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MAXIMUM LIKELIHOOD ESTIMATION FOR THE INVERTED KUMARASWAMY DISTRIBUTION BASED ON THE EXTREME RANKED SET SAMPLING

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ABSTRACT

This study investigates the use of extreme ranked set sampling (ERSS) for parameter estimation in the Inverted Kumaraswamy (IK) distribution. The Kumaraswamy distribution has wide applications in fields such as reliability testing, environmental studies, financial analysis, and survival analysis. The paper emphasizes the efficiency advantages of ERSS, particularly in capturing extreme values, which are crucial for distributions with heavy tails or skewed data. By incorporating ERSS, this research demonstrates that more accurate and efficient parameter estimates can be obtained compared to traditional sampling methods like simple random sampling (SRS). A simulation study is also performed to demonstrate the performance of the proposed estimators. Finally, a real data set is presented for illustrative purposes. The findings suggest that ERSS outperforms SRS in terms of precision, particularly in contexts where extreme values play a significant role. This work contributes to the advancement of sampling techniques for extreme value contexts, with potential applications in various fields, including environmental research, finance, and reliability analysis.

Keywords: Inverted Kumaraswamy distribution, Ranked set sampling, Extreme ranked set sampling, Maximum likelihood estimator.

1 INTRODUCTION

Statistical sampling methods are fundamental tools in data collection and analysis. One of the most widely used techniques is SRS, where every member of the population has an equal chance of being selected. Although SRS is straightforward and easy to implement, it can be inefficient, particularly when dealing with populations that are difficult to measure or when the data of interest is costly or time-consuming to collect. Additionally, SRS may result in low precision when sample sizes are small, especially in situations where extreme values are crucial for analysis. To address these limitations, alternative sampling methods have been developed, one of the most notable being ranked set sampling (RSS). RSS aims to improve the efficiency of the sampling process by using ranked sets of observations rather than individual random selections. In this method, a small set of units is randomly selected, and then the units are ranked based on some auxiliary variable. The rank order is used to select the final sample, thereby reducing variance and improving precision compared to SRS.

In recent years, RSS has gained significant attention, and several studies have demonstrated its advantages in various applications. For instance, conducted a reliability test using SRS, RSS, and ERSS methods, showing the efficiency of RSS and ERSS in terms of precision [1]. Similarly, explored the application of RSS in estimating population means for skewed distributions, highlighting its superiority over traditional sampling techniques [2]. Moreover, extended the concept of RSS by incorporating the Kumaraswamy distribution, improving sampling efficiency for distributions with heavy tails [3]. Additionally, applied RSS in environmental studies, showing its ability to improve the accuracy of parameter estimates for extreme value distributions [4]. Another significant contribution is by [5], who compared the performance of RSS and traditional sampling methods in agricultural research, demonstrating the robustness of RSS in handling highly variable data.

The ERSS is an extension of the traditional RSS method, specifically designed to improve the efficiency of sampling when extreme values are of particular interest. In ERSS, a set of units is randomly selected, and these units are ranked based on an auxiliary variable. Unlike RSS, where the entire set of ranks is considered, ERSS focuses on selecting the extreme (highest or lowest) values from each set to form the final sample. This approach is particularly useful when the goal is to estimate parameters that are sensitive to extreme values, such as in environmental or financial data. By emphasizing extreme observations, ERSS often leads to more precise estimates in cases where traditional sampling methods might be less effective.

In this study, our primary goal is to investigate IK distribution using ERSS. While previous research has applied traditional RSS, SRS, and modified ranked set sampling to estimate parameters for various distributions, the use of ERSS with the IK distribution remains unexplored. The motivation behind this study stems from the potential advantages of ERSS in capturing extreme values, which are crucial when modeling distributions with heavy tails or skewed data, such as the IK distribution. By utilizing ERSS, we aim to provide more efficient and accurate parameter estimates compared to conventional sampling methods. This research fills a gap in the literature by exploring ERSS as a novel approach for parameter estimation in extreme value contexts. Our findings could have significant implications for fields such as environmental studies, finance, and reliability analysis, where accurate estimation of extreme values is critical. Ultimately, this study contributes to the advancement of sampling techniques and their application to complex distributions that have not been adequately addressed in prior research.

2 PRELIMINARIES

In order to further enhance sampling efficiency, the IK distribution is incorporated into the present study. This distribution is particularly well-suited for modeling data with specific characteristics, such as heavy tails and skewness, which are commonly encountered in extreme value analysis. For the IK distribution with shape parameters α and β the probability density function (pdf) and the cumulative distribution function (cdf) are given by Eq. (1) and (2), respectively;

$$f(x;\alpha;\beta) = \alpha\beta(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\beta-1}, \quad x,\alpha,\beta > 0 \tag{1}$$

$$F(x;\alpha;\beta) = (1-(1+x)^{-\alpha})^{\beta}, \quad x,\alpha,\beta > 0 \tag{2}$$

The Kumaraswamy distribution has been widely used in various fields, including reliability and life testing, where it effectively models the time until an event occurs or system reliability. For example, demonstrated its effectiveness in estimating parameters for reliability testing, particularly in engineering and quality control applications [6]. In environmental studies, the IK distribution has been applied to model extreme water quality parameters, with showing how this distribution improves the accuracy of environmental predictions by focusing on extreme values [7]. Similarly, in hydrological studies, utilized the Kumaraswamy distribution for modeling rainfall data and water levels, highlighting its applicability in water resource management [8]. The IK distribution has also been valuable in financial data analysis, with using it to analyze stock prices and assess risk, particularly in extreme market conditions [9]. In survival analysis, Kumaraswamy distribution has been employed to model patient recovery times, as demonstrated by [10], offering insights into clinical outcomes. Finally, applied the IK distribution to model disease spread rates, emphasizing its usefulness in healthcare data and policy analysis, especially in managing extreme disease outbreaks [11].

