Figure 5 and detailed in Tables 7 and 8.

Table 7. Labeling for the vertices and edges of G_i in $Cl_{5,3}$

	v_i	v_{i+1}	$v_{1,1}$	e_i	$e_{i,1}$	$e_{i,2}$
G_1	1	2	10	14	16	25
G_2	2	3	9	13	17	24
G_3	3	4	8	12	18	23
G_4	4	5	7	11	19	22
G_5	5	1	6	15	20	21

Table 8. Labeling for the vertices and edges of C_5 in $Cl_{5,3}$

	v_1	v_2	v_3	v_4	v_5	e_1	e_2	e_3	e_4	e_5
C_5	1	2	3	4	5	14	13	12	11	15

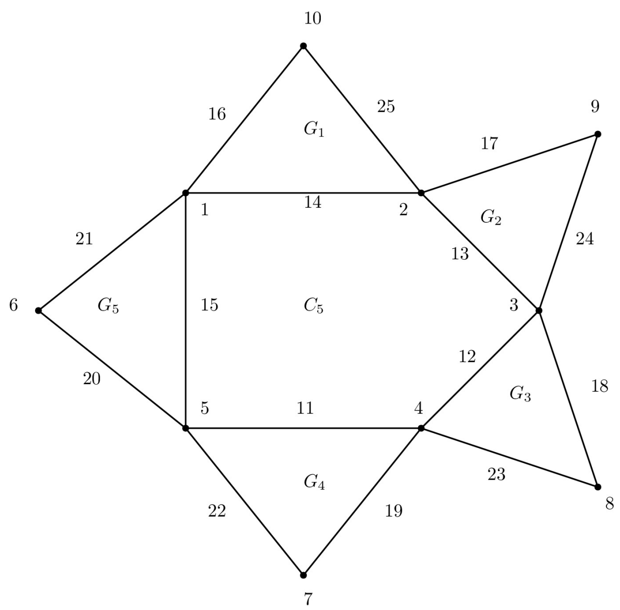


Figure 5. C_3-supermagic labeling of $Cl_{5,3}$

Example 3.5. Consider the graph $Cl_{4,5}$. By Theorem 2.5, the magic constant $\mu(\lambda) = 169$, and the C_5-supermagic labeling of $Cl_{4,5}$ is illustrated in Figure 6 and detailed in Tables 9 and 10.

Table 9. Labeling for the vertices and edges of G_i in $Cl_{4,5}$

	v_i	v_{i+1}	$v_{i,1}$	$v_{i,2}$	$v_{i,3}$	e_i	$e_{i,1}$	$e_{i,2}$	$e_{i,3}$	$e_{i,4}$
G_1	1	2	8	11	13	20	21	28	29	36
G_2	2	3	7	10	14	19	22	27	30	35
G_3	3	4	6	9	15	18	23	26	31	34
G_4	4	1	5	12	16	17	24	25	32	33

