Yuzuncu Yil University
Journal of the Institute of Natural & Applied Sciences
https://dergipark.org.tr/en/pub/yyufbed

Research Article

The Specific Energy and Specific Angular Momentum: On Special Tube Surfaces in G_3

Fatma ALMAZ

Batman University, Faculty of Science and Art, Mathematics, 72100, Batman, Türkiye
*Corresponding author e-mail: fatma.almaz@batman.edu.tr

Abstract: In this paper, some features on the tube surface generated by rectifying curves are expressed in Galilean 3-space, and Clairaut's theorem is generalized on this surface in Galilean space. Furthermore, the specific kinetic energy and the specific angular momentum are expressed on tube surface formed by rectifying curves that are geodesics obtained with the help of Clairaut's theorem.

Keywords: Clairaut's theorem, Galilean space, Rectifying curves, Specific angular momentum, Specific kinetic energy, Tubular surfaces

Spesifik Enerji ve Spesifik Açısal Momentum: G_3 de Özel Tüp Yüzeyler

Öz: Bu çalışmada, rektifiye eğriler ile oluşturulan tüp yüzeyindeki bazı özellikler Galilean 3-uzayında ifade edilmiş ve Galilean uzayında Clairaut teoremi bu yüzey üzerinde genelleştirilmiştir. Ayrıca Clairaut teoremi yardımıyla elde edilen jeodezik rektifiye eğriler ile oluşturulan tüp yüzeyinde spesifik kinetik enerji ve spesifik açısal momentum ifade edilmiştir.

Anahtar Kelimeler: Clairaut's teoremi, Galilean uzayı, Rektifiye eğriler, Spesifik açısal momentum, Spesifik kinetik enerji, Tubular yüzeyler

1. Introduction

Given Newton's third law, for every motion there is an equal and opposite reaction, such that the normal force opposes the weight of the object due to gravity. This force is always perpendicular to the contact surface. When a force and acceleration are considered together, it is known that the normal force is perpendicular to the velocity of the particle. Also, the motion is very important in terms of its energy and angular momentum, and from a physical point of view the energy of the particle is constant (Walecka, 2007; Saad & Low, 2014). Therefore, its energy and specific energy must be constant and the speed is constant.

Furthermore, the rectifying curves play some important roles in mechanics, kinematics. For example, the position vector of a rectifying curve is in the direction of the Darboux vector. Hence, the rectifying curves can be expressed kinematically as those curves whose position vector field determines the axis of rotation at each point of the curve. Also, a curve is said to be geodesic if its curvature is equally zero, the geodesic equations are expressed by motion constancy in the form of energy and the equation of motion containing the energy and angular momentum is natural topics that has been considered in many books (Röschel, 1984; Röschel, 1986; Kuhnel, 2006; Pressley, 2010; Walecka, 2013).

Received: 27.01.2025
Accepted: 10.07.2025

How to cite: Almaz, F. (2025). The specific energy and specific angular momentum: On special tube surfaces in G_3. *Yuzuncu Yil University Journal of the Institute of Natural and Applied Sciences*, *30*(2), 572-583. https://doi.org/10.53433/yyufbed.1627975

In such a sense, some mathematical characterizations on rotational surfaces are given in G_3 (Almaz & Kulahci, 2021), the definition of tube surfaces in Galilean space and differential features of tube surfaces are given (Almaz & Kulahci, 2022) and some mathematical studies are made on special tubular surface (Almaz & Kulahci, 2020). The authors analyzed the problem of constructing a family of surfaces from a given space-like (or time-like) geodesic curve using the Frenet frame of the curve in Minkowski space and the authors expressed the family of surfaces as a linear combination of the components of this frame (Kasap & Akyıldız, 2006). The authors investigated some curves in plane and in space; they stated the position vectors and gave some theorems about such curves in G_3 (Ali, 2012; Öztekin & Tatlıpınar, 2012). The authors studied a tube in Euclidean 3-space satisfying some equation in terms of the Gaussian curvature, the mean curvature and the second Gaussian curvature (Ro & Yoon, 2009). The similar studies and consequences about tubular surfaces in different spaces were given (Karacan & Yayli, 2008; Dede, 2013).

In this study, the specific energy and angular momentum on tube surfaces are expressed generated by the rectifying curves in Galilean 3-space, and given geodesic formulas with the help of Clairaut's theorem.

2. Preliminaries

A vector $\vec{U} = (u_1, u_2, u_3)$ is called non-isotropic vector if the first component u_1 is not equal to zero. All vectors $\vec{U} = (1, u_2, u_3)$ are unit non-isotropic vectors. The vectors $\vec{U} = (0, u_2, u_3)$ are isotropic vectors.

