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Research Article

Fuzzy Solutions of Fuzzy Fractional Parabolic Integro Differential Equations

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Abstract

This work primarily investigates the numerical solution of fuzzy fractional parabolic integro-differential equations of the Volterra type with the time derivative defined in the Caputo sense using the fuzzy Adomian decomposition method. Fuzzy fractional partial integro-differential equations pose significant mathematical challenges due to the interplay between fuzziness and fractional-order dynamics, while at the same time, there is a growing need for accurate and efficient methods to model real-world phenomena involving uncertainty in physics, biology, and engineering. The fuzzy Adomian decomposition method provides an alternative approach for obtaining approximate fuzzy solutions, and its applicability to such equations has not been studied in detail previously in the literature. Furthermore, existence and uniqueness theorems for the fuzzy fractional partial integro-differential equation are established by considering the differentiability type of the solution. The accuracy and efficiency of the proposed method are demonstrated through a series of numerical experiments.

1. Introduction

Fractional differential equations and fractional integral equations are powerful tools for modeling and describing the hereditary properties of various materials and processes. In recent years, the widespread use of Fractional differential equations in engineering and scientific domains has motivated researchers to pursue advancements in both theoretical and applied research methods. Many researchers have focused on establishing existence results to confirm that the mathematical models accurately describe real-world phenomena [1–3], while other research has concentrated on finding explicit or approximate solutions to these models. [4–6]. When modeling real-world phenomena using fractional differential equations, the behavior of dynamical systems can be complex and affected by errors and uncertainty. To address this, some researchers have introduced approaches that define parameters and initial conditions within a fuzzy fractional framework. Early contributions to the study of fuzzy fractional differential equations were made by Agarwal et al. [7] and Arshad et al. [8], who applied the Riemann–Liouville derivative with fuzzy initial conditions. This approach extends the classical Riemann–Liouville derivative using the Hukuhara difference ($\mathtt{H}$-difference) [9]. However, a limitation of the $\mathtt{H}$-difference is that the support of fuzzy solutions tends to increase over time (see [10–12]). Moreover, the Riemann–Liouville derivative requires knowledge of the fractional derivative of the unknown solution at the initial point, which is often difficult to measure or may not exist. To overcome these challenges, several studies have combined Caputo derivatives with generalized Hukuhara differentiability ($g\mathtt{H}$-differentiability), leading to the concept of Caputo gH-differentiability, as discussed in works by Salahshour et al. [13], Long et al. [14], Alqudah et. al. [15] and Saeed et. al. [16].
Recently, numerous authors have developed and analyzed various numerical techniques of fuzzy fractional differential equations. These include studies on the existence of global solutions using upper and lower solutions method [17], integro-differential equations with generalized Caputo differentiability [18], the fractional differential transform method [19], the Adomain decomposition method [20, 21], the Jacobi polynomial operational matrix [22], the two-dimensional Legendre wavelet method [23], the power series method [24, 25], homotopy perturbation transform method [26] and the optimal homotopy asymptotic method [27].

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The main objective of this work is to prove the uniqueness of solution of fuzzy fractional partial integro-differential equation. We also investigate the the numerical solution of fuzzy fractional parabolic integro-differential equations under Caputo generalized Hukuhara differentiability by fuzzy Adomian decomposition method. To achieve this, we convert a fuzzy fractional parabolic integro-differential equation into a system of crisp equations that can be solved bya standard numerical method.
The significance of this study from the theoretical point of view is that the current fuzzy Adomian decomposition method is developed for a general form of fuzzy fractional partial integro-differential equation under Caputo generalized Hukuhara differentiability. This can greatly help the numerical study of fuzzy fractional partial integro-differential equations and other equations in this form due to the difficulty of solving these equations analytically.
The paper is organized as follows: Section 2 introduces essential definitions and notations related to fuzzy fractional calculus. In Section 3, we present a fuzzy fractional partial integro-differential equation under Caputo generalized Hukuhara differentiability and examine the existence and uniqueness of its solutions. Section 4 discusses the convergence of the Fuzzy Adomian Decomposition Method for determining approximate solutions to the fuzzy fractional parabolic integro-differential equation (fuzzy fractional parabolic IDEs). Additionally, we explore solutions of fuzzy fractional parabolic IDEs under different differentiability types. Section 5 provides examples to illustrate the effectiveness of the proposed method.

2. Preliminary Concepts

In this section, we recall some of the basic preliminaries of fuzzy fractional calculus.
Let $\mathfrak{C}[\mathtt{I},\mathbb{R}]$ be the Banach space of all real-valued continuous functions from $\mathtt{I}=[0,a]$ into $\mathbb{R}$. For measurable real-valued function $f:\mathtt{I}\to\mathbb{R}$, define the norm $\|f\|_{L^p(\mathtt{I},\mathbb{R})}=\left(\int_{\mathtt{I}}|f(\varkappa)|^p\right)^{\frac{1}{p}}<\infty,\ 1\leq p<\infty$, where $L^p(\mathtt{I},\mathbb{R})$ denote the Banach space of all Lebesgue measurable real-valued functions f. Also, we use the notations listed below:

$\mathscr{F}_R$ is the set of all fuzzy numbers on $\mathbb{R}$.
$\mathfrak{C}[\mathtt{J},\mathscr{F}_R]$ is a space of all continuous fuzzy-valued functions which are on $\mathtt{J}=[0,a]\times[0,b]\subset\mathbb{R}^2$.
$\mathfrak{L}[\mathtt{J},\mathscr{F}_R]$ is the set of Lebesque integrable for fuzzy-valued functions on $\mathbf{B}$, where $\mathbf{B}\subset\mathbb{R}^m, m\in\mathbb{N}$.

Definition 2.1. *[28] A fuzzy number is a mapping $\alpha:\mathbb{R}\to[0,1]$ with the following features:*

(1) *For $\varkappa_0\in\mathbb{R},\alpha$ is normal. It means, $\alpha(\varkappa_0)=1$.*
(2) *For $\varkappa_1,\varkappa_2,\in\mathbb{R}$ and $\mathbf{t}\in[0,1]$, α is convex such that*

$$\alpha(\mathbf{t}\varkappa_1+(1-\mathbf{t})\varkappa_2)\geq\min\{\alpha(\varkappa_1),\alpha(\varkappa_2)\}.$$

(3) *α is upper semicontinuous.*
(4) *$cl\{\varkappa\in\mathbb{R},\alpha(\varkappa)>0\}$ is compact.*

The set of a fuzzy number $\alpha(\varkappa)\in\mathscr{F}_R$ in the ς-level form is denoted by $[\alpha]^\varsigma$ and defined as:

$$\begin{cases}\{\varkappa\in\mathbb{R}\mid\alpha(\varkappa)\geq\varsigma\} & \text{if } 0<\varsigma\leq1,\\ \text{cl}(\text{supp}\,\alpha(\varkappa)) & \text{if } \varsigma=0.\end{cases}$$

It is clear that the set of a fuzzy number $\varkappa$ in ς-level form is a closed and bounded interval $[\underline{\alpha_\varsigma},\overline{\alpha_\varsigma}]$, where $\underline{\alpha_\varsigma}$ is the left end point and $\overline{\alpha_\varsigma}$ is the right end point.
For any arbitrary elements $\alpha,\beta\in\mathscr{F}_R$ and scalar $k\in\mathbb{R}$, the operations of addition and scalar multiplication are respectively defined by their ς-level sets as follows:

$$[\alpha+\beta]^\varsigma=(\underline{\alpha_\varsigma}+\underline{\beta_\varsigma},\overline{\alpha_\varsigma}+\overline{\beta_\varsigma}),$$

$$[k\alpha]^\varsigma=\begin{cases}(k\underline{\alpha_\varsigma},k\overline{\alpha_\varsigma}) & \text{if } k\geq0,\\ (k\overline{\alpha_\varsigma},k\underline{\alpha_\varsigma}) & \text{if } k<0.\end{cases}$$

A *triangular fuzzy number* is characterized as a fuzzy set in $\mathbb{R}_F$, represented by an ordered triple $\alpha=(a,b,c)\in\mathbb{R}^3$ where $a\leq b\leq c$. The ς-level set of α is given by the endpoints:

$$\underline{\alpha_\varsigma}=a+(b-a)\varsigma,\quad \overline{\alpha_\varsigma}=c-(c-b)\varsigma,$$

for all $\varsigma\in[0,1]$.

