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Original Article

## Expected Values of Aggregation Operators on Cubic Trapezoidal Fuzzy Number and its Application to Multi-Criteria Decision Making Problems

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**Abstract** – In this paper, we define trapezoidal cubic fuzzy numbers and their operational laws. Started on these operational laws, each collection operators, with trapezoidal cubic fuzzy weighted arithmetic averaging operator and weighted geometric averaging operator are purposed. Expected values, score function, and accuracy function of trapezoidal cubic fuzzy numbers are defined. Overcoming on these, mindful of trapezoidal cubic fuzzy multi-criteria decision making program is proposed. A delineation illustration example is given to exhibit the sound judgment and openness of the procedure.

**Keywords** – Trapezoidal cubic fuzzy number, aggregation operators, multi-criteria decision making

### 1 Introduction

Their get at a considerable lot of multi-criteria decision-making (MCDM) issues in indicating sociology. Different a period past the point of no return the fuzzy set institutionalization was offered and passed down to illuminate MCDM issues by Zadeh [14]. Therefor in [1], Atanassov presented the concept of intuitionistic fuzzy set (IFS) and discussed the degree of membership as well as the degree of non-membership function. Li reachable by theories and uses of fuzzy multi-criteria decision-making [9]. Wang displayed reading on multi-criteria decision-making drawing near with divided undoubting data [11]. There are differentiating preparing on the instrument of multi-criteria decision-making issues, in which the measures' weight coefficients are obvious and the criteria's principles are changed or are fuzzy numbers in [5,7,12], and here are likewise efficient readings on multi-criteria decision making or multi-criteria group decision making in [10,13], in which the weight sizes are tight and the standards' morals are fuzzy numbers.

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Cubic set appeared by Jun in [8]. Cubic sets are the speculations of fuzzy sets and intuitionistic fuzzy sets, in which there are two portrayals, one is utilized for the degree of membership and other is utilized for the degree of non-membership. The membership function is hold as interim while non-membership is inside and out seen as the constant fuzzy set.

Aliya et al., [4] defined the triangular cubic fuzzy number and operational laws. We developed the triangular cubic fuzzy hybrid aggregation (TCFHA) administrator to total all individual fuzzy choice structure provide by the decision makers into the aggregate cubic fuzzy decision matrix. Aliya et al., [3] proposed the cubic TOPSIS method and cubic gray relation analysis (GRA) method. Finally, the proposed method is used for selection in sol-gel synthesis of titanium carbide Nano powders. Aliya et al., [2] defined weighted average operator of triangular cubic fuzzy numbers and hamming distance of the TCFN. We develop an MCDM method approach based on an extended VIKOR method using triangular cubic fuzzy numbers (TCFNS) and multi-criteria decision-making (MCDM) method using triangular cubic fuzzy numbers (TCFNs) are developed. Aliya et al., [5] defined the generalized triangular cubic linguistic hesitant fuzzy weighted geometric (GTCHFOWG) operator, generalized triangular cubic linguistic hesitant fuzzy ordered weighted average (GTCLHFOWA) operator, generalized triangular cubic linguistic hesitant fuzzy ordered weighted geometric (GTCLHFOWG) operator, generalized triangular cubic linguistic hesitant fuzzy hybrid averaging (GTCLHFHA) operator and generalized triangular cubic linguistic hesitant fuzzy hybrid geometric (GTCLHFHG) operator. Aliya et al., [6] developed Trapezoidal linguistic cubic hesitant fuzzy TOPSIS method to solve the MCDM method based on trapezoidal linguistic cubic hesitant fuzzy TOPSIS method.

Thus, it is very necessary to introduce a new extension of cubic set to address this issue. The aim of this paper is to present the notion of Trapezoidal cubic fuzzy set, which extends the cubic set to Trapezoidal cubic fuzzy environments and permits the membership of an element to be a set of several possible Trapezoidal cubic fuzzy numbers. Thus, Trapezoidal cubic fuzzy set is a very useful tool to deal with the situations in which the experts hesitate between several possible Trapezoidal cubic fuzzy numbers to assess the degree to which an alternative satisfies an attribute. In the current example, the degree to which the alternative satisfies the attribute can be represented by the Trapezoidal cubic fuzzy set. Moreover, in many multiple attribute group decision-making (MAGDM) problems, considering that the estimations of the attribute values are Trapezoidal cubic fuzzy sets, it therefore is very necessary to give some aggregation techniques to aggregate the Trapezoidal cubic fuzzy information. However, we are aware that the present aggregation techniques have difficulty in coping with group decision-making problems with Trapezoidal cubic fuzzy information. Therefore, we in the current paper propose a series of aggregation operators for aggregating the Trapezoidal cubic fuzzy information and investigate some properties of these operators. Then, based on these aggregation operators, we develop an approach to MAGDM with Trapezoidal cubic fuzzy information. Moreover, we use a numerical example to show the application of the developed approach.

The rest parts of this paper are organized as follows: Section 2, we define the definition of fuzzy set and cubic set. Section 3, we exhibit trapezoidal cubic fuzzy set and operational laws. Section 4, we exhibit Aggregation operators on trapezoidal cubic fuzzy numbers. Section 5, we define Expected values of trapezoidal cubic fuzzy numbers and comparison

between them. Section 6, we define Multi-criteria decision making method based on trapezoidal cubic fuzzy numbers. Section 7, the application of the developed approach in group decision-making problems is shown by an illustrative example. Results and discussion are given in section 8. Finally, we give the conclusions in Section 9.

## 2. Preliminaries

**Definition.2.1. [14]** Give  $p$  a chance to be a nature of talk. The possibility of fuzzy set was speak to by Zadeh, and characterized as taking after:  $j = \{p, Y_j(p) \mid p \in P\}$ . A fuzzy set in a set  $\dot{p}$  is defined  $Y_j : p \rightarrow I$ , is a membership function,  $Y_j(p)$  denoted the degree of membership of the element  $p$  to the set  $P$ , where  $I = [0,1]$ . The accumulation of every single fuzzy subset of  $P$  is meant by  $I^P$ . Characterize a connection on  $I^P$  as takes after:  $(\forall Y, \eta \in I^P)(Y \leq \eta \Leftrightarrow (\forall p \in P)(Y(p) \leq \eta(p)))$ .