Suppose that vectors $\vec{U} = (u_1, u_2, u_3)$ and $\vec{V} = (v_1, v_2, v_3)$ are two vectors in Galilean space G_3. Galilean scalar product in G_3 is

$$\langle \vec{U}, \vec{V} \rangle_{G_3} = \begin{cases} u_1 v_1 & \text{if } u_1 \neq 0 \text{ or } v_1 \neq 0 \\ u_2 v_2 + u_3 v_3 & \text{if } u_1 = 0 \text{ and } v_1 = 0 \end{cases} \tag{1}$$

(Yaglom, 1979).

The norm of the vector $\vec{U} = (u_1, u_2, u_3)$ can be written as $\|\vec{U}\| = \sqrt{< \vec{U}, \vec{U} >_{G_3}}$.

The vector product of $\vec{U} = (u_1, u_2, u_3)$ and $\vec{V} = (v_1, v_2, v_3)$ in Galilean space G_3 is defined by

$$\vec{U} \times_{G_3} \vec{V} = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} & \text{if } u_1 \neq 0 \text{ or } v_1 \neq 0 \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} & \text{if } u_1 = 0 \text{ and } v_1 = 0 \end{cases} \tag{2}$$

Let $\alpha: I \subset \mathbb{R} \to G_3$ be a curve given by $\alpha(s) = \big(s, y(s), z(s)\big)$ and this curve is called the admissible curve. The Frenet-Serret frame are expressed by

$$t(s) = \alpha'(s) = \big(1, y'(s), z'(s)\big);\ n(s) = \frac{t'(s)}{\kappa(s)};\ b(s) = \frac{n'(s)}{\tau(s)},$$

where the first curvature is given as $\kappa(s) = \| t'(s) \|$ and the second curvature function is defined as $\tau(s) = \| n'(s) \|$. Also, Frenet-Serret equations are given by follows

$$t' = \kappa n,\ \ n' = \tau b,\ \ b' = -\tau n. \tag{3}$$

Let the equation of a surface $\Omega = \Omega(s, v)$ in G_3 be given by

$$\Omega(s,v) = (x(s,v), y(s,v), z(s,v)). \tag{4}$$

Also, the unit isotropic normal vector field η on $\Omega(s,v)$ is defined as follows

$$\eta = \frac{\Omega_{,s} \times \Omega_{,v}}{\|\Omega_{,s} \times \Omega_{,v}\|}, \tag{5}$$

where the partial differentiations with respect to s and v will be denoted as follows

$$\Omega_{,s} = \frac{\partial \Omega(s,v)}{\partial s}; \;\; \Omega_{,v} = \frac{\partial \Omega(s,v)}{\partial v}. \tag{6}$$

On the other hand, the isotropic unit vector δ on the tangent plane is defined as

$$\delta = \frac{x_{,v}\Omega_{,s} - x_{,s}\Omega_{,v}}{w}, \tag{7}$$

where $x_{,s} = \frac{\partial x(s,v)}{\partial s}$, $x_{,v} = \frac{\partial x(s,v)}{\partial v}$; $w = \|\Omega_{,s} \times \Omega_{,v}\|$,

$$g_1 = x_{,s}, g_2 = x_{,v}, g_{ij} = g_i g_j; \; g^1 = \frac{x_{,v}}{w}; \; g^2 = \frac{x_{,s}}{w}; g^{ij} = g^i g^j; \; i,j = 1,2 \tag{8}$$

$$h_{11} = \langle \Omega^*_{,s}, \Omega^*_{,s} \rangle; \; h_{12} = \langle \Omega^*_{,s}, \Omega^*_{,v} \rangle; \; h_{22} = \langle \Omega^*_{,v}, \Omega^*_{,v} \rangle, \tag{9}$$

where $\Omega^*_{,s}$ and $\Omega^*_{,v}$ are the projections of the vectors $\Omega_{,s}$ and $\Omega_{,v}$ onto the yz-plane, respectively. The first fundamental form ds^2 of the surface $\Omega(s,v)$ is given as

$$ds^2 = (g_1 ds + g_2 dv)^2 + \varepsilon(h_{11} ds^2 + 2h_{12} ds dv + h_{22} dv^2), \tag{10}$$

where

$$\varepsilon = \begin{cases} 0, & ds{:}dv \;\; \text{non} - \text{isotropic} \\ 1, & ds{:}dv \;\; \text{isotropic} \end{cases} \tag{11}$$

