Konuralp Journal of Mathematics, 13 (2) (2025) 154-159

Konuralp Journal of Mathematics

Research Paper

Journal Homepage: *www.dergipark.gov.tr/konuralpjournalmath*

e-ISSN: 2147-625X

On The Coefficients of Quasi-Subordine Functions

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Abstract

In this work we define two subclasses $H(\alpha,\beta)$ and $k\left(p,\alpha,\frac{1}{2}\right)$. If $g(z)$ is in the classes then we obtain coefficient bounds for the functions $f(z)$ which is majorized by $g(z)$ and quasisubordinate to $g(z)$.

Keywords: *Mojorization; Quasi-subordination; Starlike function; Subordination; Univalent*
2010 Mathematics Subject Classification: *30C45*

1. Introduction

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.1}$$

which are analytic in the unit disk

$U = \{z \in \mathbb{C} : \lfloor z \rfloor < 1\}$

For two functions $f(z)$ and $g(z)$ analytic in U, we say that $f(z)$ is subordine to $g(z)$ in U and we write $f(z) \prec g(z)$. if there exists a function $\phi(z)$ is analiytic in U with

$$\phi(0) = 0, \quad \lfloor \phi(z) \rfloor \le \lfloor z \rfloor \text{ such that}$$
$$f(z) = g(\phi(z)) \quad (z \in U) \tag{1.2}$$

In particular if g is univalent in U, the above subordination, is equivalent to

$f(0) = g(0) = 0, f(U) \subset g(U)$

Definition 1.1. *Let $H(\alpha,\beta)$ be the subclass of the class A, which satisfy the condition*

$$\text{Re}\left[\sqrt{\frac{\text{g(z)}}{z}} + \alpha z \left(\sqrt{\frac{\text{g(z)}}{z}}\right)'\right] > \beta \tag{1.3}$$

where $\alpha \ge 0, \quad 0 \le \beta < 1$ and $g(z)$ is univalent and in the form (1.1).

Definition 1.2. *The function $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is maiorized by $g(z)$ if there exists a function $w(z)$ is analytic in U with $w(0) = 0$, $|w(z)| \le 1$ such that*

$$f(z) = w(z) \cdot g(z) \tag{1.4}$$

The class $H(\alpha,\beta)$ is defined by Altıntaş in [1].

For the definitions subordination and majorization see [4]. majorization is studied by MacGregor in [6], $H\left(\alpha,\frac{1}{2}\right)$ is studied by Altıntaş in [1, 2].

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2. The Class $H(\alpha,\beta)$

Theorem 2.1. *If $g(z)\in H(\alpha,\beta)$ and $f(z)$ is majorized by $g(z)$ then we have*

$$\lfloor a_n\rfloor \leq 1+4(1-\beta)^2\left[\frac{1}{(1+\alpha)^2}+\frac{1}{(1+2\alpha)^2}+\dots+\frac{1}{(1+(n-1)\alpha)^2}\right] \tag{2.1}$$

where $\alpha\geq 0,\quad 0\leq\beta<1,\quad n=2,3,4,\dots.$

Proof. We let

$$\sqrt{\frac{g(z)}{z}}=P(z)=1+p_1z+p_2z^2+\dots \tag{2.2}$$

If

$$\operatorname{Re}\left(P(z)+\alpha zP'(z)>\beta \text{ then } \operatorname{Re}P(z)>\beta. \qquad \text{(see [1])}. \tag{2.3}$$

Hence we have

$$|p_n|\leq 2(1-\beta). \tag{2.4}$$

Since

$$P(z)+\alpha zP'(z)=\mathrm{R}(z)=1+r_1z+r_2z^2+\dots \text{ and } \operatorname{Re}R(z)>\beta \tag{2.5}$$

we have from (2.3), (2.4) and (2.5)

$$|r_n|\leq(1+n\alpha)\,|p_n|\,,\quad |p_n|\leq\frac{2(1-\beta)}{1+n\alpha}.$$

If $\alpha=0$, then $P(z)=\dfrac{1+(1-2\beta)z}{1-z}$ is the sharp bound for the Theorem 2.1.
If $\alpha>0$ multiply the equality

$$P(z)+\alpha zP'(z)=R(z) \tag{2.6}$$

by $\frac{1}{\alpha}z^{\frac{1}{\alpha}-1}$. The first side of (2.6) is the derivative of $z^{\frac{1}{\alpha}}P(z)$.
Hence we have

$$z^{\frac{1}{\alpha}}P(z)=\frac{1}{\alpha}\int_0^z t^{\frac{1}{\alpha}-1}R(t)\,dt$$

and taking

$$R(z)=\frac{1+(1-2\beta)z}{1-z}$$

we find

$$P(z)=\frac{1}{\alpha}z^{-\frac{1}{\alpha}}\int_0^z t^{\frac{1}{\alpha}-1}\left[\frac{1+(1-2\beta)t}{1-t}\right]dt \tag{2.7}$$