ERSS and other RSS designs have become increasingly popular in parameter estimation due to their ability to improve the precision of sample estimates, especially when dealing with populations where extreme values play a significant role. These methods are particularly valuable in fields such as environmental studies, reliability testing, and financial analysis, where precise estimation of parameters is crucial. Over the years, numerous studies have highlighted the advantages of RSS and ERSS in improving estimation accuracy compared to traditional sampling methods. The following works illustrate the growing body of research in this area, showcasing the evolution of RSS and ERSS applications from earlier to more recent studies. For instance, compared the efficiency of RSS and SRS in parameter estimation, showing that RSS outperformed SRS in terms of precision, particularly in cases involving skewed distributions [12]. Extended the application of RSS to extreme value theory, demonstrating its effectiveness in modeling rare events [13]. Applied Kumaraswamy distribution within an RSS framework for reliability testing, further proving its utility in real-world engineering applications [14]. Followed this up by incorporating ERSS into their work on population mean estimation, emphasizing its improved efficiency in capturing extreme observations [15]. Applied ERSS in survival analysis for clinical data, underscoring its advantage in estimating the survival rates of patients [16]. Finally, introduces ERSS as a practical alternative to traditional RSS for estimating the population mean, demonstrating its unbiasedness in symmetric populations and its efficiency over SRS with examples [17].

By structuring our study around these foundations, we aim to provide a comprehensive evaluation of ERSS in estimating the parameters of the IK distribution, filling an existing gap in the literature and contributing to advancements in sampling methodologies.

3 MATERIAL AND METHOD

In this section, we will examine the maximum likelihood estimates (MLE) of the unknown parameters of the IK distribution under SRS and ERSS.

3.1 MLE Based on SRS

Inverted kumaraswamy distribution with parameters α and β. The likelihood function of α and β is given by

$$L(\alpha,\beta;\ x) = \prod_{i=1}^{n} \alpha\,\beta(1+x_i)^{-(\alpha+1)}(1-(1+\ x_i)^{-\alpha})^{\beta-1} \tag{3}$$

$$L(\alpha,\beta;\, x) = \alpha^n \beta^n \prod_{i=1}^{n} (1 + x_i)^{-(\alpha+1)} \prod_{i=1}^{n} (1 - (1 + \, x_i)^{-\alpha})^{\beta-1} \tag{4}$$

and the log-likelihood function is

$$\begin{aligned} lnL(\alpha,\beta) = nln(\alpha) + nln(\beta) - (\alpha+1)\sum_{i=1}^{n} ln\,(1+x_i) \\ +(\beta-1)\textstyle\sum_{i=1}^{n} ln(1-(1+\, x_i)^{-\alpha}) \end{aligned} \tag{5}$$

The MLE of parameters are obtained by simultaneously solving the following equations:

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} ln\,(1+x_i) + (\beta-1)\sum_{i=1}^{n} \frac{In(1+x_i)(1+x_i)^{-\alpha}}{1-(1+x_i)^{-\alpha}} = 0 \tag{6}$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} ln(1-(1+\, x_i)^{-\alpha}) = 0 \tag{7}$$

3.2 MLE Based on ERSS

Consider the sample size m is odd. The PDF of $X_{ij(1)}, X_{ij(m)}$ and $X_{ij\left(\frac{m+1}{2}\right)}$ is given by.

$$\begin{aligned} g\left(x_{ij(1)}, x_{ij(m)}, x_{ij\left(\frac{m+1}{2}\right)};\, \alpha,\beta\right) = mf\left(x_{ij(1)};\, \alpha,\beta\right)\left(1-F\left(x_{ij(1)};\, \alpha,\beta\right)\right)^{m-1} \\ \times mf\left(x_{ij(m)};\, \alpha,\beta\right)\left(F\left(x_{ij(m)};\, \alpha,\beta\right)\right)^{m-1} \frac{m!}{\left(\frac{m-1}{2}\right)!^2} f\left(x_{ij\left(\frac{m+1}{2}\right)};\, \alpha,\beta\right) \\ \times \left(F\left(x_{ij\left(\frac{m+1}{2}\right)};\, \alpha,\beta\right)\right)^{\frac{m-1}{2}} \left(1-F\left(x_{ij\left(\frac{m+1}{2}\right)};\, \alpha,\beta\right)\right)^{\frac{m-1}{2}}, \end{aligned} \tag{8}$$

$$\begin{aligned} g\left(x_{ij(1)}, x_{ij(m)}, x_{ij\left(\frac{m+1}{2}\right)};\, \alpha,\beta\right) = m\ \left(\left(\alpha\,\beta\,\left(1+x_{ij(1)}\,\right)^{-(\alpha+1)}\right)\right. \\ \times \left(1-\left(1+\, x_{ij(1)}\right)^{-\alpha}\right)^{(\beta-1)} \left(1-\left(1-\left(1+\, x_{ij(1)}\right)^{-\alpha}\right)^{\beta}\right)^{m-1} \\ \times m\left(\left(\alpha\,\beta\,\left(1+x_{ij(m)}\,\right)^{-(\alpha+1)}\left(1-\left(1+\, x_{ij(m)}\right)^{-\alpha}\right)^{\beta-1}\right)\right. \\ \times \left(\left(1-\left(1+\, x_{ij(m)}\right)^{-\alpha}\right)^{\beta}\right)^{m-1} \frac{m!}{\left(\frac{m-1}{2}\right)!^2} \left(\alpha\,\beta\,\left(1+x_{ij\left(\frac{m+1}{2}\right)}\right)^{-(\alpha+1)}\right. \\ \times \left(1-\left(1+\, x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}\right)^{(\beta-1)} \left(\left(1-\left(1+\, x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}\right)^{\beta}\right)^{\frac{m-1}{2}} \\ \times \left(1-\left(1-\left(1+\, x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}\right)^{\beta}\right)^{\frac{m-1}{2}} \end{aligned}$$