In this case, the coefficients of ds^2 are denoted by g^*_{ij}. The function can be represented in terms of g_i and h_{ij} as follows

$$w^2 = g_1^2 h_{22} - 2g_1 g_2 h_{12} + g_2^2 h_{11}.$$

The Gaussian curvature and the mean curvature of a surface are defined by means of the second fundamental form L_{ij} coefficients, which are the normal components of $\Omega_{,i,j} (i,j = 1,2)$. That is,

$$\Omega_{,i,j} = \sum_{k=1}^{2} \Gamma^k_{ij} \Omega_{,k} + L_{ij}\eta, \tag{12}$$

where Γ^k_{ij} is the Christoffel symbols of the surface and L_{ij} are given as

$$L_{ij} = \frac{1}{g_1} \langle g_1 \Omega^*_{,i,j} - g_{i,j} \Omega^*_{,1}, \eta \rangle = \frac{1}{g_2} \langle g_2 \Omega^*_{,i,j} - g_{i,j} \Omega^*_{,2}, \eta \rangle, \tag{13}$$

From this, the Gaussian curvature K and the mean curvature H of the surface are given as

$$K = \frac{L_{11}L_{22} - L_{12}^2}{w^2}, \quad H = \frac{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}^2}{w^2}, \tag{14}$$

(Röschel, 1984; Röschel, 1986; Milin-Šipuš & Divjak, 2012).

Definition 1 Let α be a geodesic curve with arc-length parametrized on the revolution surface given as

$$\alpha(s) = \big(x(w(s), v(s)), y(w(s), v(s)), z(w(s), v(s))\big).$$

From the Lagrangian:

$$L = \dot{w}^2 + \rho^2 \dot{v}^2,$$

and the Euler-Lagrange equations are given as

$$\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial w}{\partial s}}\right) = \frac{\partial L}{\partial w}; \quad \frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial v}{\partial s}}\right) = \frac{\partial L}{\partial v}; \quad \ddot{w} = \rho\rho' \dot{v}^2; \quad \frac{d}{ds}(\rho \dot{v}^2) = 0, \tag{15}$$

so that is a constant of the motion, (Kuhnel, 2006; Pressley, 2010).

Theorem 1 (Clairaut's Theorem) Let α a geodesic on a surface of rotation S, ρ be the distance function of a point on S from the axis of rotation and θ be the angle between α and the meridians of S. Then ρsinθ is constant along α. Conversely, if ρsinθ is constant along curve α on S, and if no part of α is part of some parallel of S then α is a geodesic, (Pressley, 2010).

3. Some Discussions on the Tube Surfaces Generated by Rectifying Curves in G_3

In this section, the tube surfaces generated by rectifying curves are investigated according to mathematical approach.

The envelope of a setting out sphere with exchanging radius is called as canal surface, the radius is defined by the orbit $\alpha(w(s))$ with its center and a radius function ρ. Also, ρ is a constant, then the canal surface is called as a tube or tubular surface. Let one expresses by ρ the vector connecting the point from the curve $\alpha(w(s))$ with the point from the surface. Therefore, position vector R of a point on the surface is given as

$$R = \alpha(w(s)) + \rho \tag{16}$$

and since ρ lies in the Euclidean normal plane of the curve $\alpha(w(s))$, the points at a distance A_1 from a point of $\alpha(w(s))$ (Kuhnel, 2006; Pressley, 2010). Therefore, one writes the equation $\rho = A_1(\cos v_1 \vec{n} + \sin v_1 \vec{b})$, where v_1 is the Euclidean angle between the isotropic vectors; $\overleftarrow{n}$ and $\overleftarrow{\rho}$ lie in the Euclidean normal plane of the curve $\alpha(w(s))$.

Also, an isotropic rectifying curve is expressed with vector fields tangential component and binormal component by using the Galilean frame in G_3. Then, the position vector of the smooth isotropic rectifying curve $\alpha: I \subset \mathbb{R} \to G_3$ with curvatures $\kappa(w) \geq 0, \tau$ in G_3 satisfies the equation

$$\alpha(w) = \Sigma_0 \overleftarrow{t} + \Sigma_1 \overleftarrow{b},$$

for some differentiable functions $\Sigma_0(w)$, $\Sigma_1(w)$ and differentiating previous equation with respect to w and using the Frenet frame equations (3), one obtains

$$\Sigma_0 = c + w; \Sigma_1 = \frac{\kappa(w)(w+c)}{\tau(w)} = d = constant.$$

Thus, the position vector is written as $\alpha(w) = (w+c)\overleftarrow{t} + d\overleftarrow{b}$, where $d = \frac{\kappa(w)(w+c)}{\tau(w)}, c, d \in \mathbb{R}_0$.

3.1. The Clairaut's theorem on tubular surfaces generated by rectifying curves in G3

In this subsection, by using the Clairaut's theorem, the tubular surfaces generated by rectifying curves are characterized in G_3.