The coefficients of this $P(z)$ satisfy

$$p_n=\frac{2(1-\beta)}{1+n\alpha} \tag{2.8}$$

and
f is majorized by g in U then we have from (1.4)

$$f(z)=w(z)\cdot g(z),\quad w(z)=\alpha_0+\alpha_1z+\dots \text{ and } \lceil w(z)\rceil\leq 1$$

see MacGregor (1967)

$$a_n=\alpha_{n-1}b_1+\alpha_{n-2}b_2+\dots\alpha_1b_{n-1}+\alpha_0b_n.$$

From Cauchy Integral formula we write

$$a_n=\frac{1}{2\pi i}\int_{|z|=r}\frac{f(z)}{z^{n+1}}dz=\frac{1}{2\pi i}\int_{[z[=r}\frac{w(z)}{z^n}\cdot\frac{g(z)}{z}dz$$

and

$$a_n=\frac{1}{2\pi i}\int_{|z|=r}\frac{\phi(z)}{z^n}\cdot\left(1+b_2z+b_2z^2+b_nz^{n-1}\right)dz. \tag{2.9}$$

Since $\frac{g(z)}{z} = P^2(z)$ and $|w(z)| \leq 1$ if we let

$q(z) = 1 + p_1 z + \ldots + p_{n-1} z^{n-1}$ then

$$\lfloor a_n \rfloor \leq \frac{1}{2\pi r^{n-1}} \int_0^{2\pi} \left\lfloor q\left(re^{i\theta}\right)\right\rfloor^2 d\theta, \quad z = re^{i\theta} \tag{2.10}$$

From parseval identity we have

$$\lfloor a_n \rfloor \leq \frac{1}{r^{n-1}} \left(1 + \lfloor p_1 \rfloor^2 + \lfloor p_2 \rfloor^2 + \ldots + \lfloor p_{n-1} \rfloor^2\right), \\ \lfloor a_n \rfloor \leq 1 + \lfloor p_1 \rfloor^2 + \lfloor p_2 \rfloor^2 + \ldots + \lfloor p_{n-1} \rfloor^2 \tag{2.11}$$

Using (2.5) we obtain

$$\lfloor a_n \rfloor \leq 1 + 4(1-\beta)^2 \left(\frac{1}{(1+\alpha)^2} + \frac{1}{(1+2\alpha)^2} + \ldots + \frac{1}{(1+(n-1)\alpha)^2}\right). \tag{2.12}$$

We show that this bound is sharp for $\alpha = 0$ we let $f(z) = g(z)$ and

$$P(z) = \frac{1+(1-2\beta)z}{1-z} \tag{2.13}$$

for $\alpha > 0$ the polinomial of $q(z)$

$$q(z) = 1 + 2(1-\beta)\left[\frac{1}{1+\alpha}z + \frac{1}{1+2\alpha}z^2 + \ldots + \frac{1}{1+(n-1)\alpha}z^{n-1}\right] \tag{2.14}$$

form a stricly decreasing sequence of positive terms. Since the KAKEYA Theorem [5] $q(z)$ does not vanish for $\lfloor z \rfloor \leq 1$, The function

$$K(z) = \frac{z^{n-1} \mathrm{Q}\left(\frac{1}{z}\right)}{Q(z)} \tag{2.15}$$

is analiytic for $\lfloor z \rfloor \leq 1$ and $\lfloor K(z) \rfloor = 1$ for $\lfloor z \rfloor = 1$ and by maximum Modulo Theorem we conclude that $\lfloor K(z) \rfloor \leq 1$ for $\lfloor z \rfloor \leq 1$.
In this case we may replace $w(z)$ in theorem by $K(z)$.
Thus

$$a_n = \frac{1}{2\pi i} \int_{[z[=r} \frac{K(z)}{z^n} q^2(z) dz$$

$$a_n = \frac{1}{2\pi i} \int_{[z[=r} \frac{1}{z} q\left(\frac{1}{z}\right) \cdot q(z)\, dz$$

$$a_n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{1}{r \cdot e^{i\theta}} \cdot q\left(r \cdot e^{i\theta}\right) \cdot q\left(r \cdot e^{i\theta}\right) r \cdot e^{i\theta} d\theta$$