**Definition.2.2. [8]** Give  $P$  a chance to be a nonempty set. By a cubic set in  $P$  we mean a structure  $F = \{q, Y(q), \chi(q) : q \in P\}$  in which  $Y$  is an IVF set in  $q$  and  $\chi$  is a fuzzy set in  $P$ . A cubic set  $\tilde{F} = \{q, Y(q), \chi(q) : q \in P\}$  is simply denoted by  $\tilde{F} = \langle Y, \chi \rangle$ . Denote by  $C^P$  the collection of all cubic sets in  $p$ . A cubic set  $\tilde{F} = \langle Y, \chi \rangle$  in which  $Y(q) = 0$  and  $\chi(q) = 1$  (resp.  $Y(q) = 1$  And  $\chi(q) = 0$  for all  $q \in P$  is denoted by  $0$  (resp.  $1$ ). A cubic set  $\tilde{D} = \langle \tilde{\lambda}, \tilde{\xi} \rangle$  in which  $\tilde{\lambda}(q) = 0$  and  $\tilde{\xi}(q) = 0$  (resp.  $\tilde{\lambda}(q) = 1$  and  $\tilde{\xi}(q) = 1$ ) for all  $q \in P$  is denoted by  $0$  (resp.  $1$ ).

**Definition.2.3.[8]** Let  $P$  be a non empty set. A cubic set  $F = (\zeta, \tilde{\lambda})$  in  $P$  is said to be an internal cubic set if  $\zeta^-(p) \leq \tilde{\lambda}(p) \leq \zeta^+(p)$  for all  $p \in P$ .

**Definition.2.4.[8]** Let  $P$  be a non empty set. A cubic set  $F = (\zeta, \tilde{\lambda})$  on  $P$  is said to be an external cubic set if  $\tilde{\lambda}(p) \notin (\zeta^-(p), \zeta^+(p))$  for all  $p \in P$ .

## 3. Trapezoidal Cubic Fuzzy Numbers

**Definition.3.1.** Let  $Y$  be trapezoidal cubic fuzzy number in the set of real numbers, its

$$\text{membership function is defined as } \mu_Y(q) = \begin{cases} f_Y^L(q) & \Gamma \leq q < \varphi \\ \mu_Y & \varphi \leq q < \Theta \\ f_Y^R(q) & \Theta \leq q < \tau \\ 0, & \text{otherwise} \end{cases}$$

Its non-membership function is defined as  $\nu_Y(q) = \begin{cases} g_Y^L(q) & \Gamma_1 \leq q < \varphi_1 \\ \nu_Y & \varphi_1 \leq q < \Theta_1 \\ g_Y^R(q) & \Theta_1 \leq q < \tau_1 \\ 0, & \text{otherwise} \end{cases}$

The trapezoidal cubic fuzzy number is denoted as  $Y = \left\langle \left\langle \left[ \Gamma, \varphi, \Theta, \tau \right]; \left[ \mu^-, \mu^+ \right], \nu \right\rangle, \left[ \Gamma_1, \varphi_1, \Theta_1, \tau_1 \right]; \left[ \mu_1^-, \mu_1^+ \right], \nu_1 \right\rangle$ . Generally from

fuzzy numbers, trapezoidal cubic fuzzy numbers have another parameter: non-membership function, which is utilized to unequivocal the admeasurements to which the decision making that the component does not have a place with  $((\Gamma, \varphi, \Theta, \tau); \nu_Y)$ . When

$\mu_Y^-(q) = 1, \mu_Y^+(q) = 1, \nu_Y = 0$ , a is called normal trapezoidal cubic fuzzy number, by method for detail, conventional fuzzy number.

$$f_Y^L(q) = \frac{x-\Gamma}{\Gamma-\Gamma}[\mu_Y^-, \mu_Y^+], f_Y^R(q) = \frac{\tau-x}{\tau-\Theta}[\mu_Y^-, \mu_Y^+], g_Y^L(q) = \frac{\varphi_1-x+\nu_Y(x-\Gamma_1)}{\varphi_1-\Gamma_1}, g_Y^R(q) = \frac{x-\Theta_1+\nu_Y(\tau_1-x)}{\tau_1-\Theta_1},$$

the cubic fuzzy number is called trapezoidal cubic fuzzy number.

**Definition.3.2.** Let  $h_1 = \left\langle \left[ \chi_1, \varpi_1, \alpha_1, \tau_1 \right]; \left[ \mu_1^-, \mu_1^+ \right], \nu_1 \right\rangle$  and  $h_2 = \left\langle \left[ \chi_2, \varpi_2, \alpha_2, \tau_2 \right]; \left[ \mu_2^-, \mu_2^+ \right], \nu_2 \right\rangle$  be two trapezoidal

cubic fuzzy numbers; then,

$$(1): h_1 + h_2 = \left\langle \left[ \chi_1 + \chi_2, \varpi_1 + \varpi_2, \alpha_1 + \alpha_2, \tau_1 + \tau_2 \right], \left[ \mu_1^- + \mu_2^- - \mu_1^- \mu_2^-, \mu_1^+ + \mu_2^+ - \mu_1^+ \mu_2^+ \right], \nu_1 \nu_2 \right\rangle,$$

$$(2): h_1 - h_2 = \left\langle \left[ \chi_1 - \chi_2, \varpi_1 - \varpi_2, \alpha_1 - \alpha_2, \tau_1 - \tau_2 \right], \left[ \mu_1^- - \mu_2^- + \mu_1^- \mu_2^-, \mu_1^+ - \mu_2^+ + \mu_1^+ \mu_2^+ \right], \nu_1 \nu_2 \right\rangle,$$

$$(3): \lambda h_1 = \left\langle \left[ \lambda \chi_1, \lambda \varpi_1, \lambda \alpha_1, \lambda \tau_1 \right]; \left[ 1 - (1 - \mu_1^-)^\lambda, 1 - (1 - \mu_1^+)^\lambda \right], (\nu_1^\lambda)^\lambda \right\rangle,$$

$$(4): h_1^\lambda = \left\langle \left[ (\chi_1)^\lambda, (\varpi_1)^\lambda, (\alpha_1)^\lambda, (\tau_1)^\lambda \right]; \left[ (\mu_1^-)^\lambda, (\mu_1^+)^\lambda \right], 1 - (1 - \nu_1)^\lambda \right\rangle$$