Theorem 2 Let $\Omega(\mathrm{w}, \mathrm{v}_1)$ be the tubular surface generated by rectifying curve and $\alpha: \mathrm{I} \subset \mathbb{R} \to \mathrm{G}_3$ be a regular isotropic curve with curvatures $\kappa(\mathrm{w}) \geq 0, \tau$ in G_3. Then the following statements hold:

1) K(the Gaussian curvature) and H(the mean curvature) of the tubular surface Ω are expressed as follows

$$K = \frac{\cos v_1(\tau'(w)d - \kappa'(w)(w+c) - 2\kappa(w))}{A_1} \text{ or } K = \frac{-\kappa(w)\cos v_1}{A_1}; \;\; H = \frac{1}{2A_1}.$$

where this family of the tube surface has constant mean curvature.

2) For the parameter $v_1 = \arccos\left(\frac{-K}{2\kappa(w)H}\right)$ the first fundamental form of the surface Ω is given by

$$I = 2\dot{w}^2 + \left(\frac{\kappa(w)\cos v_1}{K}\right)^2 \dot{v}_1^2 = 2\dot{w}^2 + \left(\frac{1}{2H}\right)^2 \dot{v}_1^2.$$

3) If the curve α is a geodesic on the surface $\Omega(w, v_1)$, then if and only if the following equations are satisfied

$$\kappa(w) = \frac{-K}{2H\cos(2H\int \sin\theta ds)}; \; \tau(w) = -\frac{(w+c)K}{2dHcos(v_1)},$$

where $d, d_i, c, c_i \in \mathbb{R}_0$.

Proof. The tube surface generated by rectifying curve is parametrized as

$$\Omega(w, v_1) = \alpha(w(s)) + A_1(\cos v_1(s)\overleftarrow{n} + \sin v_1(s)\overleftarrow{b}), \tag{17}$$

where v_1 is angle between the isotropic vectors $\overleftarrow{n}$and $\overleftarrow{R} = A_1$, one can get

$$\Omega(w, v_1) = (w+c)\overleftarrow{t} + A_1\cos v_1\overleftarrow{n} + (d + A_1\sin v_1)\overleftarrow{b} \tag{18}$$

ve

$$\Omega(w, v_1) = (w+c)\overleftarrow{t} + A_1\cos v_1\overleftarrow{n} + \left(\frac{\kappa(w)(w+c)}{\tau(w)} + A_1\sin v_1\right)\overleftarrow{b}, \tag{19}$$

then, one can get partial derivatives of $\Omega(w, v_1)$ with respect to w and v_1 as follows

$$\Omega_w = \overleftarrow{t} + \big((w+c)\kappa - \tau(d + A_1 \sin v_1)\big)\overleftarrow{n} + \tau A_1 \cos v_1 \overleftarrow{b} = N_w, \tag{20}$$

$$\Omega_{v_1} = A_1(-\sin v_1 \overleftarrow{n} + \cos v_1 \overleftarrow{b}) = A_1 N_{v_1}. \tag{21}$$

It follows that the vector cross product is obtained as

$$\Omega_w \times \Omega_{v_1} = -A_1 \cos v_1 \overleftarrow{n} - A_1 \sin v_1 \overleftarrow{b}; \tag{22}$$

$$\left\|\Omega_w \times \Omega_{v_1}\right\| = A_1, \tag{23}$$

by using (22) and (23), the normal vector η of $\Omega(w, v_1)$ is written as

$$\eta = -\cos v_1 \overleftarrow{n} - \sin v_1 \overleftarrow{b}, \tag{24}$$

from (7), the following equation is written

$$\delta = \frac{-\Omega_{v_1}}{A_1} = \sin v_1 \overleftarrow{n} - \cos v_1 \overleftarrow{b},$$

which $\overleftarrow{n}$ and $\overleftarrow{b}$ are the isotropic vectors, and by using the Galilean Frenet frame, one gets

$$x(w, v_1) = w + c; x_w = 1 = g_1; x_{v_1} = 0 = g_2;$$

$$g_{11} = 1, g_{12} = 0, g_{22} = 0; g^1 = 0, g^2 = \frac{-1}{A_1}; \tag{25}$$

$$h_{11} = 1, h_{12} = 0, h_{22} = A_1^2. \tag{26}$$

After the substitution of (25) and (26) into (10), the first fundamental form is written as

$$I = dw^2 + \varepsilon(dw^2 + A_1^2 dv_1^2) \tag{27}$$

or for $\varepsilon = 1$, one gets

$$I = 2dw^2 + A_1^2 dv_1^2$$

and for the second fundamental form of $\Omega(w, v_1)$, one has the following equations