Table 10. Labeling for the vertices and edges of C_4 in $Cl_{4,5}$

	v_1	v_2	v_3	v_4	e_1	e_2	e_3	e_4
C_4	1	2	3	4	20	19	18	17

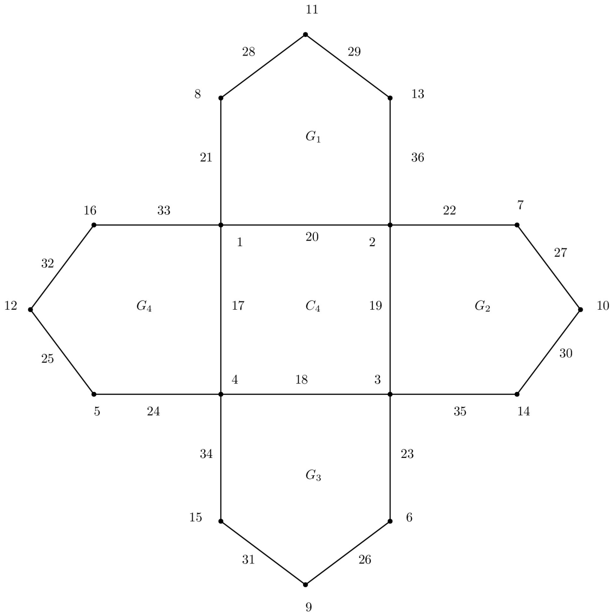


Figure 6. C_5-supermagic labeling of $Cl_{4,5}$

Example 3.6. Consider the graph $Cl_{5,4}$. By Theorem 2.6, $\mu(\lambda) = 129$ and C_4-supermagic labeling of $Cl_{5,4}$ s illustrated in Figure 7 and detailed in Tables 11 and 12.

Table 11. Labeling for the vertices and edges of G_i in $Cl_{5,4}$

	v_i	v_{i+1}	$v_{i,1}$	$v_{i,2}$	e_i	$e_{i,1}$	$e_{i,2}$	$e_{i,3}$
G_1	1	2	10	14	16	25	26	35
G_2	2	3	9	13	17	24	27	34
G_3	3	4	8	12	18	23	28	33
G_4	4	5	7	11	19	22	29	32
G_5	5	1	6	15	20	21	30	31

Table 12. Labeling for the vertices edges of C_5 in $Cl_{5,4}$

	v_1	v_2	v_3	v_4	v_5	e_1	e_2	e_3	e_4	e_5
C_5	1	2	3	4	5	16	17	18	19	20

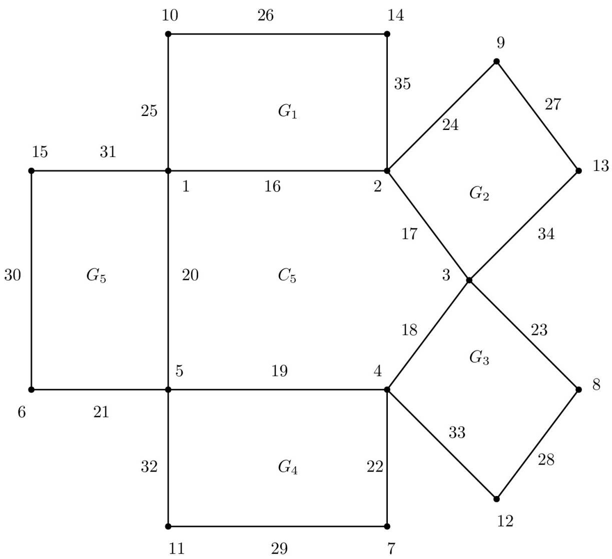
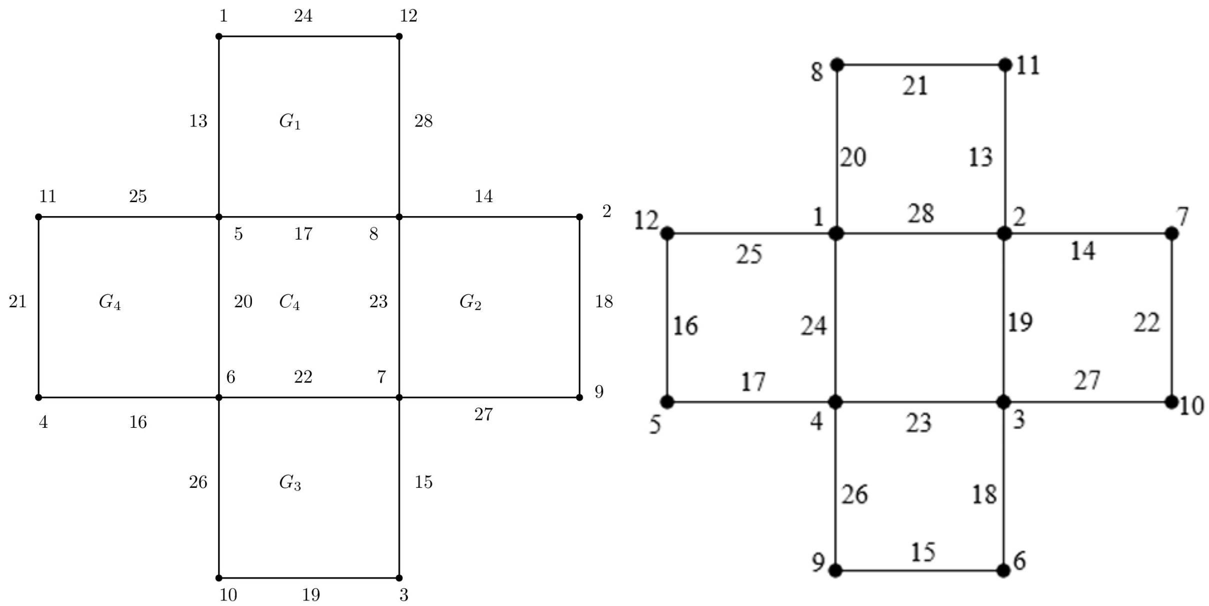