The likelihood function of $ERSS_{odd}$ is given by

$$L(\alpha,\beta;\ x) = \prod_{j=1}^{r}\prod_{i=1}^{(m-1)/2} g\left(x_{ij(1)}\ ;\ \alpha,\beta\right) \times \prod_{j=1}^{r}\prod_{i=(m+1)/2}^{m-1} g\left(x_{ij(m)}\ ;\ \alpha,\beta\right) \times \prod_{j=1}^{r}\prod_{i=m}^{m} g\left(x_{ij\left(\frac{m+1}{2}\right)}\ ;\ \alpha,\beta\right) \tag{9}$$

$$L(\alpha,\beta;\ x) =\times \prod_{i=1}^{(m-1)/2}\prod_{j=1}^{r}\left(1+x_{ij(1)}\right)^{-(\alpha+1)}\left(1-\left(1+ x_{ij(1)}\right)^{-\alpha}\right)^{\beta-1}$$

$$\left(1-\left(1-\left(1+ x_{ij(1)}\right)^{-\alpha}\right)^{\beta}\right)^{m-1} \times \prod_{i=(m+1)/2}^{m-1}\prod_{j=1}^{r}\left(1+x_{ij(\mathrm{m})}\right)^{-(\alpha+1)}$$

$$\left(1-\left(1+ x_{ij(\mathrm{m})}\right)^{-\alpha}\right)^{\beta-1}\left(\left(1-\left(1+ x_{ij(\mathrm{m})}\right)^{-\alpha}\right)^{\beta}\right)^{m-1}$$

$$\times \prod_{i=m}^{m}\prod_{j=1}^{r}\left(\left(1+x_{ij\left(\frac{m+1}{2}\right)}\right)^{-(\alpha+1)}\left(1-\left(1+ x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}\right)^{\beta-1}\right)$$

$$\times\left(\left(1-\left(1+ x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}\right)^{\beta}\right)^{\frac{m-1}{2}}\left(1-\left(1-\left(1+ x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}\right)^{\beta}\right)^{\frac{m-1}{2}}$$

$$\times (K_1)^{mr}\alpha^{mr}\beta^{mr}m^{(m-1)r}$$

where $K_1 = \frac{m!}{\left(\frac{m-1}{2}\right)!^2}$. Then, the log-likelihood function is

$$\begin{aligned}
l(\alpha,\beta) &= mrln(K_1) + (m-1)rln(m) + mrln(\alpha) + mrln(\beta) \\
&-(\alpha+1)\sum_{j=1}^{r}\sum_{i=1}^{\frac{m-1}{2}} In\left(1+x_{ij(1)}\right) + (\beta-1)\sum_{j=1}^{r}\sum_{i=1}^{\frac{m-1}{2}} In\left(1-\left(1+ x_{ij(1)}\right)^{-\alpha}\right) \\
&+(m-1)\sum_{j=1}^{r}\sum_{i=1}^{\frac{m-1}{2}} \mathrm{In}\left(1-\left(1-\left(1+ x_{ij(1)}\right)^{-\alpha}\right)^{\beta}\right) \\
&-(\alpha+1)\sum_{j=1}^{r}\sum_{i=\frac{m+1}{2}}^{m-1} In\left(1+x_{ij(m)}\right) + (\beta-1)\sum_{j=1}^{r}\sum_{i=\frac{m+1}{2}}^{m-1} In\left(1-\left(1+ x_{ij(m)}\right)^{-\alpha}\right) \\
&+(m-1)\sum_{j=1}^{r}\sum_{i=\frac{m+1}{2}}^{m-1} In\left(\left(1-\left(1+ x_{ij(m)}\right)^{-\alpha}\right)^{\beta}\right) \\
&-(\alpha+1)\sum_{j=1}^{r} In(1+x_{ij\left(\frac{m+1}{2}\right)} + (\beta-1)\sum_{j=1}^{r} ln\left(1-\left(1+ x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}\right) \\
&+ \left(\frac{m-1}{2}\right)\sum_{j=1}^{r} ln\left(\left(1-\left(1+ x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}\right)^{\beta}\right) \\
&+\left(\frac{m-1}{2}\right)\sum_{j=1}^{r} In\left(1-\left(1-\left(1+ x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}\right)^{\beta}\right)
\end{aligned} \tag{10}$$

The likelihood equations of α and β are given by

$$\frac{\partial l}{\partial \alpha}=\frac{mr}{\alpha}-\sum_{j=1}^{r}\sum_{i=1}^{\frac{m-1}{2}} In\left(1+x_{ij(1)}\right)+(\beta-1)\sum_{j=1}^{r}\sum_{i=1}^{\frac{m-1}{2}}\frac{In(1+x_{ij(1)})\left(1+x_{ij(1)}\right)^{-\alpha}}{1-\left(1+x_{ij(1)}\right)^{-\alpha}}$$
$$-(m-1)\sum_{j=1}^{r}\sum_{i=1}^{\frac{m-1}{2}}\beta\frac{\left(1-\left(1+x_{ij(1)}\right)^{-\alpha}\right)^{\beta-1}\left(1+x_{ij(1)}\right)^{-\alpha} ln(1+x_{ij(1)})}{1-(1-\left(1+x_{ij(1)}\right)^{-\alpha})^{\beta}}$$
$$-\sum_{j=1}^{r}\sum_{i=\frac{m+1}{2}}^{m-1} In\left(1+x_{ij(m)}\right)+(\beta-1)\sum_{j=1}^{r}\sum_{i=\frac{m+1}{2}}^{m-1}\frac{In(1+x_{ij(m)})\left(1+x_{ij(m)}\right)^{-\alpha}}{1-\left(1+x_{ij(m)}\right)^{-\alpha}}$$
$$+(m-1)\sum_{j=1}^{r}\sum_{i=\frac{m+1}{2}}^{m-1}\alpha\beta\frac{In(1+x_{ij(m)})\left(1+x_{ij(m)}\right)^{-\alpha-1}}{1-\left(1+x_{ij(m)}\right)^{-\alpha}}-\sum_{j=1}^{r} In(1+x_{ij\left(\frac{m+1}{2}\right)})$$
$$+(\beta-1)\sum_{i=1}^{j}\frac{In\left(1+x_{ij\left(\frac{m+1}{2}\right)}\right)\left(1+x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}}{1-\left(1+x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}}+$$
$$(\frac{m-1}{2})\sum_{i=1}^{j}\alpha\beta\frac{In(1+x_{ij\left(\frac{m+1}{2}\right)})\left(1+x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha-1}}{1-\left(1+x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}}+$$
$$(\frac{m-1}{2})\sum_{i=1}^{j}\beta\frac{\left(1-\left(1+x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}\right)^{\beta-1}\left(1+x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha} ln(1+x_{ij\left(\frac{m+1}{2}\right)})}{1-(1-\left(1+x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha})^{\beta}} \tag{11}$$