Definition 2.2. *[14] Let $\mathbb{D}:\mathscr{F}_R\times\mathscr{F}_R\longrightarrow\mathbb{R}$ be the Hausdorff distance between two fuzzy numbers α,β and defined as*

$$\begin{aligned}\mathbb{D}(\alpha,\beta)&=\sup_{0\leq\varsigma\leq1} d_H\{[\alpha]^\varsigma,[\beta]^\varsigma\}\\ &=\sup_{0\leq\varsigma\leq1} max\{|\underline{\alpha_\varsigma}-\underline{\beta_\varsigma}|,|\,\overline{\alpha_\varsigma}-\overline{\beta_\varsigma}|\},\end{aligned}$$

where the metric space $(\mathscr{F}_R,\mathbb{D})$ is complete, separable and locally compact. The supremum metric D^* on $\mathfrak{C}[\mathtt{J},\mathscr{F}_R]$ is considered as

$$D^*(\alpha,\beta)=\sup_{(\varkappa,\mathbf{t})\in\mathtt{J}}\{\,\mathbb{D}(\alpha(\varkappa,\mathbf{t}),\beta(\varkappa,\mathbf{t}))\}.\tag{2.1}$$

Definition 2.3. *[28] The Hukuhara difference* ($\mathtt{H}$*-difference) between two fuzzy numbers* α *and* β *is defined as*

$$\alpha \ominus \beta = w \quad \Leftrightarrow \quad \alpha = \beta + w,$$

where $+$ *denotes the standard fuzzy addition. Moreover, if* $\alpha \ominus \beta$ *exists, then* $\alpha \ominus \alpha = 0$.

In [14] authors have given some properties of the metric $\mathbb{D}$ in $\mathscr{F}_R$ and Hukuhara difference as:

Lemma 2.4. *For all* $\alpha, \beta, l, \gamma, \varpi \in \mathscr{F}_R$ *we have*

(1) $\mathbb{D}(\alpha + l, \beta + l) = \mathbb{D}(\alpha, \beta)$.
(2) $\mathbb{D}(\alpha + \beta, \gamma + \varpi) \leq \mathbb{D}(\alpha, \gamma) + \mathbb{D}(q, \varpi)$.
(3) $\mathbb{D}(\alpha + \beta, 0) = \mathbb{D}(\alpha, 0) + \mathbb{D}(\beta, 0)$.
(4) *If* $\alpha \ominus \beta$ *exists then* $(-1)\alpha \ominus (-1)\beta$ *exists and* $(-1)(\alpha \ominus \beta) = (-1)\alpha \ominus (-1)\beta$.
(5) *If* $\alpha \ominus \beta$ *and* $\gamma \ominus \varpi$ *exist then* $\mathbb{D}(\alpha \ominus \beta, \gamma \ominus \varpi) \leq \mathbb{D}(\alpha, \gamma) + D(\beta, \varpi)$.

Definition 2.5. *[28] The generalized Hukuhara difference of two fuzzy numbers* $\alpha, \beta \in \mathscr{F}_R$ *(*$g\mathtt{H}$*-difference for short) is defined as follows:*

$$\alpha \ominus_{g\mathtt{H}} \beta = w \quad \Leftrightarrow \quad \begin{cases} (i)\ \alpha = \beta + w, \\ or \\ (ii)\ \beta = \alpha + (-1)w. \end{cases}$$

It is easy to show that (i) and (ii) are both valid if and only if w is a crisp number.
For the ς-levels, the generalized Hukuhara difference ($g\mathtt{H}$-difference) between α and β is given by:

$$[\alpha \ominus_{g\mathtt{H}} \beta]^{\varsigma} = \left[\min\{\underline{\alpha_\varsigma} - \underline{\beta_\varsigma}, \overline{\alpha_\varsigma} - \overline{\beta_\varsigma}\}, \max\{\underline{\alpha_\varsigma} - \underline{\beta_\varsigma}, \overline{\alpha_\varsigma} - \overline{\beta_\varsigma}\}\right].$$

If the Hukuhara difference (H-difference) exists, then $\alpha \ominus \beta = \alpha \ominus_{g\mathtt{H}} \beta$. The conditions for the existence of $\alpha \ominus_{g\mathtt{H}} \beta \in \mathscr{F}_R$ are shown in [10, 29].

Remark 2.6. *Throughout the remainder of this paper, we assume that* $\alpha \ominus_{g\mathtt{H}} \beta \in \mathscr{F}_R$.

Definition 2.7. *[30] A fuzzy number* α *can be represented in parametric form as* $[\underline{\alpha}(\varsigma), \overline{\alpha}(\varsigma)]$*, for* $0 \leq \varsigma \leq 1$*, if and only if*

(i) $\underline{\alpha}(\varsigma)$ *is increasing bounded function and left continuous over (0,1].*
(ii) $\overline{\alpha}(\varsigma)$ *is decreasing bounded function and right continuous over (0,1].*
(iii) $\underline{\alpha}(\varsigma) \leq \overline{\alpha}(\varsigma)$.

Allahviranloo [28] introduced the definition of fuzzy partial derivative as follows:

Definition 2.8. *Let* $\nu : \mathtt{J} \longrightarrow \mathscr{F}_R$*, then* $g\mathtt{H}$*-partial derivative of first order at the point* $(\varkappa_0, \mathbf{t}_0) \in \mathtt{J}$ *with respect to variables* $\varkappa$, $\mathbf{t}$ *are denoted by* $\frac{\partial \nu(\varkappa_0, \mathbf{t}_0)}{\partial \varkappa}$, $\frac{\partial \nu(\varkappa_0, \mathbf{t}_0)}{\partial \mathbf{t}}$ *and given by*

$$\frac{\partial \nu(\varkappa_0, \mathbf{t}_0)}{\partial \varkappa} = \lim_{h \to 0} \frac{\nu(\varkappa_0 + h, \mathbf{t}_0) \ominus_{g\mathtt{H}} \nu(\varkappa_0, \mathbf{t}_0)}{h},$$

$$\frac{\partial \nu(\varkappa_0, \mathbf{t}_0)}{\partial \mathbf{t}} = \lim_{k \to 0} \frac{\nu(\varkappa_0, \mathbf{t}_0 + k) \ominus_{g\mathtt{H}} \nu(\varkappa_0, \mathbf{t}_0)}{k},$$

provided that $\frac{\partial \nu(\varkappa_0, \mathbf{t}_0)}{\partial \varkappa}$ *and* $\frac{\partial \nu(\varkappa_0, \mathbf{t}_0)}{\partial \mathbf{t}} \in \mathscr{F}_R$.