**Example.3.3.** Let  $h_1 = \left\langle [0.4, 0.8, 0.12, 0.16]; \begin{matrix} [0.7, 0.9], 0.8 \end{matrix} \right\rangle$  and  $h_2 = \left\langle [0.3, 0.5, 0.7, 0.11]; \begin{matrix} [0.1, 0.5], 0.3 \end{matrix} \right\rangle$  be two

trapezoidal cubic fuzzy numbers; then,

$$(1): h_1 + h_2 = \left\langle \begin{matrix} [0.4+0.3, 0.8+0.5, 0.12+0.7, 0.16+0.11] \\ [0.7+0.1-(0.7)(0.1), (0.9+0.5-(0.9)(0.5))], \\ ((0.8)(0.3)) = [0.7, 1.3, 0.82, 0.27][0.73, 0.95], 0.24 \end{matrix} \right\rangle,$$

$$(2): h_1 - h_2 = \left\langle \begin{matrix} [0.4-0.3, 0.8-0.5, 0.12-0.7, 0.16-0.11], \\ [0.7-0.1+(0.7)(0.1), \\ (0.9-0.5+(0.9)(0.5))], ((0.8)(0.3)) \\ = [0.1, 0.3, 0.58, 0.05][0.67, 0.85], 0.24 \end{matrix} \right\rangle.$$

### 4. Aggregation Operators on Trapezoidal Cubic Fuzzy Numbers

**Definition.4.1.** Let  $\check{\alpha}_j (j=1, \dots, n)$  be a set of trapezoidal cubic fuzzy numbers, and  $TrC - WAA : \Omega_n \rightarrow \Omega$ ; if  $TrC - WAA_{\omega}(\check{\alpha}_1, \check{\alpha}_2, \dots, \check{\alpha}_n) = \sum_{j=1}^n w_j \check{\alpha}_j$  where  $\Omega$  is the set of all trapezoidal cubic fuzzy numbers, and  $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$  is the weight vector of  $\check{\alpha}_j (j=1, \dots, n), \omega_j \in [0, 1], \sum_{j=1}^n w_j = 1$ , then,  $TrC - WAA$  is called the weighted arithmetic average operator on trapezoidal cubic fuzzy numbers.

Specially, if  $\omega = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$ .  $TrC - WAA$  is the arithmetic average operator ( $TrC - WA$ ) on trapezoidal cubic fuzzy numbers.

**Theorem.4.2.** Let  $h = \langle [((\alpha, \Upsilon, \Theta, \tau); \mu^-, \mu^+), \nu] \rangle$  be a set of trapezoidal cubic fuzzy numbers; then, the results aggregated from Definition 4.1 are still trapezoidal cubic fuzzy numbers, and even

$$TrC - WAA_{\omega}(h_1, h_2, \dots, h_n) = \sum_{j=1}^n \omega_j h_j = \left\langle \begin{matrix} [\prod_{j=1}^n (\alpha)^{\omega_j}, \prod_{j=1}^n (\Upsilon)^{\omega_j}, \prod_{j=1}^n (\Theta)^{\omega_j}, \prod_{j=1}^n (\tau)^{\omega_j}]; \\ [1 - \prod_{j=1}^n (1 - \mu^-)^{\omega_j}, 1 - \prod_{j=1}^n (1 - \mu^+)^{\omega_j}], \prod_{j=1}^n (\nu)^{\omega_j} \end{matrix} \right\rangle$$

where  $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$  is the weight vector of  $h_j (j=1, \dots, n), \omega_j \in [0, 1], \sum_{j=1}^n \omega_j = 1$ .

**Example.4.3.** Let

$$\begin{aligned} & \frac{[0.5, 0.6, 0.7, 0.8][0.9, 0.15], 0.13}{[0.1, 0.2, 0.3, 0.4][0.23, 0.27], 0.25} \\ & \frac{[0.4, 0.6, 0.8, 0.10][0.6, 0.10], 0.8}{[0.5, 0.6, 0.7, 0.8][0.9, 0.15], 0.13} \\ & \frac{[0.8, 0.10, 0.12, 0.14][0.22, 0.28], 0.24}{[0.25, 0.26, 0.27, 0.28][0.39, 0.45], 0.40} \\ & = [0.002, 0.0012, 0.0039, 0.0010], \langle [0.9985, \\ & 0.8121], 0.0003 \rangle \end{aligned}$$

**Definition.4.4.** Let  $h_j (j=1, \dots, n)$  be a set of trapezoidal cubic fuzzy numbers, and  $TrC - WGA : \Omega_n \rightarrow \Omega$  ; if  $TrC - WGA_{\omega}(h_1, h_2, \dots, h_n) = \sum_{j=1}^n h_j^{\omega_j}$  where  $\Omega$  is the set of all trapezoidal cubic fuzzy numbers, and  $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$  is the weight vector of  $h_j (j=1, \dots, n), \omega_j \in [0, 1], \sum_{j=1}^n \omega_j = 1$ , then,  $TrC - WAA$  is called the weighted arithmetic average operator on trapezoidal cubic fuzzy numbers. Specially, if  $\omega = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$ .  $TrC - WAA$  is the arithmetic average operator ( $TrC - WA$ ) on trapezoidal cubic fuzzy numbers.