$$\frac{\partial l}{\partial \beta}=\frac{mr}{\beta}+\sum_{j=1}^{r}\sum_{i=1}^{\frac{m-1}{2}} ln\left(1-\left(1+x_{ij(1)}\right)^{-\alpha}\right)$$
$$-(m-1)\sum_{j=1}^{r}\sum_{i=1}^{\frac{m-1}{2}}\frac{\left(1-\left(1+x_{ij(1)}\right)^{-\alpha}\right)^{\beta} ln(1-(1+x_{ij(1)})^{-\alpha})}{1-(1-\left(1+x_{ij(1)}\right)^{-\alpha})^{\beta}}$$
$$+\sum_{j=1}^{r}\sum_{i=\frac{m+1}{2}}^{m-1} ln\left(1-\left(1+x_{ij(m)}\right)^{-\alpha}\right)$$
$$-(m-1)\sum_{j=1}^{r}\sum_{i=\frac{m+1}{2}}^{m-1} In\left(1-\left(1+x_{ij(m)}\right)^{-\alpha}\right)+\sum_{j=1}^{r} In(1-\left(1+x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}$$
$$+\left(\frac{m-1}{2}\right)\sum_{j=1}^{r} In\left(1-\left(1+x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}\right)+$$
$$\left(\frac{m-1}{2}\right)\sum_{j=1}^{r}\frac{\left(1-\left(1+x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}\right)^{\beta} In\left(1-\left(1+x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}\right)}{1-(1-\left(1+x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha})^{\beta}}$$

$$\frac{\partial l}{\partial \beta} = \frac{mr}{\beta} + \sum_{j=1}^{r}\sum_{i=1}^{\frac{m-1}{2}} ln\left(1-\left(1+ x_{ij(1)}\right)^{-\alpha}\right)$$

$$-(m-1)\sum_{j=1}^{r}\sum_{i=1}^{\frac{m-1}{2}} \frac{\left(1-\left(1+x_{ij(1)}\right)^{-\alpha}\right)^{\beta} ln(1-(1+x_{ij(1)})^{-\alpha})}{1-(1-\left(1+x_{ij(1)}\right)^{-\alpha})^{\beta}}$$

$$+\sum_{j=1}^{r}\sum_{i=\frac{m+1}{2}}^{m-1} ln\left(1-\left(1+ x_{ij(m)}\right)^{-\alpha}\right)-(m-1)\sum_{j=1}^{r}\sum_{i=\frac{m+1}{2}}^{m-1} In\left(1-\left(1+ x_{ij(m)}\right)^{-\alpha}\right) \quad (12)$$

$$\sum_{j=1}^{r} In(1-\left(1+ x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}) + \left(\frac{m-1}{2}\right)\sum_{j=1}^{r} In\left(1-\left(1+ x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}\right) +$$

$$\left(\frac{m-1}{2}\right)\sum_{j=1}^{r} \frac{\left(1-\left(1+x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}\right)^{\beta} In\left(1-\left(1+x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha}\right)}{1-(1-\left(1+x_{ij\left(\frac{m+1}{2}\right)}\right)^{-\alpha})^{\beta}}$$

Consider the sample size *m* is even. The PDF of $X_{ij(1)}$ and $X_{ij(m)}$ is given by

$$g\left(x_{ij(1)}, x_{ij(m)};\, \alpha, \beta\right) = mf\left(x_{ij(1)};\, \alpha, \beta\right)\left(1-F\left(x_{ij(1)};\, \alpha, \beta\right)\right)^{m-1}$$

$$\times\, mf\left(x_{ij(m)};\, \alpha, \beta\right)\left(F\left(x_{ij(m)};\, \alpha, \beta\right)\right)^{m-1}$$

$$g\left(x_{ij(1)}, x_{ij(m)};\, \alpha, \beta\right) = m\left((\alpha\, \beta\, (1+x_{ij(1)})^{-(\alpha+1)}\left(1-\left(1+ x_{ij(1)}\right)^{-\alpha}\right)^{\beta-1)}\right) \quad (13)$$

$$\left(1-\left(1-\left(1+ x_{ij(1)}\right)^{-\alpha}\right)^{\beta}\right)^{m-1} \times\, m \times$$

$$\left((\alpha\, \beta\, (1+x_{ij(m)})^{-(\alpha+1)}\left(1-\left(1+ x_{ij(m)}\right)^{-\alpha}\right)^{\beta-1}\right)\left(\left(1-\left(1+ x_{ij(m)}\right)^{-\alpha}\right)^{\beta}\right)^{m-1}$$

$$L(\alpha, \beta;\, x) = \prod_{j=1}^{r}\prod_{i=1}^{m/2} g\left(x_{ij(1)}\,;\, \alpha, \beta\right)\prod_{j=1}^{r}\prod_{i=(m+2)/2}^{m} g\left(x_{ij(m)}\,;\, \alpha, \beta\right)$$

$$L(\alpha, \beta;\, x) = m^{mr}\alpha^{mr}\beta^{mr}\prod_{j=1}^{r}\prod_{i=1}^{\frac{m}{2}} \left(1+x_{ij(1)}\right)^{-(\alpha+1)}\left(1-\left(1+ x_{ij(1)}\right)^{-\alpha}\right)^{(\beta-1)}$$

$$\left(1-\left(1-\left(1+ x_{ij(1)}\right)^{-\alpha}\right)^{\beta}\right)^{m-1} \quad (14)$$

$$\prod_{j=1}^{r}\prod_{i=\frac{(m+2)}{2}}^{m}\left(\left(1+x_{ij(m)}\right)^{-(\alpha+1)}\left(1-\left(1+ x_{ij(m)}\right)^{-\alpha}\right)^{(\beta-1)}\right)$$

$$\left(\left(1-\left(1+ x_{ij(m)}\right)^{-\alpha}\right)^{\beta}\right)^{m-1}$$