Definition 2.9. *Let* $\nu : \mathtt{J} \longrightarrow \mathscr{F}_R$ *be* $g\mathtt{H}$*-partial differentiable with respect to* $\varkappa$ *at* $(\varkappa_0, \mathbf{t}_0) \in \mathtt{J}$*. We say that*

(1) ν *is* (i) $g\mathtt{H}$*-partial differentiable with respect to* $\varkappa$ *at* $(\varkappa_0, \mathbf{t}_0) \in \mathtt{J}$*. If*

$$\left[\frac{\partial \nu(\varkappa_0, \mathbf{t}_0, \varsigma)}{\partial \varkappa}\right] = \left[\frac{\partial \underline{\nu}(\varkappa_0, \mathbf{t}_0, \varsigma)}{\partial \varkappa}, \frac{\partial \overline{\nu}(\varkappa_0, \mathbf{t}_0, \varsigma)}{\partial \varkappa}\right], \quad \forall \varsigma \in [0,1].$$

(2) ν *is* (ii) $g\mathtt{H}$*-partial differentiable with respect to* $\varkappa$ *at* $(\varkappa_0, \mathbf{t}_0) \in \mathtt{J}$*. If*

$$\left[\frac{\partial \nu(\varkappa_0, \mathbf{t}_0, \varsigma)}{\partial \varkappa}\right] = \left[\frac{\partial \overline{\nu}(\varkappa_0, \mathbf{t}_0, \varsigma)}{\partial \varkappa}, \frac{\partial \underline{\nu}(\varkappa_0, \mathbf{t}_0, \varsigma)}{\partial \varkappa}\right], \quad \forall \varsigma \in [0,1].$$

Definition 2.10. *[14] For a fixed* $\varkappa_0$*, the point* $(\varkappa_0, \mathbf{t}) \in \mathtt{J}$ *is called a* switching point for the differentiability of ν with respect to $\varkappa_0$ *if, in every neighborhood* V *of* $(\varkappa_0, \mathbf{t})$*, there exist points* $A_1(\varkappa_1, \mathbf{t})$ *and* $A_2(\varkappa_2, \mathbf{t})$ *with* $\varkappa_1 < \varkappa_0 < \varkappa_2$ *such that either:*

1. ν *is (i)-*$g\mathtt{H}$ *differentiable at* A_1 *and (ii)-*$g\mathtt{H}$ *differentiable at* A_2 *for all* $\mathbf{t}$*, or*
2. ν *is (i)-*$g\mathtt{H}$ *differentiable at* A_2 *and (ii)-*$g\mathtt{H}$ *differentiable at* A_1 *for all* $\mathbf{t}$.

Lemma 2.11. *[14] (Newton–Leibniz formula) Let* $\nu \in \mathfrak{C}(\mathbb{R}^2, \mathscr{F}_R)$.

(1) If ν *is* (i)*-*$g\mathtt{H}$ *differentiable with respect to* $\mathbf{t}$*, with no switching point on* $\mathbb{R} \times [b, \mathbf{t}]$*, then*

$$\int_b^{\mathbf{t}} \frac{\partial \nu(\varkappa, \delta)}{\partial \delta}\, d\delta = \nu(\varkappa, \mathbf{t}) \ominus \nu(\varkappa, b).$$

(2) If ν is (ii)-gH differentiable with respect to $\mathbf{t}$, with no switching point on $\mathbb{R}\times[b,\mathbf{t}]$, then

$$\int_b^{\mathbf{t}} \frac{\partial \nu(\varkappa,\delta)}{\partial \delta} d\delta = (-1)\nu(\varkappa,b) \ominus (-1)\nu(\varkappa,\mathbf{t}).$$

In [13, 28], authors have defined the concepts of Riemann-Liouville integral and Caputo's gH-derivative of fuzzy valued functions as follows:

Definition 2.12. *Let $\nu(\varkappa)\in\mathfrak{C}[\mathscr{I},\mathscr{F}_R]\cap\mathfrak{L}[\mathscr{I},\mathscr{F}_R]$, $\mathscr{I}\in\mathbb{R}$. The fuzzy fractional integral in Riemann-Liouville sense of order $\theta>0$ is defined as*

$$\mathfrak{I}^{\theta}\nu(\varkappa,\varsigma) = [\mathfrak{I}^{\theta}\underline{\nu}(\varkappa,\varsigma), \mathfrak{I}^{\theta}\overline{\nu}(\varkappa,\varsigma)], \ \ \varsigma\in[0,1],$$

where

$$\mathfrak{I}^{\theta}\underline{\nu}(\varkappa,\varsigma) = \frac{1}{\Gamma(\theta)}\int_0^{\varkappa}(\varkappa-\tau)^{\theta-1}\underline{\nu}(\tau,\varsigma)d\tau, \ \varkappa>0,$$

$$\mathfrak{I}^{\theta}\overline{\nu}(\varkappa,\varsigma) = \frac{1}{\Gamma(\theta)}\int_0^{\varkappa}(\varkappa-\tau)^{\theta-1}\overline{\nu}(\tau,\varsigma)d\tau, \ \varkappa>0.$$

Definition 2.13. *Let $\nu(\varkappa)\in\mathfrak{C}[\mathscr{I},\mathscr{F}_R]\cap\mathfrak{L}[\mathscr{I},\mathscr{F}_R]$. Then the fuzzy fractional Caputo's gH-derivative under (i) gH-differentiability is defined as*

$${}^{c}_{gH}\mathfrak{D}^{\theta}_{\varkappa}\nu(\varkappa,\varsigma) = [\,{}^{c}\mathfrak{D}^{\theta}_{\varkappa}\underline{\nu}(\varkappa,\varsigma), \ {}^{c}\mathfrak{D}^{\theta}_{\varkappa}\overline{\nu}(\varkappa,\varsigma)]$$

and under (ii) gH-differentiability is given as:

$${}^{c}_{gH}\mathfrak{D}^{\theta}_{\varkappa}\nu(\varkappa,\varsigma) = [\,{}^{c}\mathfrak{D}^{\theta}_{\varkappa}\overline{\nu}(\varkappa,\varsigma), \ {}^{c}\mathfrak{D}^{\theta}_{\varkappa}\underline{\nu}(\varkappa,\varsigma)],$$

where

$${}^{c}\mathfrak{D}^{\theta}_{\varkappa}\underline{\nu}(\varkappa,\varsigma)] = \frac{1}{\Gamma(\mathtt{m}-\theta)}\int_0^{\varkappa}(\varkappa-\tau)^{\mathtt{m}-\theta-1}\ \underline{\nu}^{(\mathtt{m})}(\tau,\varsigma)d\tau,$$

$${}^{c}\mathfrak{D}^{\theta}_{\varkappa}\overline{\nu}(\varkappa,\varsigma) = \frac{1}{\Gamma(\mathtt{m}-\theta)}\int_0^{\varkappa}(\varkappa-\tau)^{\mathtt{m}-\theta-1}\ \overline{\nu}^{(\mathtt{m})}(\tau,\varsigma)d\tau.$$