**Theorem.4.5.** Let  $h = \langle [(\alpha, \kappa, \varpi, \tau); [\mu^-, \mu^+]], \nu \rangle$  be a set of trapezoidal cubic fuzzy numbers; then, the results aggregated from Definition 4.4 are still trapezoidal cubic fuzzy numbers, and even

$$TrC - WGA_{\omega}(h_1, h_2, \dots, h_n) = \sum_{j=1}^n h_j \omega_j = \left\langle \left[ \prod_{j=1}^n (\alpha)^{\omega_j}, \prod_{j=1}^n (\kappa)^{\omega_j}, \prod_{j=1}^n (\varpi)^{\omega_j}, \prod_{j=1}^n (\tau)^{\omega_j} \right]; \left[ \prod_{j=1}^n (\mu_{\alpha}^-)^{\omega_j}, \prod_{j=1}^n (\mu_{\alpha}^+)^{\omega_j} \right], 1 - \prod_{j=1}^n (1 - \nu)^{\omega_j} \right\rangle$$

where  $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$  is the weight vector of  $h_j (j=1, \dots, n), \omega_j \in [0, 1], \sum_{j=1}^n \omega_j = 1$ .

**Example.4.6.** Let

	table
$h_1$	$[0.3, 0.5, 0.7, 0.9][0.29, 0.35], 0.33$
$h_2$	$[0.21, 0.22, 0.23, 0.24][0.2, 0.7], 0.5$
$h_3$	$[0.44, 0.46, 0.48, 0.50][0.12, 0.18], 0.16$
$h_4$	$[0.1, 0.3, 0.7, 0.9][0.21, 0.26], 0.23$
$h_5$	$[0.3, 0.4, 0.5, 0.6][0.2, 0.8], 0.4$
$h_6$	$[0.5, 0.6, 0.7, 0.8][0.9, 0.15], 0.13$
	$= [0.0004, 0.0036, 0.0189, 0.0466], \langle [0.0002, 0.0005], 0.8868 \rangle$

### 5. Expected Values of Trapezoidal Cubic Fuzzy Numbers and Comparison between them

For trapezoidal cubic fuzzy numbers,  $f_L(p)$ , are strictly linear increasing function, and  $f_R(p)$  is strictly linear decreasing function in Definition 3.1. There in lay functions are respectively,

$$p_{\varpi}^L(p) = \varpi + \frac{y}{[\mu_{\varpi}^-, \mu_{\varpi}^+]} \times (Y - \varpi), p_{\varpi}^R(Y) = \tau + \frac{y}{[\mu_{\varpi}^-, \mu_{\varpi}^+]} \times (\Theta - \tau),$$

The assurance degree of trapezoidal cubic fuzzy number  $\tilde{a}$  is between  $\langle [\mu_{\varpi}^-, \mu_{\varpi}^+], 1 - \nu_{\varpi} \rangle$ .

**Definition.5.1.**

$$I_{\lambda}(\varpi) = \frac{1}{3} \int_0^{[\mu_{\varpi}^-, \mu_{\varpi}^+]} (1 - \lambda) \times g_{\varpi}^L(p) + \lambda \times g_{\varpi}^R(p) dy + \int_0^{1 - \nu_{\varpi}} (1 - \lambda) \times g_{\varpi}^L(p) + \lambda \times g_{\varpi}^R(p) dy$$

is called the expected value of trapezoidal cubic fuzzy number  $\tilde{a}$ .

**Theorem.5.2.** The trapezoidal cubic fuzzy number  $\tilde{a} = \langle [(\varpi, \lambda, \zeta, \tau); \mu^-, \mu^+], \nu \rangle$ ,  
 $I(\varpi) = \frac{1}{12} \langle [(\varpi + \lambda + \zeta + \tau) \times [1 + \mu^- - \nu) \times (1 + \mu^+ - \nu)] \rangle$

**Example.5.3.**

table

$h_1$	[0.25, 0.26, 0.27, 0.28]⟨[0.29, 0.35], 0.33⟩
$h_2$	[0.41, 0.42, 0.43, 0.44]⟨[0.53, 0.57], 0.55⟩
$h_3$	[0.44, 0.46, 0.48, 0.50]⟨[0.56, 0.60], 0.58⟩
$h_4$	[0.55, 0.56, 0.57, 0.58]⟨[0.59, 0.65], 0.63⟩
$h_5$	[0.88, 0.91, 0.92, 0.94]⟨[0.92, 0.98], 0.94⟩
$h_6$	[0.25, 0.26, 0.27, 0.28]⟨[0.39, 0.45], 0.40⟩

$$I_1(\varpi) = \frac{1}{12} [1.06] \times \langle [0.96 \times 1.02] \rangle$$

$$= 0.0864,$$

$$I_2(\varpi) = \frac{1}{12} [1.7] \times \langle [0.98 \times 1.02] \rangle$$

$$= 0.1416,$$

$$I_3(\varpi) = \frac{1}{12} [1.88] \times \langle [0.98 \times 1.02] \rangle$$

$$= 0.1566,$$

$$I_4(\varpi) = \frac{1}{12} [2.26] \times \langle [0.96 \times 1.02] \rangle$$

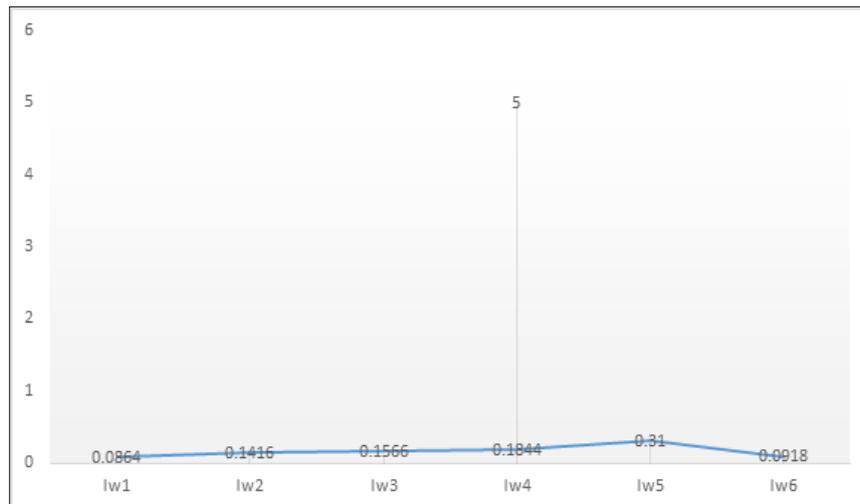
$$= 0.1844,$$

$$I_5(\varpi) = \frac{1}{12} [3.65] \times \langle [0.98 \times 1.04] \rangle$$

$$= 0.3100,$$

$$I_6(\varpi) = \frac{1}{12} [1.06] \times \langle [0.99 \times 1.05] \rangle$$

$$= 0.0918.$$



The score function and accuracy function of trapezoidal cubic fuzzy numbers.