$$\begin{gathered}\Omega_{ww} = (2\kappa(w) + (w+c)\kappa'(w) - \tau'(w)(d + A_1 \sin v_1) - \tau^2(w) A_1 \cos v_1)\overleftarrow{n} \\ +(\tau(w)\kappa(w)(w+c) - \tau^2(w)(d + A_1 \sin v_1) + \tau'(w) A_1 \cos v_1)\overleftarrow{b}; \\ \Omega_{v_1 v_1} = A_1(-\cos v_1 \overleftarrow{n} - \sin v_1 \overleftarrow{b});\ \ \Omega_{w v_1} = -\tau(w) A_1 \cos v_1 \overleftarrow{n} - \tau(w) A_1 \sin v_1 \overleftarrow{b}\end{gathered} \tag{28}$$

and from (13) and (24), (28) the coefficients of the second fundamental form are given as follows

$$\begin{gathered}L_{11} = \big(-2\kappa(w) + \tau'(w)d - \kappa'(w)(w+c)\big)\cos v_1 + \tau^2(w) A_1; \\ L_{22} = A_1;\ L_{12} = \tau(w) A_1.\end{gathered} \tag{29}$$

Thus, the Gaussian curvature K and the mean curvature H are expressed as

$$K=\frac{\cos v_1(\tau'(w)d-\kappa'(w)(w+c)-2\kappa(w))}{A_1}=\frac{-\kappa(w)\cos v_1}{A_1}(\tau(w)d \\ =\kappa(w)(w+c)); \tag{30}$$

$$H=\frac{1}{2A_1}. \tag{31}$$

Therefore, from the previous equation, one can write

$$A_1=\frac{-\kappa(w)\cos v_1}{K}=\frac{1}{2H} \quad \text{or} \quad -2\kappa(w)\cos v_1=\frac{K}{H}$$

and hence the following equation is satisfied

$$v_1=\arccos\left(\frac{-K}{2\kappa(w)H}\right).$$

Furthermore, one can write the first fundamental form as follows

$$I=2\dot{w}^2+A_1^2\dot{v}_1^2=2\dot{w}^2+\left(\frac{\kappa(w)\cos v_1}{K}\right)^2\dot{v}_1^2=2\dot{w}^2+\left(\frac{1}{2H}\right)^2\dot{v}_1^2.$$

Since, $\tau\neq 0$, for $\frac{(w+c)\kappa(w)}{d}=\tau(w)$, the first fundamental form has two variable parameters and since the first fundamental form is diagonal the parametrization coordinates are orthogonal. Then, the Lagrangian equation is written as

$$L=2\dot{w}^2+\left(\frac{1}{2H}\right)^2\dot{v}_1^2. \tag{32}$$

Then, a geodesic on the surface $\Omega(w,v_1)$ is expressed by using the Euler-Lagrangian equations

$$\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial w}{\partial s}}\right)=\frac{\partial L}{\partial w};\quad \frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial v_1}{\partial s}}\right)=\frac{\partial L}{\partial v_1}. \tag{33}$$

1) For $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial w}{\partial s}}\right)=\frac{\partial L}{\partial w}=0$, one obtains $\frac{\partial L}{\frac{\partial w}{\partial s}}=4\dot{w}=$constant, which means

$$w=\frac{c_1}{4}s+d_1. \tag{34}$$

2) For $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial v_1}{\partial s}}\right)=\frac{\partial L}{\partial v_1}=0$, one can obtain $\frac{\partial}{\partial s}\left(2\left(\frac{1}{H}\right)^2\dot{v}_1\right)=0$, where $2\left(\frac{1}{H}\right)^2\dot{v}_1$ is constant along the geodesic and leading to

$$v_1=\frac{c_2}{2A_1^2}s+d_2 \quad \text{or} \quad v_1=2H^2c_2s+d_2 \tag{35}$$

and since $v_1=\arccos\left(\frac{-K}{2\kappa(w)H}\right)$, one gets

$$\arccos\left(\frac{-K}{2\kappa(w)H}\right) = 2H^2c_2s + d_2.$$

Let $\alpha(w)$ be a geodesic on the surface of $\Omega(w, v_1)$ so it is given as $(w(s), v_1(s))$. In the mean time, the angle between the meridian $\dot{\alpha}$ and N_w is θ; the vector pointing along paralells of Ω is N_{v_1}. Hence, one can say that $\{N_w, N_{v_1}\}$ has orthonormal basis, and a unit vector $\dot{\alpha}$ tangent to $\Omega(w, v_1)$ can be written as,

$$\dot{\alpha} = N_w\cos\theta + N_{v_1}\sin\theta = \dot{w}\Omega_w + \dot{v}_1\Omega_{v_1} = \dot{w}N_w + \dot{v}_1\frac{1}{2H}N_{v_1}.$$