Proposition 2.14. *[28] If $\nu(\varkappa):[0,a]\to E_f$ is an integrable fuzzy function and $\theta_1>0$, $\theta_2>0$. Then,*

$$(\mathfrak{I}^{\theta_1})(\mathfrak{I}^{\theta_2})\nu(\varkappa) = (\mathfrak{I}^{\theta_1+\theta_2})\nu(\varkappa), \ \ \varkappa\in[0,a].$$

Theorem 2.15. *[31](Holder's Inequality) If q_1 and q_2 are positive numbers satisfying the relation $\frac{1}{q_1}+\frac{1}{q_2}=1$ and if $\mathtt{f}\in L^{q_1}(0,a)$, $\mathtt{g}\in L^{q_2}(0,a)$, then $\mathtt{fg}\in L(0,a)$ and*

$$\int_0^a|\mathtt{f}(\varkappa)\mathtt{g}(\varkappa)|d\varkappa \le \left(\int_0^a|\mathtt{f}(\varkappa)|^{q_1}\right)^{\frac{1}{q_1}}\left(\int_0^a|\mathtt{g}(\varkappa)|^{q_2}\right)^{\frac{1}{q_2}}.$$

3. Fuzzy Fractional Partial Integro-Differential Equations (FFPIDEs)

In the current section, we establish that the following FFPIDEs of Volterra type have a unique solution in $\mathfrak{C}(\mathtt{J},\mathscr{F}_R)$.

$$\begin{aligned} &{}^{c}\mathfrak{D}^{\theta}_{\mathbf{t}}\nu(\varkappa,\mathbf{t}) = \Upsilon(\varkappa,\mathbf{t},\nu,\nu_{\varkappa},\nu_{\varkappa\varkappa},S\nu),\\ &\nu(\varkappa,0)=\pounds(\varkappa), \end{aligned} \tag{3.1}$$

where ${}^{c}\mathfrak{D}^{\theta}_{\mathbf{t}}$ is the fuzzy Caputo derivative with respect to $\mathbf{t}$, $0<\theta<1,(\varkappa,\mathbf{t})\in\mathtt{J}$, $\Upsilon:\mathfrak{I}\times\mathscr{F}_R\times\mathscr{F}_R\times\mathscr{F}_R\times\mathscr{F}_R\to\mathscr{F}_R$ and S is a linear integral operator given by:

$$S\nu = \int_0^{\mathbf{t}} k(\varkappa,\mathbf{t},s)\nu(s,\mathbf{t})ds,$$

where k is a sufficiently smooth crisp function.
The following lemma provides the equivalent formulations to equation (3.1).

Lemma 3.1. *The fuzzy initial value problem (3.1) is equivalent to one of the following integrals equations:*

Case (I): *If ν is $(i)-$gH differentiable, then*

$$\nu(\varkappa,\mathbf{t}) = \pounds(\varkappa) + \mathfrak{I}^{\theta}_{\mathbf{t}}\left[\Upsilon(\varkappa,\mathbf{t},\nu,\nu_{\varkappa},\nu_{\varkappa\varkappa},S\nu)\right]. \tag{3.2}$$

Case (II): *If ν is $(ii)-$gH differentiable, then*

$$\nu(\varkappa,\mathbf{t}) = \pounds(\varkappa,)\ominus(-1)\mathfrak{I}^{\theta}_{\mathbf{t}}\left[\Upsilon(\varkappa,\mathbf{t},\nu,\nu_{\varkappa},\nu_{\varkappa\varkappa},S\nu)\right]. \tag{3.3}$$

Proof. Applying integral operator $\mathfrak{I}^{\theta}_{\mathbf{t}}$ on both the sides of equation (3.1) and from the Proposition 2.14 and the Lemma 2.11 we get (3.2) and (3.3). Thus (3.1) and (3.2) - (3.3) are equivalent. ☐

Now we establish the existence and uniqueness of the fuzzy solution to the problem (3.1) using the Banach contraction principle.

Theorem 3.2. *Let $\mathfrak{C}(\mathtt{J},\mathscr{F}_R)$ be the Banach space of all continuous fuzzy-valued functions. Assume that the following hypotheses are fulfilled*

- $H1$: *For any $\nu,\omega\in\mathfrak{C}(\mathtt{J},\mathscr{F}_R)$, there exists a constant $\theta_1\in(0,\theta)$ and real-valued positive functions $\mathfrak{K}_1(\varkappa,\mathbf{t}),\mathfrak{K}_2(\varkappa,\mathbf{t})\in L^{\frac{1}{\theta}}(\mathtt{J},\mathbb{R}^+)$ such that*
$$\mathbb{D}\Big(\Upsilon(\varkappa,\mathbf{t},\nu,\nu_{\varkappa},\nu_{\varkappa\varkappa},S\nu),\Upsilon(\omega,\mathbf{t},\omega,\omega_{\varkappa},\omega_{\varkappa\varkappa},S\omega)\Big)\leq\mathfrak{K}_1(\varkappa,\mathbf{t})\mathbb{D}\Big(\nu(\varkappa,\mathbf{t}),\omega(\varkappa,\mathbf{t})\Big)$$
$$+\mathfrak{K}_2(\varkappa,\mathbf{t})\mathbb{D}\Big(S\nu(\varkappa,\mathbf{t}),S\omega(\varkappa,\mathbf{t},)\Big).$$
- $H2$: *For the set of all non negative continuous function on $I=\{(\varkappa,\mathbf{t},s)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}:0\leq s\leq t\leq b\}$ there exist η_o such that*
$$\eta_o=\sup_{(\varkappa,\mathbf{t})\in J}\int_0^{\mathbf{t}}|k(\varkappa,\mathbf{t},s)|ds.<+\infty$$
and $\mathfrak{M}=\{\mathfrak{K}_1(\varkappa,\mathbf{t},s)+\eta_0 d\mathfrak{K}_2(\varkappa,\mathbf{t},s)\}_{L^{\frac{1}{\theta_1}}(\mathtt{J},\mathbb{R}^+)}$.

If
$$l^*=\frac{\mathfrak{M}d^{\theta-\theta_1}}{\Gamma(\theta)(\frac{\theta-\theta_1}{1-\theta_1})^{1-\theta_1}}<1,$$
then the problem (3.1) has a unique solution defined on $\mathtt{J}$.