**Definitio.5.4.** Let  $\tilde{a} = \langle [((\Gamma, \Upsilon, \Theta, \tau); \mu^-, \mu^+)], \nu \rangle$  be the trapezoidal cubic fuzzy number; then,  $S(h) = I_h \times [(\mu_h^-, \mu_h^+) - (1 - \nu_h)]$  is called the score function of  $\tilde{a}$ , where  $I_h$  is the expected value of trapezoidal cubic fuzzy number  $h$ .

**Example.5.5.** Let

table

$h_1$	$[0.5, 0.6, 0.7, 0.8] \langle [0.9, 0.15], 0.13 \rangle$
$h_2$	$[0.1, 0.2, 0.3, 0.4] \langle [0.9, 0.15], 0.13 \rangle$
$h_3$	$[0.4, 0.6, 0.8, 0.10] \langle [0.6, 0.10], 0.8 \rangle$
$h_4$	$[0.15, 0.16, 0.17, 0.18] \langle [0.9, 0.15], 0.13 \rangle$
$h_5$	$[0.8, 0.10, 0.12, 0.14] \langle [0.9, 0.28], 0.14 \rangle$
$h_6$	$[0.25, 0.26, 0.27, 0.28] \langle [0.39, 0.45], 0.40 \rangle$

$$S_1(h) = \frac{1}{12} [2.6 * 1.77 * 1.02] * [1.05 - 0.87] = 0.3911 * 0.18 = 0.0703,$$

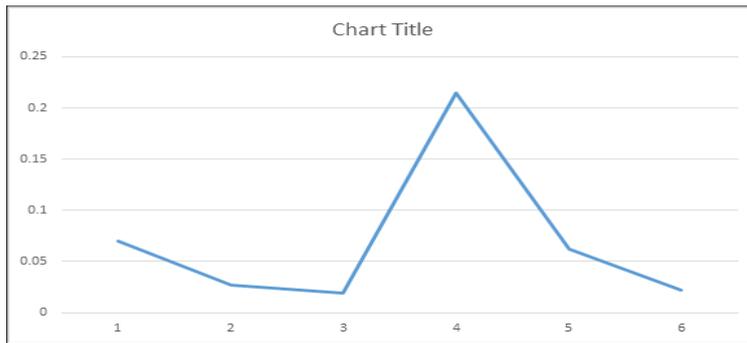
$$S_2(h) = \frac{1}{12} [1 * 1.77 * 1.02] * [1.05 - 0.87] = 0.1504 * 0.18 = 0.0270,$$

$$S_3(h) = \frac{1}{12} [1.9 * 0.8 * 0.3] * [0.7 - 0.2] = 0.038 * 0.5 = 0.019,$$

$$S_4(h) = \frac{1}{12} [0.66 * 1.77 * 1.02] * [1.05 - 0.87] = 1.1915 * 0.18 = 0.2144,$$

$$S_5(h) = \frac{1}{12} [1.16 * 1.76 * 1.14] * [1.18 - 0.86] = 0.1939 * 0.32 = 0.0620,$$

$$S_6(h) = \frac{1}{12} [1.06 * 0.99 * 1.05] * [0.84 - 0.6] = 0.0918 * 0.24 = 0.0220$$



**Definition.5.6.** Let  $\tilde{a} = \langle [(\Gamma, \Upsilon, \Theta, \tau); [\mu^-, \mu^+], \nu] \rangle$  be the trapezoidal cubic fuzzy number; then,  $p(h) = I \times [(\mu^-, \mu^+) + (1 - \nu)]$  is called the accuracy function of  $\tilde{a}$ , where  $I_\Gamma$  is the expected value of trapezoidal cubic fuzzy number  $\Gamma$ .

**Example.5.7.** Let  $I(\varpi) = \frac{1}{12} \langle [((\varpi + \lambda + \zeta + \tau) \times [1 + \mu_\varpi^- - \nu_\varpi]) \times (1 + \mu_\varpi^+ - \nu_\varpi)] \rangle$

Table

$h_1$	[0.15, 0.16, 0.17, 0.18]	$\langle [0.9, 0.15], 0.13 \rangle$
$h_2$	[0.21, 0.22, 0.23, 0.24]	$\langle [0.23, 0.27], 0.25 \rangle$
$h_3$	[0.44, 0.46, 0.48, 0.50]	$\langle [0.6, 0.10], 0.8 \rangle$
$h_4$	[0.55, 0.56, 0.57, 0.58]	$\langle [0.9, 0.15], 0.13 \rangle$
$h_5$	[0.88, 0.90, 0.92, 0.94]	$\langle [0.22, 0.28], 0.24 \rangle$
$h_6$	[0.35, 0.36, 0.37, 0.38]	$\langle [0.39, 0.45], 0.40 \rangle$