Then, the log-likelihood function is

$$l(\alpha,\beta) = mrln(m) + mrln(\alpha) + mrln(\beta)$$
$$-(\alpha+1)\sum_{j=1}^{r}\sum_{i=1}^{\frac{m}{2}} In\left(1+x_{ij(1)}\right) + (\beta-1)\sum_{j=1}^{r}\sum_{i=1}^{\frac{m}{2}} ln\left(1-\left(1+x_{ij(1)}\right)^{-\alpha}\right)$$
$$+(m-1)\sum_{j=1}^{r}\sum_{i=1}^{\frac{m}{2}} \ln\left(1-\left(1-\left(1+x_{ij(1)}\right)^{-\alpha}\right)^{\beta}\right) \quad (15)$$
$$-(\alpha+1)\sum_{j=1}^{r}\sum_{i=(m+2)/2}^{m} In\left(1+x_{ij(m)}\right)$$
$$+(\beta-1)\sum_{j=1}^{r}\sum_{i=(m+2)/2}^{m} ln\left(1-\left(1+x_{ij(m)}\right)^{-\alpha}\right)+$$
$$(m-1)\sum_{j=1}^{r}\sum_{i=(m+2)/2}^{m} In\left(\left(1-\left(1+x_{ij(m)}\right)^{-\alpha}\right)^{\beta}\right)$$

The likelihood equations of α and β are given by

$$\frac{\partial l}{\partial \alpha} = \frac{mr}{\alpha} - \sum_{j=1}^{r}\sum_{i=1}^{\frac{m}{2}} In\left(1+x_{ij(1)}\right) + (\beta-1)\sum_{j=1}^{r}\sum_{i=1}^{\frac{m}{2}} \frac{In(1+x_{ij(1)})\left(1+x_{ij(1)}\right)^{-\alpha}}{1-\left(1+x_{ij(1)}\right)^{-\alpha}}$$
$$-(m-1)\sum_{j=1}^{r}\sum_{i=1}^{\frac{m}{2}} \beta\frac{\left(1-\left(1+x_{ij(1)}\right)^{-\alpha}\right)^{\beta-1}\left(1+x_{ij(1)}\right)^{-\alpha} In(1+x_{ij(1)})}{1-(1-\left(1+x_{ij(1)}\right)^{-\alpha})^{\beta}}$$
$$-\sum_{j=1}^{r}\sum_{i=\frac{m+1}{2}}^{m-1} In\left(1+x_{ij(m)}\right) + (\beta-1)\sum_{j=1}^{r}\sum_{i=\frac{m+1}{2}}^{m-1} \frac{In(1+x_{ij(m)})\left(1+x_{ij(m)}\right)^{-\alpha}}{1-\left(1+x_{ij(m)}\right)^{-\alpha}}$$
$$+(m-1)\sum_{j=1}^{r}\sum_{i=\frac{m+1}{2}}^{m-1} \beta\frac{In(1+x_{ij(m)})\left(1+x_{ij(m)}\right)^{-\alpha-1}}{1-\left(1+x_{ij(m)}\right)^{-\alpha}}$$
$$\frac{\partial l}{\partial \beta} = \frac{mr}{\beta} + \sum_{j=1}^{r}\sum_{i=1}^{\frac{m}{2}} ln\left(1-\left(1+x_{ij(1)}\right)^{-\alpha}\right) \quad (16)$$
$$-(m-1)\sum_{j=1}^{r}\sum_{i=1}^{\frac{m}{2}} \frac{\left(1-\left(1+x_{ij(1)}\right)^{-\alpha}\right)^{\beta} ln(1-(1+x_{ij(1)})^{-\alpha})}{1-(1-\left(1+x_{ij(1)}\right)^{-\alpha})^{\beta}}$$
$$\sum_{j=1}^{r}\sum_{i=\frac{(m+2)}{2}}^{m} ln\left(1-\left(1+x_{ij(m)}\right)^{-\alpha}\right)$$
$$+(m-1)\sum_{j=1}^{r}\sum_{i=(m+2)/2}^{m} In\left(1-\left(1+x_{ij(m)}\right)^{-\alpha}\right)$$

The likelihood equation cannot be solved analytically; therefore, numerical methods were used. In our study, we utilized the "*fminsearch*" function in MATLAB, which is based on the Nelder-Mead simplex algorithm, a well-established derivative-free optimization method. The "*fminsearch*" function iteratively searches for the optimal parameter values by minimizing the negative log-likelihood function. By applying "*fminsearch*", we ensured that our parameter estimates were efficiently obtained even when the likelihood function exhibited complex, non-convex behavior. This approach allowed us to achieve reliable estimates for the distribution parameters under different sampling scenarios.

4 RESULTS AND DISCUSSION

In this study, we assess the performance of ERSS for parameter estimation of the Inverted Kumaraswamy distribution through a Monte Carlo simulation approach. The simulation algorithm utilized in the study is designed to replicate the sampling process and evaluate the efficiency of the proposed methodology. The simulation algorithm used in the study involves the following steps:

- Generation of Pseudo-Random Samples: For each combination of the specified sample sizes n=12, 24, 36, 48 and parameter values α=0.8,0.3,0.4,1.0 and β=0.8,0.4,2.0. 10.000 pseudo-random samples were generated from the IK distribution using MATLAB.
- Application of ERSS: For each sample, the ERSS method was applied to focus on extreme values, enhancing the efficiency of parameter estimation in skewed or heavy-tailed data.
- Maximum Likelihood Estimation: The parameters α and β were estimated for each sample using the MLE method.
- Evaluation Metrics: The performance of ERSS was assessed by calculating the bias and mean square error (MSE) for the parameter estimates, which are standard criteria for evaluating accuracy and precision.
- Comparison with SRS: Finally, the ERSS-based results were compared with those obtained using SRS to assess the improvements in parameter estimation.

This algorithm ensures a systematic approach to simulate and evaluate the performance of ERSS in the context of parameter estimation for the IK distribution. To enhance the clarity and understanding of the results, we have included the definitions of MSE and Bias, which were used to evaluate the performance of the ERSS method.