Proof. We define the operator $\Xi:\mathfrak{C}(\mathtt{J},\mathscr{F}_R)\to\mathfrak{C}(\mathtt{J},\mathscr{F}_R)$ by
$$\Xi(\nu(\varkappa,\mathbf{t}))=\pounds(\varkappa)+\frac{1}{\Gamma(\theta)}\int_0^{\mathbf{t}}(\mathbf{t}-\rho)^{\theta-1}\Upsilon(\varkappa,\rho,\nu,\nu_{\varkappa},\nu_{\varkappa\varkappa},S\nu)d\rho,$$
for all $(\varkappa,\mathbf{t})\in\mathtt{J}$.
Assume that $\nu\in\mathfrak{C}(\mathtt{J},\mathscr{F}_R)$. First we show that Ξ is a fuzzy continuous operator. Let us assume that $\{\nu_n\}$ be a sequence such that $\nu_n\to\nu$ as $n\to\infty$ in $\mathfrak{C}[\mathtt{J}]$. Then for each $(\varkappa,\mathbf{t})\in\mathtt{J}$. We have
$$\mathbb{D}\Big((\Xi\nu_n)(\varkappa,\mathbf{t}),(\Xi\nu)(\varkappa,\mathbf{t})\Big)$$
$$\leq\frac{1}{\Gamma(\theta)}\int_0^{\mathbf{t}}(\mathbf{t}-\rho)^{\theta-1}\mathbb{D}\Big(\Upsilon(\varkappa,\rho,\nu_n,\nu_{n\varkappa},\nu_{n\varkappa\varkappa},S\nu_n),\Upsilon(\varkappa,\rho,\nu,\nu_{\varkappa},\nu_{\varkappa\varkappa},S\nu)\Big)d\rho.$$
From (2.1) and $H1$ we have
$$\mathbb{D}\Big((\Xi\nu_n)(\varkappa,\mathbf{t}),(\Xi\nu)(\varkappa,\mathbf{t})\Big)$$
$$\leq\frac{D^*\Big(\Upsilon(\varkappa,\rho,\nu_n,\nu_{n\varkappa},\nu_{n\varkappa\varkappa},S\nu_n),\Upsilon(\varkappa,\rho,\nu,\nu_{\varkappa},\nu_{\varkappa\varkappa},S\nu)\Big)}{\Gamma(\theta)}\int_0^{\mathbf{t}}(\mathbf{t}-\rho)^{\theta-1}d\rho.$$
$$\leq\frac{\mathbf{t}^{\theta}D^*\Big(\Upsilon(\varkappa,\rho,\nu_n,\nu_{n\varkappa},\nu_{n\varkappa\varkappa},S\nu_n),\Upsilon(\varkappa,\rho,\nu,\nu_{\varkappa},\nu_{\varkappa\varkappa},S\nu)\Big)}{\Gamma(\theta+1)}.$$
Since ν is a fuzzy continuous function, we have
$$\mathbb{D}\Big((\Xi\nu_n)(\varkappa,\mathbf{t}),(\Xi\nu)(\varkappa,\mathbf{t})\Big)$$
$$\leq\frac{\mathbf{t}^{\theta}D^*\Big(\Upsilon(\varkappa,\rho,\nu_n,\nu_{n\varkappa},\nu_{n\varkappa\varkappa},S\nu_n),\Upsilon(\varkappa,\rho,\nu,\nu_{\varkappa},\nu_{\varkappa\varkappa},S\nu)\Big)}{\Gamma(\theta+1)}$$
$$\to0\text{ as }n\to\infty.$$

Hence, N^* is a fuzzy continuous operator.
Now, we transform the problem (3.1) into a fixed-point problem. Suppose that $\nu(\varkappa,\mathbf{t})$ is a (i)-gH differentiable. We shall prove that Ξ is a contraction mapping using Banach contraction principle theorem. For this, let $\nu,\omega\in\mathfrak{C}(\mathtt{J},\mathscr{F}_R)$ and $(\varkappa,\mathbf{t})\in\mathtt{J}$. Using Lemma 2.11, $H1$ and Theorem 2.15 we have that
$$\mathbb{D}\Big((\Xi\nu)(\varkappa,\mathbf{t}),(\Xi\omega)(\varkappa,\mathbf{t})\Big)$$
$$\leq\frac{1}{\Gamma(\theta_1)}\int_0^{\mathbf{t}}(\mathbf{t}-\rho)^{\theta-1}\mathbb{D}\Big(\Upsilon(\varkappa,\mathbf{t},\nu,\nu_{\varkappa},\nu_{\varkappa\varkappa},S\nu),\Upsilon(\omega,\mathbf{t},\omega,\omega_{\varkappa},\omega_{\varkappa\varkappa},S\omega)\Big)d\rho$$
$$\leq\frac{1}{\Gamma(\theta)}\int_0^{\mathbf{t}}(\mathbf{t}-\rho)^{\theta-1}$$
$$\big[\mathfrak{K}_1(\varkappa,\mathbf{t})\mathbb{D}\Big(\nu(\varkappa,\mathbf{t}),\omega(\varkappa,\mathbf{t})\Big)+\mathfrak{K}_2(\varkappa,\mathbf{t})\mathbb{D}\Big(S\nu(\varkappa,\mathbf{t}),S\omega(\varkappa,\mathbf{t},)\Big)\big]d\rho$$

$$\leq \frac{D^*\Big(\nu(\varkappa,\mathbf{t}),\omega(\varkappa,\mathbf{t})\Big)}{\Gamma(\theta)}\int_0^{\mathbf{t}}|(\mathbf{t}-\rho)|^{\theta-1}\big[\mathfrak{K}_1(\varkappa,\mathbf{t})+\eta_0 d\mathfrak{K}_2(\varkappa,t)\big]d\rho$$

$$\leq \frac{D^*\Big(\nu(\varkappa,\mathbf{t}),\omega(\varkappa,\mathbf{t})\Big)}{\Gamma(\theta)}\Big(\int_0^{\mathbf{t}}(\mathbf{t}-\rho)^{\frac{\theta-1}{1-\theta_1}}d\rho\Big)^{1-\theta_1}\Big(\int_0^{\mathbf{t}}\big[\mathfrak{K}_1(\varkappa,\mathbf{t})+\eta_0 d\mathfrak{K}_2(\varkappa,\mathbf{t})\big]^{\frac{1}{\theta_1}}d\rho\Big)^{\theta_1}.$$

This implies that

$$D^*\Big((\Xi\nu)(\varkappa,\mathbf{t}),(\Xi\omega)(\varkappa,\mathbf{t})\Big)\leq\frac{\mathfrak{M}d^{\theta-\theta_1}}{\Gamma(\theta)(\frac{\theta-\theta_1}{1-\theta_1})^{1-\theta_1}}D^*\Big(\nu(\varkappa,\mathbf{t}),\omega(\varkappa,\mathbf{t})\Big)$$

$$\leq l^* D^*\Big(\nu(\varkappa,\mathbf{t}),\omega(\varkappa,\mathbf{t})\Big).$$

Since $l^*<1$, the operator Ξ is a contraction mapping. Thus, according to Banach fixed point theorem, the problem (3.1) has a unique fuzzy solution ν defined on $\mathtt{J}$ which is the unique $(i)-g\mathtt{H}$ differentiable solution of the problem (3.1).
Let $\nu(\varkappa,\mathbf{t})$ be $(ii)-g\mathtt{H}$ differentiable. In this case, we define the operator $\Xi:\mathfrak{C}(\mathtt{J},\mathscr{F}_R)\to\mathfrak{C}(\mathtt{J},\mathscr{F}_R)$ by

$$\Xi(\nu(\varkappa,\mathbf{t}))=\pounds(\varkappa)\ominus(-1)\frac{1}{\Gamma(\theta)}\int_0^{\mathbf{t}}(\mathbf{t}-\rho)^{\theta-1}\Upsilon(\varkappa,\rho,\nu,\nu_\varkappa,\nu_{\varkappa\varkappa},S\nu)d\rho.$$

Similarly, this type of differentiability can be demonstrated and therefore, it is not included in the proof. □