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \left|q\left(e^{i\theta}\right)\right|^2 d\theta$$

$a_n = 1 + |p_1|^2 + |p_2|^2 + \ldots + |p_{n-1}|^2$ and we obtain

$$a_n = 1 + 4(1-\beta)^2 \left|\frac{1}{(1+\alpha)^2} + \frac{1}{(1+2\alpha)^2} + \ldots + \frac{1}{(1+(n-1)\alpha)^2}\right|. \tag{2.16}$$

Consequently for $\alpha > 0$ we take $g(z) = 2P^2(z)$ in the equality $f(z) = w(z) \cdot g(z)$ replace $P(z)$ in (2.7) and replace $w(z)$ by the function $K(z)$ in (2.15) and then obtain the function $f(z)$ for the sharp bound. The proof of the theorem is completed. □

Corollary 2.2. *If $g \in H\left(\alpha, \frac{1}{2}\right)$ and f is majorized by g in U. Then we have*

$$|a_n| \leq 1 + \frac{1}{(1+\alpha)^2} + \frac{1}{(1+2\alpha)^2} + \ldots + \frac{1}{(1+(n-1)\alpha)^2} \qquad (see\ [1])$$

Corollary 2.3. *If $g \in H\left(0, \frac{1}{2}\right)$ and f in majorized by g in U.Then we have $|a_n| \leq n$.*
We let $\alpha = 0$ in Corollary 2.2 (see [1])

Note :
If $g \in H\left(0, \frac{1}{2}\right)$, then g is starlike in u (see [4])
If g is starlike and f is majorized by g, then $|a_n| \leq n$ (see [6, Theorem 2])

Corollary 2.4. *If $g \in H(0,0)$ and f in majorized by g in U.*
Then we have

$|a_n| \leq 1 + 4(n-1)$

3. The Class $K\left(p,\alpha,\frac{1}{2}\right)$

Definition 3.1. *$f(z)$, $g(z)$, $s(z)$ are in the form equation (1.1)*

$$g(z)\in k\left(p,\alpha,\frac{1}{2}\right)\Leftrightarrow \operatorname{Re}\left\{\sqrt[p]{\frac{g(z)}{s(z)}}+\alpha z\left(\sqrt[p]{\frac{g(z)}{s(z)}}\right)'\right\}>\frac{1}{2} \tag{3.1}$$

where $p\in IN^{+},\alpha\geq 0, s(z)$ is starlike and univalent in U.

Definition 3.2. *Suppese that $f(z)$, $g(z)$, $w(z)$, $\phi(z)$ are analytic in U,*

$w(0)=\phi(0)=0,\quad |w(z)|\leq 1,\quad |\phi(z)|\leq |z|$ and

if the condition

$$f(z)=w(z)\cdot g(\phi(z)) \tag{3.2}$$

is satisfy then we say that $f(z)$ is quasi-subordine to $g(z)$ in U and we write

$$f(z)\underset{q}{\prec} g(z). \qquad \text{(see [3])} \tag{3.3}$$

It is clear that
If $\phi(z)=z\Rightarrow f(z)<g(z)$, f is subordina to g.
If $w(z)=1\Rightarrow |f(z)|\leq |g(z)|, f$ is majorized by g.