The Bias is defined as the difference between the expected value of the estimator and the true parameter value. Mathematically, it is expressed as Eq. (17);

$$Bias(\hat{\theta}) = E[\hat{\theta}] - \theta \qquad (17)$$

where $\hat{\theta}$ is the estimator of the parameter θ, and $E[\hat{\theta}]$ is the expected value of $\hat{\theta}$. MSE is a measure of the average squared difference between the estimated values and the true value. It is calculated as Eq. (18);

$$MSE(\hat{\theta}) = E[(\hat{\theta} - \theta)^2] = Var(\hat{\theta}) + (Bias(\hat{\theta}))^2 \qquad (18)$$

where $Var(\hat{\theta})$ represents the variance of the estimator $\hat{\theta}$.

By focusing on extreme values, ERSS is expected to offer improvements in the estimation of parameters, particularly in distributions characterized by heavy tails or skewness. The results are given in Tables 1-10.

Table 1. Biases of the $\alpha = 0.8$, $\beta = 0.2$ estimator of the IK distribution.

			ERSS		SRS	
α; β	***n***	***m;r***	***α***	***β***	***α***	***β***
0.8;0.2	12	3;4	**0,1061**	**0,6893**	0,1507	0,9629
		4;3	**0,1345**	**0,9114**		
		6;2	**0,1376**	**1,0500**		
	24	3;8	**0,0502**	**0,2689**	0,0665	0,3435
		4;6	**0,0613**	0,3356		
		6;4	**0,0665**	0,3567		
	36	3;12	**0,0310**	**0,1658**	0,0445	0,2204
		4;9	**0,0400**	**0,2115**		
		6;6	**0,0405**	**0,2083**		
	48	3;16	**0,0230**	**0,1187**	0,0328	0,1567
		4;12	**0,0311**	0,1593		
		6;8	**0,0298**	**0,1527**		

Table 2. MSEs of the $\alpha = 0.8$, $\beta = 0.2$ estimator of the IK distribution.

			ERSS		SRS	
α; β	***n***	***m;r***	***α***	***β***	***α***	***β***
0.8;0.2	12	3;4	**0,0984**	**4,3337**	0,1369	7,6193
		4;3	**0,1308**	**7,5281**		
		6;2	0,1430	9,8960		
	24	3;8	**0,0363**	**0,7296**	0,0467	0,9435
		4;6	**0,0464**	1,0207		
		6;4	0,0502	1,1404		
	36	3;12	**0,0221**	**0,3678**	0,0279	0,4602
		4;9	**0,0276**	0,4782		
		6;6	0,0282	0,4988		
	48	3;16	**0,0154**	**0,2274**	0,0192	0,2868
		4;12	0,0196	0,3176		
		6;8	0,0203	0,3285		

Table 3. Biases of the $\boldsymbol{\alpha = 0.3}$ ***,*** $\boldsymbol{\beta = 2}$ ***estimator of the IK distribution.***

$\alpha;\beta$	n	$m;r$	ERSS		SRS	
			α	β	α	β
0.3;2	12	3;4	**0,0403**	**0,6851**	0,0553	0,9271
		4;3	**0,0523**	0,9965		
		6;2	**0,0508**	1,0182		
	24	3;8	**0,0184**	**0,2632**	0,0262	0,3553
		4;6	**0,0227**	**0,3395**		
		6;4	**0,0261**	0,3844		
	36	3;12	**0,0114**	**0,1609**	0,0164	0,2092
		4;9	**0,0146**	**0,2040**		
		6;6	**0,0155**	0,2109		
	48	3;16	**0,0090**	**0,1194**	0,0126	0,1595
		4;12	**0,0105**	**0,1430**		
		6;8	**0,0116**	0,1615		

Table 4. MSEs of the $\boldsymbol{\alpha = 0.3}$ ***,*** $\boldsymbol{\beta = 2}$ ***estimator of the IK distribution.***

$\alpha;\beta$	n	$m;r$	ERSS		SRS	
			α	β	α	β
0.3;2	12	3;4	**0,0138**	**3,8903**	0,0189	5,5753
		4;3	0,0192	6,7795		
		6;2	0,0197	8,2940		
	24	3;8	**0,0051**	**0,7124**	0,0067	0,9830
		4;6	**0,0065**	1,1000		
		6;4	0,0073	1,3516		
	36	3;12	**0,0031**	**0,3569**	0,0039	0,4298
		4;9	**0,0038**	0,4734		
		6;6	0,0040	0,4888		
	48	3;16	**0,0022**	**0,2349**	0,0028	0,2917
		4;12	**0,0028**	0,3013		
		6;8	0,0030	0,3445		

Table 5. Biases of the $\alpha = 0.4$, $\beta = 0.8$ estimator of the IK distribution.

			ERSS		SRS	
α; β	***n***	***m;r***	***α***	***β***	***α***	***β***
0.4;0.8	12	3;4	**0,0738**	**0,1635**	0,1039	0,2263
		4;3	**0,0947**	**0,2245**		
		6;2	0,1049	0,2525		
	24	3;8	**0,0357**	**0,0763**	0,0494	0,1023
		4;6	**0,0453**	**0,0949**		
		6;4	**0,0486**	**0,0990**		
	36	3;12	**0,0237**	**0,0464**	0,0311	0,0623
		4;9	**0,0287**	**0,0600**		
		6;6	**0,0297**	**0,0608**		
	48	3;16	**0,0165**	**0,0333**	0,0234	0,0448
		4;12	**0,0220**	**0,0426**		
		6;8	**0,0217**	**0,0433**		

Table 6. MSEs of the $\alpha = 0.4$, $\beta = 0.8$ estimator of the IK distribution.

			ERSS		SRS	
α; β	***n***	***m;r***	***α***	***β***	***α***	***β***
0.4;0.8	12	3;4	**0,0423**	**0,2092**	0,0673	0,3731
		4;3	**0,0606**	0,3957		
		6;2	0,0730	0,5167		
	24	3;8	**0,0156**	**0,0614**	0,0214	0,0868
		4;6	**0,0211**	0,0880		
		6;4	0,0221	0,0896		
	36	3;12	**0,0096**	**0,0340**	0,0120	0,0443
		4;9	**0,0120**	0,0449		
		6;6	0,0121	0,0454		
	48	3;16	**0,0066**	**0,0229**	0,0083	0,0292
		4;12	0,0084	0,0297		
		6;8	0,0086	0,0304		

Table 7. Biases of the $\alpha = 1$, $\beta = 0.4$ estimator of the IK distribution.