One can see that $\frac{1}{2H}\dot{v}_1 = \sin\theta$, and hence one can write $2\left(\frac{1}{2H}\right)^2\dot{v}_1 = \frac{1}{H}\sin\theta$ being a constant along $\alpha(w)$. On the contrary, $\alpha(w)$ is a rectifying curve with $2\left(\frac{1}{2H}\right)^2\dot{v}_1 = \frac{1}{H}\sin\theta$ which is a constant, by using the second Euler-Lagrange equation and by differentiating L and by substituting this into the second equation, one gets the first Euler Lagrange equation

$$v_1 = \int\frac{\sin\theta}{A_1}ds \quad \text{or} \quad -\int\frac{K\sin\theta}{\kappa(w)\cos v_1}ds = 2\int H\sin\theta ds = v_1 \tag{36}$$

and since $v_1 = \arccos\left(\frac{-K}{2\kappa(w)H}\right)$, one has

$$\arccos\left(\frac{-K}{2\kappa(w)H}\right) = \int\frac{\sin\theta}{A_1}ds \Rightarrow \cos\left(2\int H\sin\theta ds\right) = \frac{-K}{2\kappa(w)H}$$

and hence, for the rectifying curve the curvatures of the curve can be written as

$$\kappa(w) = \frac{-K}{2H\cos(2\int H\sin\theta ds)}; \quad \tau(w) = -\frac{(w+c)K}{2dHcos(v_1)}.$$

Furthermore, for $w = \frac{c_1}{4}s + d_1$, one can obtain $\dot{w} = \frac{c_1}{4}$. Also, since one gets $4\dot{w} = 4\cos\theta$ being a constant along $\alpha(w)$. If $\alpha(w)$ is a rectifying curve given as $4\cos\theta$ =constant, then from the first Euler Lagrange equation and the second Euler Lagrange equation, one has

$$w = \int\cos\theta ds \text{ (or } \; w = \int\cos\theta ds + c_8), \tag{37}$$

where $c_i, d_i \in \mathbb{R}_0$.

4. The Physics Approach on Tube Surfaces Generated by Rectifying Curves in G3

In this section, one considers a geodesic movement by reaching the time-dependent parameter $w(s)$, from here one can clearly say that one will try to express some characterizations on surfaces with this path called the trajectory of the particle. Then, a parametrized curve $\Omega(w(s), v_1(s))$ is given as

$$\Omega(w(s), v_1(s)) = (w(s)+c)\overleftarrow{t} + \frac{1}{2H}\cos v_1(s)\overleftarrow{n} + (d + \frac{1}{2H}\sin v_1(s))\overleftarrow{b} \tag{38}$$

or

$$\Omega(w(s), v_1(s)) = (w(s) + c)\overleftarrow{t} + \frac{\cos v_1(s)}{2H}\overleftarrow{n} + (\frac{\kappa(w(s))(w(s) + c)}{\tau(s)} + \frac{\sin v_1(s)}{2H})\overleftarrow{b}. \tag{39}$$

To calculate the derivative of this tangent vector along the curve using the chain rule, the tangent vector of the curve $\alpha(w)$ can be written as follows:

$$\frac{d\Omega(w(s), v_1(s))}{ds} = \frac{dw(s)}{ds}\Omega_w + \frac{dv_1(s)}{ds}\Omega_{v_1}, \tag{40}$$

$$\dot{\alpha} = N_w \cos\theta + N_{v_1}\sin\theta = \dot{w}\Omega_w + \dot{v_1}\Omega_{v_1} = \dot{w}N_w + \dot{v_1}\frac{1}{2H}N_{v_1}. \tag{41}$$

Since the velocity is tangent vector of the geodesic curve, one gets

$$\overleftarrow{V} = \frac{d\Omega(w(s), v_1(s))}{ds} = V^w\Omega_w + V^{v_1}\Omega_{v_1}.$$

One thinks that $V^{w^*} = \sqrt{2}V^w = V\cos\theta$ is the first axis, which is the radial velocity; since the horizontal angular velocity is V^{v_1}, $V^{v_1^*} = \frac{V^{v_1}}{2H} = V\sin\theta$ is the second axis which is the horizontal component of the velocity vector. One can also express the velocity with respect to polar coordinates in the tangent plane to find the slope and norm with respect to the given radial direction on the surface. Also, the angle θ expresses the side of the velocity relative to the side Ω_{w^*} in the same plane, and the speed is constant along the geodesic for parametrized geodesics. These features, which physically require energy and momentum, is given as follows.

$$\begin{aligned} E = \frac{V^2}{2} = \frac{\left(\left(\sqrt{2E}\cos\theta\right)^2 + \left(\sqrt{2E}\sin\theta\right)^2\right)}{2} &= \left(\frac{dw}{ds}\right)^2 + \frac{1}{2}\left(\frac{1}{2H}\right)^2\left(\frac{dv_1}{ds}\right)^2 \\ = \frac{1}{2}(V^2\cos^2\theta + V^2\sin^2\theta) \end{aligned} \tag{42}$$

from the right side of (42), both the specific energy and speed are constant along geodesic.