4. Fuzzy Adomian Decomposition Method (FADM)

The Adomian decomposition method, introduced by G. Adomian in 1984 [32], is a straightforward and effective approach for solving both linear and nonlinear differential equations. This method serves as a powerful tool for approximating solutions to fuzzy differential equations by representing the solution as an infinite series, often converging to the exact solution. Although the Adomian decomposition method may have some limitations, such as being computationally intensive for complex problems, it is particularly valuable for problems that are challenging or unsolvable by other means. Recently, several researchers have employed this method to solve various linear and nonlinear systems within fuzzy frameworks. For instance, Pandit et al studied a population dynamic model of two species and solved it using the FADM [33]. Saeed et al. [34] applied FADM to solve some nonlinear FFPDE. Further we can see [35–39].
Consider the following FFPIDE:

$${}^c\mathfrak{D}_{\mathbf{t}}^{\theta}\nu(\varkappa,\mathbf{t})=\Upsilon(\varkappa,\mathbf{t},\nu,\nu_\varkappa,\nu_{\varkappa\varkappa},S\nu)=L\nu(\varkappa,\mathbf{t})+A\nu(\varkappa,\mathbf{t})+I\nu(\varkappa,\mathbf{t}),\tag{4.1}$$

with fuzzy initial condition $\nu(\varkappa,0)=\pounds(\varkappa)$. Where L is a linear operator, A represents the nonlinear operator and I is an integral operator. The Adomian supposes that the unknown function $\nu(\varkappa,\mathbf{t})$ can be written by a series as

$$\nu(\varkappa,\mathbf{t})=\sum_{k=o}^{\infty}\nu_k(\varkappa,\mathbf{t}).\tag{4.2}$$

The nonlinear operator is represented by an infinite series as

$$A\nu=\sum_{k=0}^{\infty}M_k,$$

where M_k is Adomian polynomials given by

$$M_k=\frac{1}{k!}\frac{\partial^k}{\partial\beta^k}\Big[A(\sum_i^{\infty}\beta^i\nu_i)\Big]_{\beta=0}.$$

Finally, to compute the terms of the series $\sum_{k=o}^{\infty}\nu_k$, we use the following iterated scheme

- If ν is $(i)-g\mathtt{H}$ differentiable, then

$$\begin{aligned}&\nu_0(\varkappa,\mathbf{t})=\pounds(\varkappa),\\&\nu_{k+1}(\varkappa,\mathbf{t})=\mathfrak{I}_{\mathbf{t}}^{\theta}\big(L\nu_k+\sum_{k=0}^{\infty}M_k+I\nu_k\big).\end{aligned}\tag{4.3}$$

- If ν is $(ii)-g\mathtt{H}$ differentiable, then

$$\begin{aligned}&\nu_0(\varkappa,\mathbf{t})=\pounds(\varkappa),\\&\nu_{k+1}(\varkappa,\mathbf{t})=\ominus(-1)\mathfrak{I}_t^{\theta}\big(L\nu_k+\sum_{k=0}^{\infty}M_k+I\nu_k\big).\end{aligned}\tag{4.4}$$

4.1. Convergence of FADM

Here, we aim to thoroughly demonstrate the convergence of the series solution that is obtained from equation (4.2), by analyzing its structure and applying appropriate mathematical techniques to ensure that the solution behaves as expected under the given conditions.

Theorem 4.1. *Assume that the operators L, N and I defined in Equation (4.1) satisfy the following Lipschitz conditions with constants L_1, L_2 and L_3.*

$$\mathbb{D}(Lv_k(\varkappa,\mathbf{t}),Lv_{k-1}(\varkappa,\mathbf{t}))\leq L_1,$$
$$\mathbb{D}(Av_k(\varkappa,\mathbf{t}),Av_{k-1}(\varkappa,\mathbf{t}))\leq L_2,$$
$$\mathbb{D}(Iv_k(\varkappa,\mathbf{t}),Iv_{k-1}(\varkappa,\mathbf{t}))\leq L_3.$$

The series solution (4.2) of Equations (4.1) converges to the exact solution if $0<L_1+L_2+L_3<1$ and $D(v_k,0)<\infty, k\geq 0$, where v_k are given by (4.3) and (4.4).

Proof. Here we will prove the theorem for case v is $(i)-g\mathtt{H}$ differentiable. The proof of case v $(ii)-g\mathtt{H}$ differentiable is similar, so it will omitted.
Let S_n be the partial sum of the series $S_n=\sum_{k=o}^{n}v_k(\varkappa,t)$. We prove that S_n is a Cauchy sequence in the Banach space $\mathfrak{C}(\mathtt{J},\mathscr{F}_R)$. By hypothesis, we get

$$\begin{aligned}
&D^*(S_n(\varkappa,\mathbf{t}),S_m(\varkappa,\mathbf{t}))\\
=&\sup_{(\varkappa,\mathbf{t})\in\mathtt{J}}\mathbb{D}\left(\sum_{k=m+1}^{n}v_k(\varkappa,\mathbf{t}),0\right)\\
=&\sup_{(\varkappa,\mathbf{t})\in\mathtt{J}}\mathbb{D}\left(\frac{1}{\Gamma(\theta)}\int_0^{\mathbf{t}}(\mathbf{t}-\rho)^{\theta-1}\big(LS_{n-1}(\varkappa,\rho)+AS_{n-1}(\varkappa,\rho)\right.\\
&+IS_{n-1}(\varkappa,\rho)\big)d\rho,\frac{1}{\Gamma(\theta)}\int_0^{\mathbf{t}}(\mathbf{t}-\rho)^{\theta-1}\big(LS_{m-1}(\varkappa,\rho)+AS_{m-1}(\varkappa,\rho)\\
&\left.+IS_{m-1}(\varkappa,\rho)\big)d\rho\right)\\
\leq&\frac{1}{\Gamma(\theta)}\sup_{(\varkappa,\mathbf{t})\in\mathtt{J}}\Big(\mathbb{D}\big(LS_{n-1}(\varkappa,\mathbf{t}),LS_{m-1}(\varkappa,\mathbf{t})\big)+\mathbb{D}\big(AS_{n-1}(\varkappa,\mathbf{t}),AS_{m-1}(\varkappa,\mathbf{t})\big)\\
&+\mathbb{D}\big(IS_{n-1}(\varkappa,\mathbf{t}),IS_{m-1}(\varkappa,\mathbf{t})\big)\Big)\int_0^{\mathbf{t}}|\mathbf{t}-\rho|^{\theta-1}d\rho\\
\leq&CD^*\big(S_{n-1}(\varkappa,\mathbf{t}),S_{m-1}(\varkappa,\mathbf{t})\big),
\end{aligned}$$

where $C=(L_1+L_2+L_3)\frac{b^\theta}{\Gamma(\theta+1)}$.