$$p(h_1) = \frac{1}{12} [0.66 * 1.77 * 1.02] * [1.05 + 0.87]$$

$$= [0.0992] * 1.92 = 0.1904,$$

$$p(h_2) = \frac{1}{12} [0.9 * 0.98 * 1.02] * [0.5 + 0.75]$$

$$= [0.0749] * [1.25] = 0.0936,$$

$$p(h_3) = \frac{1}{12} [1.88 * 0.8 * 0.3] * [0.7 + 0.2]$$

$$= [0.0376] * 0.9 = 0.0338,$$

$$p(h_4) = \frac{1}{12} [2.26 * 1.77 * 1.02] * [1.05 + 0.87]$$

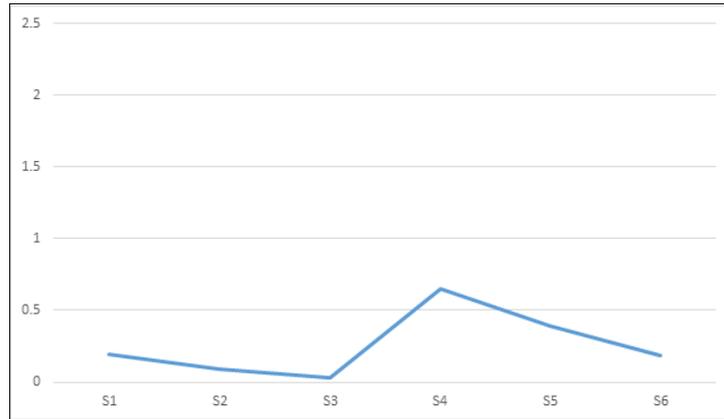
$$= [0.3400] * 1.92 = 0.6528,$$

$$p(h_5) = \frac{1}{12} [3.64 * 0.98 * 1.04] * [0.5 + 0.76]$$

$$= [0.3091] * 1.26 = 0.3894,$$

$$p(h_6) = \frac{1}{12} [1.46 * 0.99 * 1.05] * [0.84 + 0.6]$$

$$= 0.1264 * 1.44 = 0.1820.$$



### 6. Multi-Criteria Decision Making Method Based on Trapezoidal Cubic Fuzzy Numbers

For a brief fuzzy multi-criteria decision making issue, guess that there are  $m$  choices  $A = \{h_1, h_2, \dots, h_n\}$ ,  $l$  decision criteria  $C = \{\Theta_1, \Theta_2, \dots, \Theta_n\}$ , and the relating weight coefficients are  $\omega = \{\omega_1, \omega_2, \dots, \omega_l\}$ ,  $\omega_j \in [0, 1]$ ,  $\omega_1 + \omega_2 + \dots + \omega_l = 1$ . The value of alternative  $h_i$  on the criteria  $\Theta_j$  is trapezoidal cubic fuzzy number

$$h_{ij} = \langle ([m_{1j}(h_i), m_{2j}(h_i), m_{3j}(h_i), m_{4j}(h_i)]; [\mu_j^-(h_i), \mu_j^+(h_i)], \nu_j(h_i)) \rangle$$

The local sorts of criteria are benefit and cost in multi criteria decision making problems. To dispense with the impact from various physical measurements to choice outcomes, the matrix

$$T = (t_{ij})_{m \times n}, \quad (t_{ij})_{m \times n} = \langle ([m_{1j}(h_i), m_{2j}(h_i), m_{3j}(h_i), m_{4j}(h_i)]; [\mu_j^-(h_i), \mu_j^+(h_i)], \nu_j(h_i)) \rangle$$

created by trapezoidal fuzzy numbers of fuzzy decision matrix  $D = (h_{ij})_{n \times l}$  is revamp into homogenized matrix  $R = (r_{ij})_{n \times l}$ ,  $r_{ij} = \langle [r_{ij}^1, r_{ij}^2, r_{ij}^3, r_{ij}^4] \rangle$  using formulas to homogenize the fuzzy decision matrix.

For cost criteria:

Decision steps:

(1) homogenize decision matrix

(2) Using weighted arithmetic average operator

$$h_i = TrC - WAA_{\omega}(\Theta_1(h_i), \Theta_2(h_i), \dots, \Theta_l(h_i)), i = 1, 2, \dots, n$$

or using weighted geometric average operator

$$h_i = TrC - WGA_{\omega}(\Theta_1(h_i), \Theta_2(h_i), \dots, \Theta_l(h_i)), i = 1, 2, \dots, n$$

Aggregate criteria's weights and qualities to land at the mixed trapezoidal cubic fuzzy numbers  $h_i$ ,  $i = 1, 2, \dots, n$  of alternative  $h_i$ .

(3) Enumerate the score value and the accuracy, respectively.

(4) Reeking the alternatives by Definition 5.4.

**7. Example.** There are 4 options  $h_1, h_2, \dots, h_4$  and 4 criteria  $\Theta_1, \Theta_2, \dots, \Theta_4$  in a multi-criteria decision making problem; the weight vector of criteria is  $\omega = (0.20, 0.30, 0.40, 0.10)$ , and the choice data is given as Table 1 by chiefs, extreme to impact positioning of the 4 options.

Steps applying the ways and means in this unit are as continue from

(1) homogenize material in Table 1;

	$\Theta_1$	$\Theta_2$	$\Theta_3$	$\Theta_4$
$A_1$	$\left\{ \begin{array}{l} [0.2, 0.4, \\ 0.6, 0.8], \\ \langle [0.2, 0.6], \\ 0.4 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.4, 0.8, \\ 0.10, 0.18], \\ \langle [0.8, 12], \\ 0.10 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.12, 0.14, \\ 0.16, 0.18], \\ \langle [0.1, 9], \\ 0.6 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.12, 0.14, \\ 0.16, 0.18], \\ \langle [0.52, 60], \\ 0.56 \rangle \end{array} \right\}$
$A_2$	$\left\{ \begin{array}{l} [0.4, 0.6, \\ 0.8, 0.10], \\ \langle [0.14, 0.22], \\ 0.19 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.21, 0.41, \\ 0.61, 0.81], \\ \langle [0.22, 28], \\ 0.25 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.11, 0.14, \\ 0.19, 0.28], \\ \langle [0.2, 0.7], \\ 0.5 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.9, 0.14, \\ 0.19, 0.28], \\ \langle [0.42, 0.50], \\ 0.45 \rangle \end{array} \right\}$
$A_3$	$\left\{ \begin{array}{l} [0.12, 0.14, \\ 0.16, 0.18], \\ \langle [0.2, 0.6], \\ 0.4 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.14, 0.17, \\ 0.20, 0.28], \\ \langle [0.8, 0.12], \\ 0.10 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.21, 0.31, \\ 0.41, 0.51], \\ \langle [0.1, 0.9], \\ 0.6 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.4, 0.8, \\ 0.16, 0.18], \\ \langle [0.52, 0.60], \\ 0.56 \rangle \end{array} \right\}$
$A_4$	$\left\{ \begin{array}{l} [0.22, 0.24, \\ 0.26, 0.28], \\ \langle [0.14, 0.22], \\ 0.19 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.42, 0.44, \\ 0.46, 0.48], \\ \langle [0.22, 0.28], \\ 0.25 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.10, 0.14, \\ 0.16, 0.18], \\ \langle [0.2, 0.7], \\ 0.5 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.20, 0.22, \\ 0.26, 0.29], \\ \langle [0.42, 0.50], \\ 0.45 \rangle \end{array} \right\}$