Theorem 3. Let $\Omega(\mathrm{w}, \mathrm{v}_1)$ be the tube surface generated by isotropic rectifying curve $\alpha(\mathrm{w})$, then the following statements hold:

• For the parameter $v_1 = 2c_2H^2s + d_2$ or $v_1 = 2H\int \sin\theta ds$, the specific angular momentum ℓ is given by

$$\ell = \frac{1}{2H}V\sin\theta = -\frac{\kappa(w)}{K}V\cos v_1 \sin\theta$$

and the specific energy E is written as

$$E = \frac{1}{2}\left(\frac{c_5}{2} + 4H^2\ell^2\right) = \cos^2\theta + 2H^2\ell^2 \quad \text{or} \quad \frac{2H^2\ell^2}{\sin^2\theta} = E$$

• For the parameter $w = \int \cos\theta ds$(or $w = \frac{c_1}{4}s + d_1$), the specific angular momentum ℓ is given by

$$\ell = \frac{1}{\sqrt{2}} V\cos\theta$$

and the specific energy E is given by

$$E = 4\ell^2 + c_7 H^2 = \frac{1}{2}(8\ell^2 + \sin^2\theta) \quad \text{or} \quad \frac{\ell^2}{\cos^2\theta} = E,$$

where the curve $\alpha(w)$ is a geodesic on the surface Ω and $c_i \in \mathbb{R}_0$.

Proof. 1) For $v_1 = 2c_2H^2 s + d_2(\ v_1 = 2H\int \sin\theta ds)$ from circular movement around an axis with radius $\|\overleftarrow{R}\| = \frac{1}{2H}$ or $\overleftarrow{R} = \frac{1}{2H}\overleftarrow{e_1}$, that is to say the velocity $V^{v_1^*} = \frac{1}{2H}\frac{dv_1}{ds} = \frac{-\kappa(w)\cos v_1}{K}\frac{dv_1}{ds}$ in the angular side multiplied by the radius $\frac{1}{2H}$ of the circle. Physically, we can write the specific angular momentum ℓ as the following equations

$$\ell = \overleftarrow{e_3}.\left(\overleftarrow{R} \times_{G_3} \overleftarrow{V}\right) = \frac{1}{2H} V\sin\theta \ \text{ or } \ \ell = \frac{-\kappa(w)\cos v_1}{K} V\sin\theta. \tag{43}$$

Also, since $V^{v_1^*} = V\sin\theta = \sqrt{2E}\sin\theta$, one gets $\left(\frac{1}{2H}\right)^2 \frac{dv_1}{ds} = \frac{1}{2H}V\sin\theta$, and since the specific angular momentum is constant along a geodesic, one has

$$\ell = \left(\frac{1}{2H}\right)^2 \frac{dv_1}{ds} \Rightarrow \frac{dv_1}{ds} = 4H^2\ell. \tag{44}$$

This expression can be rewritten in the form of the changeable angular velocity dv_1/ds according to the specific energy formula where the constant angular momentum, the specific energy E is expressed by the radial motion with another of the motion as

$$E = \left(\frac{dw}{ds}\right)^2 + 2H^2\ell^2 = \frac{1}{2}\left(\frac{c_5}{2} + 4H^2\ell^2\right) = \cos^2\theta + 2H^2\ell^2 \quad \text{or} \quad \ell = \frac{1}{2H}\sqrt{2E}\sin\theta,$$

then, we get $\frac{2H^2\ell^2}{\sin^2\theta} = E$.

2) For the $w = \int \cos\theta ds(\ w = \frac{c_1}{4}s + d_1)$ one writes $\frac{1}{\sqrt{2}}\dot{w} = \frac{1}{\sqrt{2}}\cos\theta =$ constant along $\alpha(w)$, then from circular movement round an axis with radius $\|\overleftarrow{R}\| = \frac{1}{\sqrt{2}}$ or $\overleftarrow{R} = -\frac{1}{\sqrt{2}}\overleftarrow{e_2}$, that is to say the velocity $V^{w^*} = \frac{1}{\sqrt{2}}V^w = V\cos\theta = \frac{1}{\sqrt{2}}\frac{dw}{ds} = \sqrt{2E}\cos\theta$ in the angular direction is multiplied by the radius $\frac{1}{\sqrt{2}}$ of the circle. The first geodesic equation has the specific angular momentum, which is constant along a geodesic and hence, one can write it as follows

$$\ell = \overleftarrow{e_3}.\left(\overleftarrow{R} \times_{G_3} \overleftarrow{V}\right) = \frac{1}{\sqrt{2}} V\cos\theta. \tag{45}$$