Figure 7. C_4-supermagic labeling of $Cl_{5,4}$

Remark 3.7. To illustrate the difference between the labeling functions in this paper and [9], we provide Figure 8. It can be observed that the edges and vertices are labeled differently.



(a) In this paper **(b)** In [9]

Figure 8. C_4-supermagic labelings of $Cl_{4,4}$

4. Conclusion

This study successfully investigated the C_m-supermagic properties of the graph $Cl_{n,m}$, constructed by combining a cycle graph C_n with n copies of the cycle graph C_m. Through rigorous analysis and explicit constructions, we demonstrated that $Cl_{n,m}$ exhibits C_m-supermagic labeling under specific conditions, depending on the parity and relationship between n and m. The key results of this study are summarized as follows:

i. When $n = m$,

- For even n, $Cl_{n,n}$ admits a C_n-supermagic labeling, as proven in Theorem 2.1.
- For odd n, $Cl_{n,n}$ also admits a C_n-supermagic labeling, as shown in Theorem 2.2.

ii. When $n \neq m$,

- If $m = 3$, then $Cl_{n,3}$ has a C_3-supermagic labeling, as established in Theorem 2.4.
- If $m > 3$ and m is odd, then $Cl_{n,m}$ has a C_m-supermagic labeling, as proven in Theorem 2.5.
- If $m > 3$ and m is even, then $Cl_{n,m}$ also has a C_m-supermagic labeling, as demonstrated in Theorem 2.6.

To illustrate the practical application of these theoretical results, Section 3 provided detailed constructions of C_m-supermagic labelings for various graphs, including $Cl_{4,4}$, $Cl_{6,6}$, $Cl_{5,5}$, $Cl_{5,3}$, $Cl_{4,5}$, and $Cl_{5,4}$. Each example includes the following:

i. Explicit labeling of vertices and edges, presented in tables for clarity.

ii. Visualization of the labeled graphs through figures, demonstrating the structure and labeling patterns.

iii. Calculation of the magic constant $\mu(\lambda)$, verifying the consistency of the labeling across all subgraphs isomorphic to C_m.

These examples validate the theoretical findings and provide a practical guide for constructing C_m-supermagic labelings in similar graph structures. They highlight the proposed labeling methods' versatility and applicability to graphs of varying sizes and configurations.

The findings of this study open up numerous avenues for future research, both in theoretical graph theory and practical applications. One of these directions is the extension of C_m-supermagic labeling to more complex graph structures, such as graphs constructed by combining paths, trees, or complete graphs with cycles, as well as higher-dimensional or multi-layered graph structures like grid graphs, toroidal graphs, or hypergraphs. Additionally, investigating H-supermagic labeling for graph products, such as Cartesian products or tensor products of graphs, could yield new insights into the properties of these combined structures. Furthermore, the study could be extended to other types of magic labelings, such as edge-magic or vertex-magic labelings. Their properties in $Cl_{n,m}$ and related graphs could be investigated alongside irregular labelings, where vertex or edge weights are distinct, and labelings that satisfy additional constraints, such as parity conditions, modular arithmetic, or bounds on label values.

Author Contributions

All the authors contributed equally to this work. They also read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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