(2) Total every one of the components  $a_{ij}$  ( $j = 1, \dots, 4$ ) in the  $i$ th row of decision matrix  $D$  using TrC-WAA; then, the coordinated trapezoidal cubic fuzzy numbers

$h_i, i = 1, 2, \dots, 4$  of alternative  $\Gamma_i$  are achieved.

$$h_1 = \langle [(0.22, 0.39, 0.23, 0.304), [0.4742, 0.7089], 0.3209] \rangle$$

$$h_2 = \langle [0.324, 0.447, 0.716, 0.147][0.2199, 0.5029], 0.3311 \rangle$$

$$h_3 = \langle [0.19, 0.219, 0.272, 0.342], [0.4742, 0.7089], 0.3209 \rangle$$

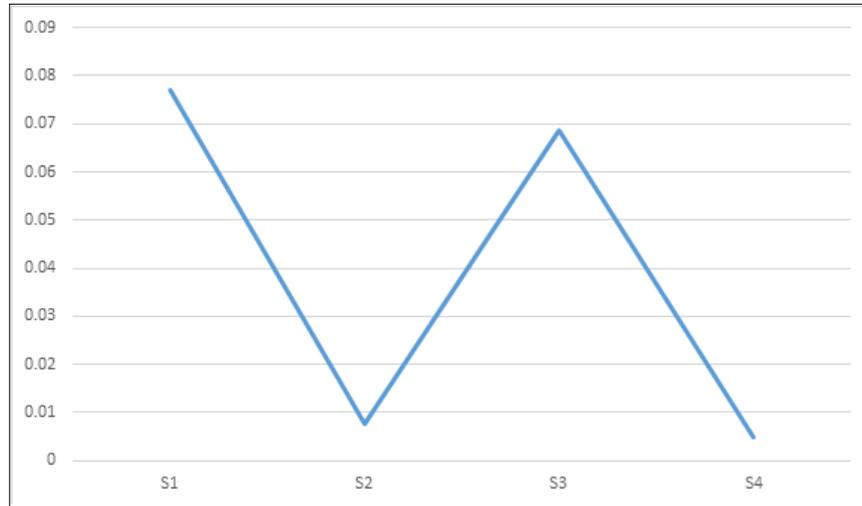
$$h_4 = \langle [(0.23, 0.258, 0.28, 0.298), [0.2199, 0.5029], 0.3311] \rangle$$

(3) Calculate the score values  $S(h_i)$  of  $h_i$

$$S_1 = 0.0769, S_2 = 0.0076,$$

$$S_3 = 0.0687, S_4 = 0.0049.$$

Step 4 : Rank the score value  $S_1 > S_3 > S_2 > S_4$



$$H_1 = 0.2841, H_2 = 0.1974, H_3 = 0.5975, H_4 = 0.1287.$$

	$\Theta_1$	$\Theta_2$	$\Theta_3$	$\Theta_4$
$A_1$	$\left\{ \begin{array}{l} [0.2, 0.4, \\ 0.6, 0.8], \\ \langle [0.2, 0.6], \\ 0.4 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.4, 0.8, \\ 0.10, 0.18], \\ \langle [0.8, 12], \\ 0.10 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.12, 0.14, \\ 0.16, 0.18], \\ \langle [0.1, 9], \\ 0.6 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.12, 0.14, \\ 0.16, 0.18] \\ \langle [0.52, 60], \\ 0.56 \rangle \end{array} \right\}$
$A_2$	$\left\{ \begin{array}{l} [0.4, 0.6, \\ 0.8, 0.10], \\ \langle [0.2, 0.6], \\ 0.4 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.21, 0.41, \\ 0.61, 0.81], \\ \langle [0.8, 12], \\ 0.10 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.11, 0.14, \\ 0.19, 0.28], \\ \langle [0.1, 0.9], \\ 0.6 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.9, 0.14, \\ 0.19, 0.28] \\ \langle [0.52, 0.60], \\ 0.56 \rangle \end{array} \right\}$
$A_3$	$\left\{ \begin{array}{l} [0.12, 0.14, \\ 0.16, 0.18], \\ \langle [0.2, 0.6], \\ 0.4 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.14, 0.17, \\ 0.20, 0.28], \\ \langle [0.8, 0.12], \\ 0.10 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.21, 0.31, \\ 0.41, 0.51], \\ \langle [0.1, 0.9], \\ 0.6 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.4, 0.8, \\ 0.16, 0.18], \\ \langle [0.52, 0.60], \\ 0.56 \rangle \end{array} \right\}$
$A_4$	$\left\{ \begin{array}{l} [0.22, 0.24, \\ 0.26, 0.28], \\ \langle [0.2, 0.6], \\ 0.4 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.42, 0.44, \\ 0.46, 0.48], \\ \langle [0.8, 0.12], \\ 0.10 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.10, 0.14, \\ 0.16, 0.18] \\ \langle [0.1, 0.9], \\ 0.6 \rangle \end{array} \right\}$	$\left\{ \begin{array}{l} [0.20, 0.22, \\ 0.26, 0.29], \\ \langle [0.52, 0.60], \\ 0.56 \rangle \end{array} \right\}$

In the event that all components  $h_{ij}$  ( $j=1, \dots, 4$ ) in the  $i$ th row of decision matrix  $D$  are aggregated using  $TC - WGA$ , the coordinated trapezoidal cubic fuzzy numbers  $\Gamma_i$ ,  $i=1, 2, \dots, 4$  of alternative  $h_i$  are as per the following

(2) Total every one of the components  $h_{ij}$  ( $j=1, \dots, 4$ ) in the  $i$ th row of decision matrix  $D$  using  $TrC - WGA$ ; then, the coordinated trapezoidal cubic fuzzy numbers  $h_i, i=1, 2, \dots, 4$  of alternative  $h_i$  are accomplished.