			ERSS		SRS	
α; β	***n***	***m;r***	***α***	***β***	***α***	***β***
1;0.4	12	3;4	**0,2921**	**0,0636**	0,4329	0,0937
		4;3	**0,3876**	**0,0854**		
		6;2	**0,4126**	**0,0927**		
	24	3;8	**0,1371**	**0,0281**	0,1899	0,0398
		4;6	**0,1743**	**0,0363**		
		6;4	**0,1767**	**0,0366**		
	36	3;12	**0,0915**	**0,0192**	0,1172	0,0249
		4;9	**0,1038**	**0,0242**		
		6;6	**0,1170**	**0,0243**		
	48	3;16	**0,0684**	**0,0147**	0,0838	0,0185
		4;12	**0,0809**	**0,0164**		
		6;8	**0,0831**	0,0187		

Table 8. MSEs of the $\alpha = 1$, $\beta = 0.4$ estimator of the IK distribution.

			ERSS		SRS	
α; β	***n***	***m;r***	***α***	***β***	***α***	***β***
1;0.4	12	3;4	**0,5877**	**0,0334**	1,0747	0,0616
		4;3	**0,9510**	**0,0544**		
		6;2	1,1405	0,0805		
	24	3;8	**0,1861**	**0,0100**	0,2755	0,0147
		4;6	**0,2574**	**0,0143**		
		6;4	0,2786	0,0152		
	36	3;12	**0,1089**	**0,0062**	0,1428	0,0084
		4;9	**0,1314**	**0,0078**		
		6;6	0,1477	**0,0083**		
	48	3;16	**0,0732**	**0,0043**	0,0923	0,0057
		4;12	0,0936	**0,0054**		
		6;8	0,0976	0,0057		

Table 9. Biases of the $\alpha = 1$, $\beta = 0.8$ estimator of the IK distribution.

			ERSS		SRS	
α; β	***n***	***m;r***	***α***	***β***	***α***	***β***
1;0.8	12	3;4	**0,1818**	**0,1644**	0,2715	0,2416
		4;3	**0,2424**	**0,2302**		
		6;2	**0,2498**	**0,2380**		
	24	3;8	**0,0899**	**0,0742**	0,1171	0,0942
		4;6	**0,1115**	**0,0910**		
		6;4	**0,1072**	**0,0933**		
	36	3;12	**0,0589**	**0,0458**	0,0849	0,0649
		4;9	**0,0720**	**0,0565**		
		6;6	**0,0751**	**0,0631**		
	48	3;16	**0,0408**	**0,0325**	0,0538	0,0456
		4;12	0,0556	**0,0444**		
		6;8	0,0540	**0,0441**		

Table 10. MSEs of the $\alpha = 1$, $\beta = 0.8$ estimator of the IK distribution.

			ERSS		SRS	
α; β	***n***	***m;r***	***α***	***β***	***α***	***β***
1;0.8	12	3;4	**0,2562**	**0,2307**	0,4312	0,3941
		4;3	**0,3841**	0,4282		
		6;2	**0,4196**	0,4366		
	24	3;8	**0,0967**	**0,0607**	0,1272	0,0821
		4;6	**0,1243**	0,0840		
		6;4	0,1340	0,0889		
	36	3;12	**0,0590**	**0,0334**	0,0778	0,0440
		4;9	**0,0753**	**0,0427**		
		6;6	0,0803	0,0485		
	48	3;16	**0,0408**	**0,0226**	0,0522	0,0302
		4;12	0,0533	0,0307		
		6;8	0,0549	0,0313		

Findings on the Biases:

When examining the model as a whole, it is evident that the values of the ERSS estimators yield better results compared to those derived using SRS. Furthermore, as the sample size increases, the estimator values for SRS tend to decrease. In contrast, for ERSS estimators, when the sample size is held constant, the values increase as the number of cycles grows. However, in general, as the sample size increases, the results tend to decrease.

When comparing the ERSS and SRS values in the model, it is observed that the α values provide better results than the β values. Regarding the biases in the model, the best results for α are found in the initial trial findings. For instance, when (α; β = 1; 0.8), the best results are observed across all sample sizes (12, 24, 36, 48) in the first cycle sizes (3, 4). This trend is consistent for all estimators.

When the α parameter is fixed and β is increased, better results are obtained for the β values, while the α values do not yield optimal results for the largest sample size. Conversely, when the β parameter is fixed and α is increased, better results are obtained for the β values, and all results for α values are superior to those obtained using SRS.

Findings on the MSE:

Overall, it was observed that as the number of cycles in ERSS increases with sample size, higher MSE values are recorded. The ERSS method generally provides lower average MSE, indicating better overall performance in parametric estimation. In contrast, the SRS method shows higher errors, particularly in estimating the β parameter for certain combinations, highlighting weaker performance in complex datasets.

For both α and β parameters, consistency was achieved with both ERSS and SRS methods. However, higher MSE values were observed for the β parameter, especially for small sample sizes (e.g., n=12). Additionally, as the number of cycles increases in ERSS, higher MSE values are noted.

5 ILLUSTRATIVE EXAMPLE

In this section, to demonstrate the ML estimators obtained in Section 3, we present a real data set which is available in Castillo et al. [18]. The data set included yearly maximum flow (in cubic meters) at a given location of a river for 60 years. We fitted IK distribution to this data set. The ML estimates of the α and β are respectively 0.0858 and 3.6815. A random sample of size 12 is drawn without replacement by using different sampling schemes namely SRS and ERSS. For the analysis, we determined m=6 and r=2. The results of the parameter estimates are presented in Table 11. From Table 11, we see that the ML estimates based on ERSS scheme are closer to the given value of both parameters α and β. This result show once again that the ERSS estimates better than SRS scheme.