Theorem 3.3. *If $g(z)\in K\left(p,\alpha,\frac{1}{2}\right)$ and $f(z)$ is quasi-subordine to $g(z)$ in U then we have*

$$|a_n|\leq \frac{(p+2)(p+3)\dots(p+n)}{(n-1)!}. \tag{3.4}$$

Proof. If $g\in K\left(p,\alpha,\frac{1}{2}\right)$, then $\operatorname{Re}\sqrt[p]{\frac{g(z)}{s(z)}}>\frac{1}{2}$. (see [1]).
We let $\sqrt[p]{\frac{g(z)}{s(z)}}=h(z)$, since $\operatorname{Re}h(z)>\frac{1}{2}$ $h(z)<\frac{1}{1-z}$. If we let

$$H=\left\{h:h\prec\frac{1}{1-z}\right\} \text{ then}$$
$$H^p=\{h^p:h\in H\} \tag{3.5}$$
$$\text{ex}\,\overline{CO}H^p=\left\{P(z):P(z)=\frac{1}{(1-z)^p}\right\}.$$

where $ex\overline{\mathrm{CO}}H^p$ means the set of extreme points of the closed convex hull of H^p.
If we set $\frac{g(z)}{s(z)}=K(z)$, then we have from Definition 3.2.

$$f(z)=w(z)\cdot s(\phi(z)\cdot K(\phi(z)). \tag{3.6}$$

Let us $R(z)=K(\phi(z))$ and $Q(z)=w(z)\cdot s(\phi(z))$

$$R(z)\prec K(z),\quad R(z)\prec\frac{1}{(1-z)^p}. \tag{3.7}$$

If $R(z)=1+r_1z+r_2z^2+\dots$ then we have

$$|r_n|\leq\frac{p(p+1)\dots(p+n-1)}{n!}\qquad n\geq 1. \tag{3.8}$$

$$Q(z)=w(z).s(\phi(z))\Rightarrow Q(z)\underset{q}{\prec} s(z). \tag{3.9}$$

If $Q(z)=q_1z+q_2z^2+\dots$ since $s(z)$ is starlike in U, then we have

$|q_n|\leq n$ see [3]

and from

$f(z)=Q(z)R(z)$ we find

$$a_n=q_n+q_{n-1}r_1+q_{n-2}r_2+\dots+q_1r_{n-1} \text{ and}$$
$$|a_n|\leq n+(n-1)p+(n-2)\frac{p(p+1)}{2!}+\dots+\frac{p(p+1)\dots(p+n-2)}{(n-1)!}. \tag{3.10}$$

Using the induction method on the right-hand side in this inequality (3.10). We obtain

$$|a_n| \leq \frac{(p+2)(p+3)\dots(p+n)}{(n-1)!} \qquad \text{(see [3])} \tag{3.11}$$

and the proof of Theorem is completed. The sharp bound for the Theorem we have in the equality

$$f(z) = w(z) \cdot g(\varnothing(z))$$
$$w(z) = 1, \varnothing(z) = z$$

and

$$f(z) = g(z) = \frac{z}{(1-z)^{p+2}} \tag{3.12}$$

□

Corollary 3.4. *If*

$$\text{Re}\left[\sqrt{\frac{g(z)}{s(z)}} + \alpha z\left(\sqrt{\frac{g(z)}{s(z)}}\right)'\right] > \frac{1}{2}$$

and
$f(z) \underset{q}{\prec} g(z)$ in U. Then we have

$$|a_n| \leq \frac{4.5.6\dots(n+2)}{(n-1)!}.$$

For the sharp bound $f(z) = g(z) = \dfrac{z}{(1-z)^4}$.

Corollary 3.5. *If*

$$\text{Re}\left[\sqrt[p]{\frac{g(z)}{z}} + \alpha z\left(\sqrt[p]{\frac{g(z)}{z}}\right)'\right] > \frac{1}{2}$$

and
$f(z) \underset{q}{\prec} g(z)$ in U. Then we have

$$|a_n| \leq \frac{p(p+1)\dots(p+n-2)}{(n-1)!}.$$

For the sharp bound $f(z) = g(z) = \dfrac{z}{(1-z)^p}$.

Corollary 3.6. *If*

$$\text{Re}\left[\sqrt{g'(z)} + z\left(\sqrt{g'(z)}\right)'\right] > \frac{1}{2}$$

and
$f(z) \underset{q}{\prec} g(z)$ in U. Then we have

$$|a_n| \leq 1.$$

for the sharp bound $f(z) = g(z) = \dfrac{z}{1-z}$. (see [2])

Note:
On the other hand, if CV is the class convex functions in U, then

$$\text{Re}\sqrt{f'(z)} > \frac{1}{2} \Rightarrow f \in CV \qquad \text{(Goodman 1.129 pr.13)}$$

$g \in CV$ and $f(z)$ is majorized by $g(z)$ then $|a_n| \leq 1$. (see [6, Theorem 2])

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable

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