Furthermore, since $\frac{1}{\sqrt{2}}\frac{dw}{ds} = V\cos\theta$, one can write $\frac{1}{2}\frac{dw}{ds} = \frac{1}{\sqrt{2}}V\cos\theta$, and that the specific angular momentum is constant along a geodesic, one gets $2\ell = \frac{dw}{ds}$. Hence, from the changeable angular velocity dw/ds in the specific energy formula E according to the constant angular momentum. Hence, E is given by

$$E=\frac{1}{2}\left(\left(\frac{dw}{ds}\right)^2+\left(\frac{1}{2H}\right)^2\left(\frac{dv_1}{ds}\right)^2\right)=4\ell^2+c_8H^2=\frac{1}{2}(8\ell^2+\sin^2\theta)\quad \text{or}\quad \ell=\sqrt{E}\cos\theta,$$

then, we get $\frac{\ell^2}{\cos^2\theta}=E.$

5. Discussion and Conclusion

In this paper, it is explored that the conditions of being geodesic, in which the curves can be chosen to be rectifying curves, allows one to constitute the specific energy and specific angular momentum. Also, the tube surfaces generated by rectifying curves are expressed and the some certain results of describing the rectifying geodesics are examined on the tubular surfaces in detail. Furthermore, the specific energy and the angular momentum are expressed on these tube surfaces in Galilean 3-space.

Acknowledgements

The author wishes to express their thanks to the authors of literatures for the supplied scientific aspects and idea for this study.

References

Ali, A. T. (2012). Position vectors of curves in the Galilean space G_3, *Matematicki Vesnik*, *64(249)*, 200-210.

Almaz, F., & Kulahci, M. A. (2020). Some characterizations on the special tubular surfaces in Galilean space. *Prespacetime Journal*, *11*(7), 626-636.

Almaz, F., & Kulahci, M. A. (2021). The notes on rotational surfaces in Galilean space, *International Journal of Geometric Methods in Modern Physics, 18(02)*, 2150017. https://doi.org/10.1142/S0219887821500171

Almaz, F., & Külahci, M. (2022). A survey on tube surfaces in Galilean 3-space. *Politeknik Dergisi*, *25*(3), 1133-1142. https://doi.org/10.2339/politeknik.747869

Dede, M. (2013). Tubular surfaces in Galilean space. *Mathematical Communications*, *18*(1), 209-217.

Kasap, E., & Akyildiz, F. T. (2006). Surfaces with common geodesic in Minkowski 3-space. *Applied Mathematics and Computation*, *177*(1), 260-270. https://doi.org/10.1016/j.amc.2005.11.005

Karacan, M. K., & Yayli, Y. (2008). On the geodesics of tubular surfaces in Minkowski 3-space. *Bulletin of the Malaysian Mathematical Sciences Society. Second Series*, *31*(1), 1-10.

Kuhnel, W. (2006). D*ifferential geometry curves-surfaces and manifolds*, (2nd ed.). Providence, RI, United States, American Math. Soc., 16.

Milin Šipuš, Ž., & Divjak, B. (2012). Surfaces of Constant Curvature in the Pseudo-Galilean Space. *International Journal of Mathematics and Mathematical Sciences*, *2012*(1), 375264. https://doi.org/10.1155/2012/375264

Öztekin, H. B., & Tatlipinar, S. (2012). On some curves in Galilean plane and 3-dimensional Galilean space. *Journal of Dynamical Systems and Geometric Theories*, *10*(2), 189-196. https://doi.org/10.1080/1726037X.2012.10698620

Pressley, A. (2010). *Elementary differential geometry*, second edition. London, UK. Springer-Verlag London Limited.

Ro, J. S., & Yoon, D. W. (2009). Tubes of Weingarten types in a Euclidean 3-space. *Journal of the Chungcheong Mathematical Society*, *22*(3), 359-359.

Röschel, O. (1984). *Die Geometrie des Galileischen Raumes*, Bericht der Mathematisch Statistischen Sektion in der Forschungs-Gesellschaft Joanneum, Bericht Nr. 256, Habilitationsschrift, Leoben.

Röschel, O. (1986). *Die Geometrie des Galileischen Raumes*, Forschungszentrum Graz Research Centre, Austria.
Saad, A., & Low, R. J. (2014). A generalized Clairaut's theorem in Minkowski space, *Journal of Geometry and Symmetry in Physics, 35*, 103-111.
Walecka, J. D. (2007). *Introduction to general relativity*. World Scientific, Singapore.
Walecka, J. D. (2013). *Topics in modern physics*: *Theoretical foundations*. World Scientific.
Yaglom, I. M. (1979). *A simple non-Euclidean geometry and its physical basis.* Springer-Verlag, New York Inc.