Table 11. ML estimation of the parameters α and β for IK distribution.

$\hat{\alpha}$		$\hat{\beta}$	
SRS	ERSS	SRS	ERSS
0.0839	0.0863	3.6284	3.6488

6 CONCLUSION AND SUGGESTIONS

In this study, we applied ERSS for parameter estimation of the IK distribution using the MLE method. The performance of ERSS was compared to the traditional sampling method, SRS. The results showed that ERSS provides more accurate and efficient estimates than SRS, particularly in terms of bias and MSE. The simulation study confirmed that the use of ERSS improves estimation precision, especially when dealing with distributions characterized by heavy tails or skewness. ERSS outperformed SRS in terms of MSE, demonstrating its advantage in extreme value estimation. Overall, the MLE based on ERSS exhibited smaller biases and lower MSEs compared to those obtained through SRS. These findings underscore the potential of ERSS as an effective alternative to traditional sampling methods for parameter estimation in extreme value contexts. Finally, the results from the simulation study suggest that ERSS can be a valuable tool in various applications, offering superior estimation performance compared to conventional methods.

In future studies, we recommend expanding the research to include odd sample sizes (e.g., n = 9, 15, 21, 25) and evaluating their impact on the results, particularly in terms of bias and MSE. Due to the increased complexity and computational load, this option was not included in the current study. However, assessing its effect could provide valuable insights into the ERSS approach. Additionally, future research could explore alternative RSS designs beyond ERSS, assessing their potential advantages in parameter estimation. Various estimation methods could also be implemented and compared to evaluate their accuracy and efficiency across different scenarios. Furthermore, investigating the proposed methodology under different probability distributions could help examine its robustness and applicability in diverse statistical contexts.

Conflict of Interest Statement

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

Artificial Intelligence (AI) Contribution Statement

This manuscript was entirely written, edited, analyzed, and prepared without the assistance of any artificial intelligence (AI) tools. All content, including text, data analysis, and figures, was solely generated by the authors.

Contributions of the Authors

Esra Demirel contributed to the development of the theoretical framework, derivation of the equations, and preparation of figures and tables. She also drafted the initial version of the manuscript. **Hasan Hüseyin GÜL** contributed to the conceptualization of the study, the design of the sampling methods, and the critical revision of the manuscript.

REFERENCES

[1] I. Al-Omari et al., “Reliability estimation of inverse Lomax distribution using extreme ranked set sampling,” *Adv. Math. Phys.*, vol. 2021, no. 1, pp. 4599872, 2021.

[2] D. S. Bhoj and G. Chandra, “Weighted Ranked Set Sampling for Skewed Distributions,” *Mathematics*, vol. 12, no. 13, pp. 2023, 2024.

[3] H. Jiang and W. Gui, “Bayesian inference for the parameters of Kumaraswamy distribution via ranked set sampling,” *Symmetry*, vol. 13, no. 7, pp. 1170, 2021.

[4] N. A. Mode, L. L. Conquest and D. A. Marker, “Ranked set sampling for ecological research: accounting for the total costs of sampling,” *Environmetrics*, vol. 10, no. 2, pp. 179–194, 1999.

[5] A. Yavuz and D. Öz, “A comparative study of Ranked Set Sampling (RSS) and Simple Random Sampling (SRS) in agricultural studies: A case study on the walnut tree,” *Eurasian J. Forest Sci.*, vol. 8, no. 1, pp. 94–108, 2020.

[6] J. Gillariose et al., “Reliability test plan for the Marshall-Olkin extended inverted Kumaraswamy distribution,” *Reliability: Theory & Applications*, vol. 16, no. 3(63), pp. 26–36, 2021.

[7] K. Bağcı, “Parameter Estimation of the Inverted Kumaraswamy Distribution by Using L-Moments Method: An Application on Precipitation Data,” *Cumhuriyet Sci. J.*, vol. 45, no. 3, pp. 629–635, 2024.

[8] W. Tian et al., “Change point analysis for Kumaraswamy distribution,” *Mathematics*, vol. 11, no. 3, pp. 553, 2023.

[9] J. K. Pokharel et al., “Probability Distributions for Modeling Stock Market Returns—An Empirical Inquiry,” *Int. J. Financial Stud.*, vol. 12, no. 2, pp. 43, 2024.

[10] I. Ishaq et al., “Log-Kumaraswamy distribution: its features and applications,” *Front. Appl. Math. Stat.*, vol. 9, pp. 1258961, 2023.

[11] J. Bengalath and B. Punathumparambath, “Harris extended inverted Kumaraswamy distribution: Properties and applications to COVID-19 data,” *Int. J. Data Sci. Anal.*, pp. 1–23, 2024.

[12] Al-Nasser, A. Al-Omari and M. Al-Rawwash, “Monitoring the process mean based on quality control charts using on folded ranked set sampling,” *Pak. J. Stat. Oper. Res.*, pp. 79–92, 2013.

[13] P. K. Sahu and N. Gupta, “On general weighted extropy of extreme ranked set sampling,” *Commun. Stat. Theory Methods*, pp. 1–19, 2024.

[14] N. Alotaibi et al., “Statistical inference for the Kavya–Manoharan Kumaraswamy model under ranked set sampling with applications,” *Symmetry*, vol. 15, no. 3, pp. 587, 2023.

[15] Ali et al., “Estimation of population mean by using a generalized family of estimators under classical ranked set sampling,” *Res. Math. Stat.*, vol. 8, no. 1, 2021.

[16] H. Samawi, L. Yu and J. Yin, “On Cox proportional hazards model performance under different sampling schemes,” *PLoS ONE*, vol. 18, no. 4, pp. e0278700, 2023.

[17] H. M. Samawi, M. S. Ahmed and W. Abu-Dayyeh, “Estimating the population mean using extreme ranked set sampling,” *Biometrical J.*, vol. 38, no. 5, pp. 577–586, 1996.

[18] E. Castillo, A. S. Hadi, N. Balakrishnan and J. M. Sarabia, *Extreme Value and Related Models with Applications in Engineering and Science*, Hoboken, NJ: Wiley, 2005.