If $n=m+1$. We get

$$\begin{aligned}
D^*(S_n(\varkappa,\mathbf{t}),S_m(\varkappa,\mathbf{t})\leq&CD^*\big(S_m(\varkappa,\mathbf{t}),S_{m-1}(\varkappa,\mathbf{t})\big)\\
\leq&C^2D^*\big(S_{m-1}(\varkappa,\mathbf{t}),S_{m-2}(\varkappa,\mathbf{t})\big)\\
&\vdots\\
\leq&C^mD^*\big(S_1(\varkappa,\mathbf{t}),S_0(\varkappa,\mathbf{t})\big).
\end{aligned}$$

Now, for $n>m$, we have

$$\begin{aligned}
D^*(S_n(\varkappa,\mathbf{t}),S_m(\varkappa,\mathbf{t}))\leq&D^*\big(S_m(\varkappa,\mathbf{t}),S_{m+1}(\varkappa,\mathbf{t})\big)+\cdots+D^*\big(S_n(\varkappa,\mathbf{t}),S_{n+1}(\varkappa,\mathbf{t})\big)\\
\leq&\frac{C^m}{1-C}D^*\big(v_1(\varkappa,\mathbf{t}),0\big).
\end{aligned}$$

Since v is bounded, as $m\to\infty$, then $D^*(S_n(\varkappa,\mathbf{t}),S_m(\varkappa,\mathbf{t}))\to\infty$. Hence, S_n is a Cauchy sequence in $\mathfrak{C}(\mathfrak{I},\mathscr{F}_R)$ and therefore, the series converges and the proof is complete. □

4.2. FADM for solving fuzzy fractional parabolic IDEs

Now, we employ the FADM to analyze the following fuzzy fractional parabolic IDEs

$$\begin{aligned}
&{}^c\mathfrak{D}_{\mathbf{t}}^{\theta}v(\varkappa,\mathbf{t})=v_{\varkappa\varkappa}(\varkappa,\mathbf{t})+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)v(s,\mathbf{t})ds+\mathfrak{h}(\varkappa,\mathbf{t}),\\
&v(\varkappa,0)=\pounds(\varkappa),
\end{aligned}\tag{4.5}$$

where ${}^c\mathfrak{D}_{\mathbf{t}}^{\theta}$ is the fuzzy Caputo derivative with respect to $\mathbf{t}$, $0<\theta<1,(\varkappa,t)\in\mathtt{J}$, k is a crisp function whose sign does not change in $\mathtt{J}$ and $\pounds,\mathfrak{h}$ are known crisp or fuzzy valued functions.
By applying the operator $\mathfrak{I}_{\mathbf{t}}^{\theta}$ to both side of equation (4.5), we get

$$v(\varkappa,\mathbf{t})\ominus v(\varkappa,0)=\mathfrak{I}_{\mathbf{t}}^{\theta}\left[v_{\varkappa\varkappa}(\varkappa,\mathbf{t})+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)v(s,\mathbf{t})ds\oplus\mathfrak{h}(\varkappa,\mathbf{t})\right].$$

If ν is $(i)-g\mathtt{H}$ differentiable, then

$$\nu(\varkappa,\mathbf{t})=\nu(\varkappa,0)+\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\nu_{\varkappa\varkappa}(\varkappa,\mathbf{t})+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\nu(s,\mathbf{t})ds\oplus\mathfrak{h}(\varkappa,\mathbf{t})\right] \tag{4.6}$$

and if ν is $(ii)-g\mathtt{H}$ differentiable, then

$$\nu(\varkappa,\mathbf{t})=\nu(\varkappa,0)\ominus(-1)\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\nu_{\varkappa\varkappa}(\varkappa,\mathbf{t})+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\nu(s,\mathbf{t})ds\oplus\mathfrak{h}(\varkappa,\mathbf{t})\right]. \tag{4.7}$$

Now we study four cases to find the numerical solution:

Case (1): Let ν be $(i)-g\mathtt{H}$ differentiable and $k(\varkappa,\mathbf{t},s)$ be a positive real function, then the parametric form of equation (4.6) is:

$$\underline{\nu}(\varkappa,\mathbf{t},\varsigma)=\underline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\underline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\underline{\nu}(s,\mathbf{t},\varsigma)ds+\underline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma)\right],$$

$$\overline{\nu}(\varkappa,\mathbf{t},\varsigma)=\overline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\overline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\overline{\nu}(s,\mathbf{t},\varsigma)ds+\overline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma)\right].$$

The standard Adomian Method assumes that the solution $\nu(\varkappa,\mathbf{t},\varsigma)$ can be written as the following series

$$\underline{\nu}(\varkappa,\mathbf{t},\varsigma)=\sum_{k=o}^{\infty}\underline{\nu}_k(\varkappa,\mathbf{t},\varsigma),$$

$$\overline{\nu}(\varkappa,\mathbf{t},\varsigma)=\sum_{k=o}^{\infty}\overline{\nu}_k(\varkappa,\mathbf{t}\varsigma).$$

Finally, to calculate the terms of the above series, we use the following iterated scheme

$$\underline{\nu}_0(\varkappa,\mathbf{t},\varsigma)=\underline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\underline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma),$$

$$\underline{\nu}_{k+1}(\varkappa,\mathbf{t},\varsigma)=\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\underline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\underline{\nu}(s,\mathbf{t},\varsigma)ds\right]$$

and

$$\overline{\nu}_0(\varkappa,\mathbf{t},\varsigma)=\overline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\overline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma),$$

$$\overline{\nu}_{k+1}(\varkappa,\mathbf{t},\varsigma)=\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\overline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\overline{\nu}(s,\mathbf{t},\varsigma)ds\right].$$

Case (2): Let ν be $(i)-g\mathtt{H}$ differentiable and $k(\varkappa,\mathbf{t},s)$ be a negative real function, then the parametric form of equation (4.6) is:

$$\underline{\nu}(\varkappa,\mathbf{t},\varsigma)=\underline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\underline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\overline{\nu}(s,\mathbf{t},\varsigma)ds+\underline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma)\right],$$

$$\overline{\nu}(\varkappa,\mathbf{t},\varsigma)=\overline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\overline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\underline{\nu}(s,\mathbf{t},\varsigma)ds+\overline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma)\right].$$

According to the above process , we get the solutions $\underline{\nu}(\varkappa,\mathbf{t},\varsigma)$ and $\overline{\nu}(\varkappa,\mathbf{t},\varsigma)$.

Case (3): Let ν be $(ii)-g\mathtt{H}$ differentiable and $k(\varkappa,\mathbf{t},s)$ be positive real function, then the parametric form of equation (4.7) is:

$$\underline{\nu}(\varkappa,\mathbf{t},\varsigma)=\underline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\overline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\overline{\nu}(s,\mathbf{t},\varsigma)ds+\overline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma)\right],$$

$$\overline{\nu}(\varkappa,\mathbf{t},\varsigma)=\overline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\underline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\underline{\nu}(s,\mathbf{t},\varsigma)ds+\underline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma)\right].$$

Then in the same way to previous case, we obtain the solutions $\underline{\nu}(\varkappa,\mathbf{t},\varsigma)$ and $\overline{\nu}(\varkappa,\mathbf{t},\varsigma)$.

Case (4): Let ν be $(ii)-g\mathtt{H}$ differentiable and $k(\varkappa,\mathbf{t},s)$ be negative real function, then the parametric form of equation (4.7) is:

$$\underline{\nu}(\varkappa,\mathbf{t},\varsigma)=\underline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\overline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\underline{\nu}(s,\mathbf{t},\varsigma)ds+\overline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma)\right],$$

$$\overline{\nu}(\varkappa,\mathbf{t},\varsigma)=\overline{\pounds}(\varkappa,\varsigma)+\mathfrak{I}_{\mathbf{t}}^{\theta}\left[\underline{\nu}_{\varkappa\varkappa}(\varkappa,\mathbf{t},\varsigma)+\int_0^{\mathbf{t}}k(\varkappa,\mathbf{t},s)\overline{\nu}(s,\mathbf{t},\varsigma)ds+\underline{\mathfrak{h}}(\varkappa,\mathbf{t},\varsigma)\right].$$

Therefore, by applying the method discussed in detail in the previous case, we get the solutions $\underline{\nu}(\varkappa,\mathbf{t},\varsigma)$ and $\overline{\nu}(\varkappa,\mathbf{t},\varsigma)$.