$$h_1 = \langle [(0.2583, 0.2183, 0.0749, 0.5846), [0.2527, 0.4354], 0.4414] \rangle$$

$$h_2 = \langle [0.2134, 0.2585, 0.3594, 0.3134), [0.2527, 0.4354], 0.4414] \rangle$$

$$h_3 = \langle [0.1773, 0.2427, 0.2492, 0.3117]), [0.2527, 0.4354], 0.4414] \rangle$$

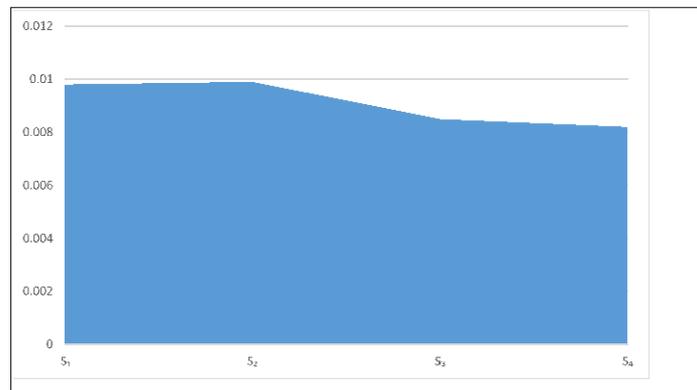
$$h_4 = \langle [(0.1930, 0.2300, 0.2540, 0.2767), [0.2527, 0.4354], 0.4414] \rangle$$

(3) Calculate the score values  $S(\Gamma_i)$  of

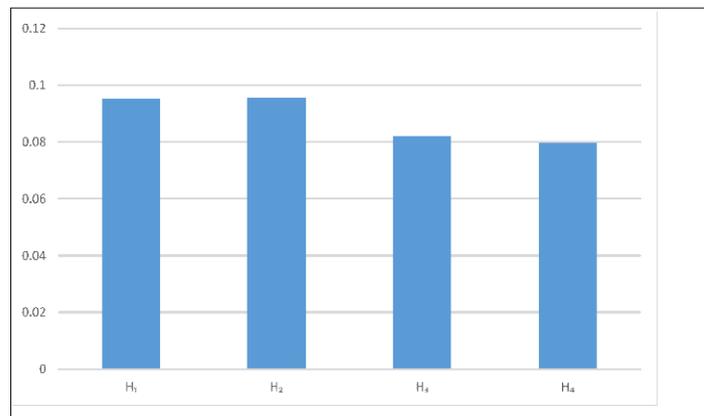
$$S_1 = 0.0098, S_2 = 0.0099,$$

$$S_3 = 0.0085, S_4 = 0.0082.$$

Step 4: Rank the alternatives by  $S_2 > S_1 > S_3 > S_4$



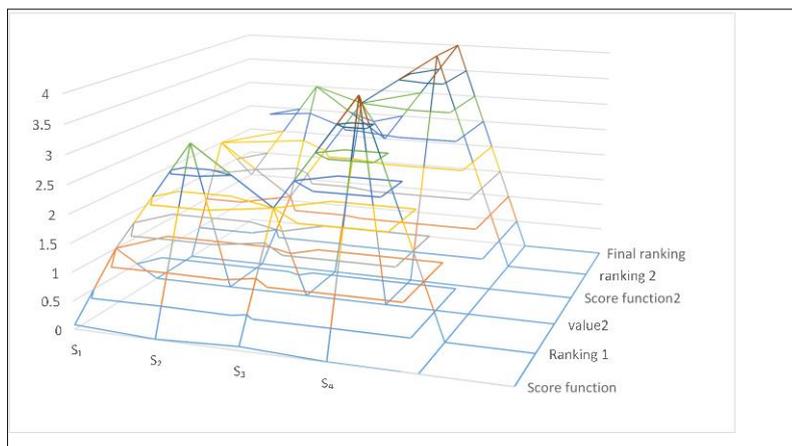
$$H_1 = 0.0951, H_2 = 0.0958, H_3 = 0.0821, H_4 = 0.0797.$$



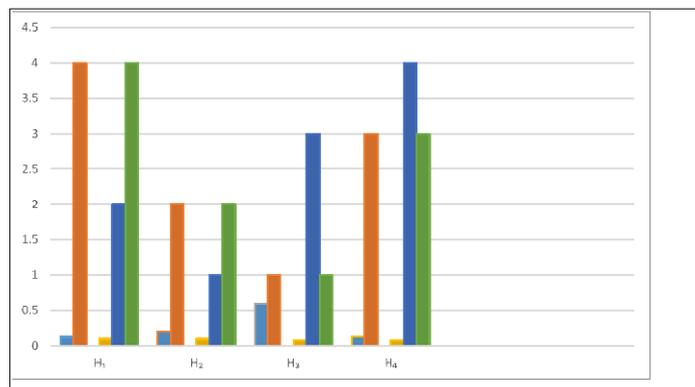
### 8. Comparison Analyses

The result of the score value 1 and score value 2 of the numerical examples are tabulated below

	Score function	Ranking 1	Score function	ranking 2	Final ranking
$S_1$	0.0769	1	$S_1$	0.0098	2
$S_2$	0.0076	3	$S_2$	0.0099	1
$S_3$	0.0687	2	$S_3$	0.0085	3
$S_4$	0.0049	4	$S_4$	0.0082	4



	Accuracy function	Ranking 1	Accuracy function	ranking 2	Final ranking
$H_1$	0.1241	4	$H_1$	0.0951	2
$H_2$	0.1974	2	$H_2$	0.0958	1
$H_3$	0.5975	1	$H_3$	0.0821	3
$H_4$	0.1287	3	$H_4$	0.0797	4



## 9. Conclusion

In this paper, we define trapezoidal cubic fuzzy numbers and their operational laws. Started on these operational laws, each collection operators, with trapezoidal cubic fuzzy weighted arithmetic averaging operator and weighted geometric averaging operator are purposed. Expected values, score function, and accuracy function of trapezoidal cubic fuzzy numbers are defined.

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