Remark 4.2. *In this subsection, the fuzzy fractional parabolic IDEs were converted into a system of scalar differential equations via ς-level representations and subsequently solved using the FADM. While the scalarized problems yield unique solutions under standard conditions, it is important to note that this does not necessarily guarantee the uniqueness of the original fuzzy solution. This discrepancy arises due to the inherent properties of fuzzy arithmetic, particularly in the multiplication of fuzzy numbers, which may result in non-uniqueness or bifurcation of solutions. In such cases, multiple fuzzy-valued functions can correspond to the same scalar ς-level solutions. A similar observation has been made in [40], where a numerical scheme for fuzzy fractional models resulted in bifurcated solutions depending on the nature of fuzzy multiplication. To address this challenge, we adopt the concept of maximal solutions as proposed in [29]. A maximal fuzzy solution is one that dominates all other admissible fuzzy solutions pointwise, providing an upper bound to the solution set. This approach not only accommodates the possibility of non-uniqueness but also strengthens the interpretation and reliability of the obtained fuzzy solution.*
Hence, while our method guarantees the uniqueness of the scalar components, we emphasize that the full fuzzy solution may admit multiple interpretations. The incorporation of maximal solutions allows for a well-defined framework within which these solutions can be understood and compared.

5. Applications and Simulations

In this chapter, we provide a series of numerical examples corresponding to each of the cases discussed in the previous chapter, with the aim of illustrating the applicability and effectiveness of the proposed methods.

Example 5.1. *Consider the following fuzzy fractional parabolic IDEs:*

$$\begin{cases} {}^{c}\mathfrak{D}_{\mathbf{t}}^{\theta}\nu(\varkappa,\mathbf{t}) = \nu_{\varkappa\varkappa}(\varkappa,\mathbf{t}) + \int_0^{\mathbf{t}}(\mathbf{t}-s)\nu(\varkappa,s)ds + \mathfrak{h}(\varkappa,\mathbf{t}), \ \ 0 \leq \varkappa,\mathbf{t} \leq 1, \\ \nu(\varkappa,0) = 0, \end{cases} \tag{5.1}$$

where

$$\mathfrak{h}(\varkappa,\mathbf{t}) = [\varsigma+1, 5-3\varsigma]cos\varkappa[\frac{\mathbf{t}^{1-\theta}}{\Gamma(2-\theta)} + \mathbf{t} - \frac{\mathbf{t}^3}{6}].$$

The exact solution of Equation (5.1) is $\nu(\varkappa,\mathbf{t}) = [\varsigma+1, 5-3\varsigma]\mathbf{t}cos\varkappa$. Figure 5.1 represent (a) the exact solutions and its ${}^{c}\mathfrak{D}_{\mathbf{t}}^{\theta}\nu(\varkappa,t)$ plotted in (b) with $\theta = \frac{3}{4}$ and different values of uncertainty $\varsigma \in [0,1]$. As it is seen, $\nu(\varkappa,t)$ is $(i)-g\mathrm{H}$-differentiable, this problem has been solved by FADM for $k = 2$. Figure 5.2 clearly shows that the numerical solution converges to the exact solution. This demonstrates that the proposed method is highly efficient for obtaining numerical solutions to these problems.

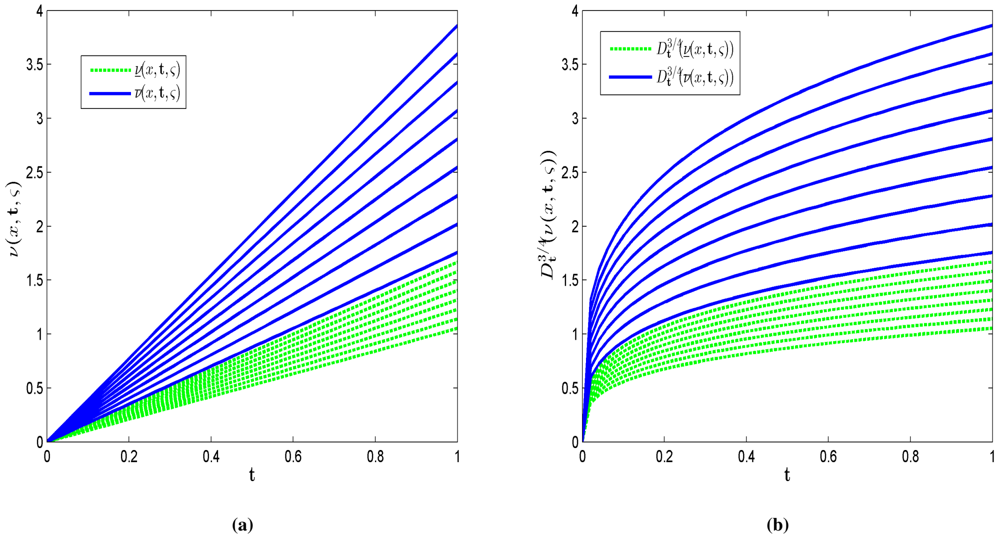

Figure 5.1: 2D plots of $\nu(\varkappa,\mathbf{t})$ and ${}^{c}\mathfrak{D}_{\mathbf{t}}^{\frac{3}{4}}\nu(\varkappa,\mathbf{t})$ of ς-level of Example 5.1 at $\varkappa = \frac{1}{2}$.

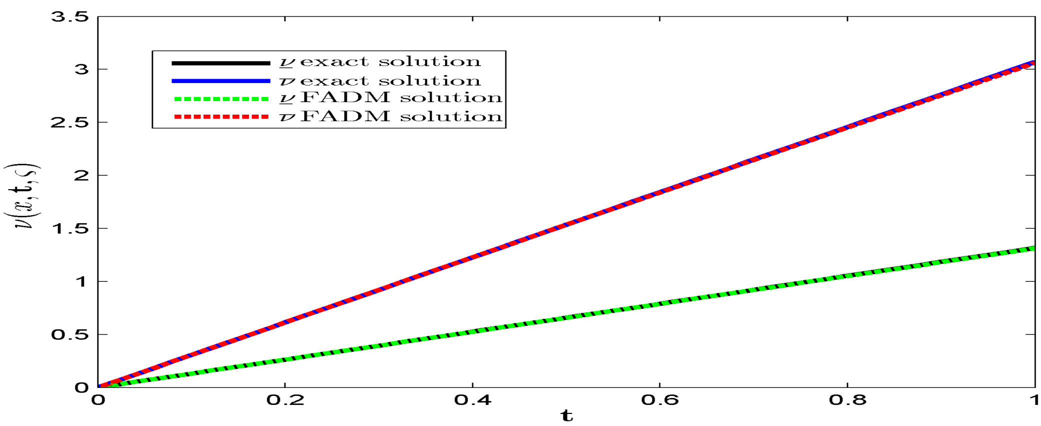

Figure 5.2: 2D plot of ς-level representations of exact and FADM solution of Example 5.1 at $\varkappa = \frac{1}{2}$ and $\varsigma = \frac{1}{2}$.

Example 5.2. *Let us Consider the fuzzy parabolic IDEs of fractional order as:*

$$\begin{cases} {}^{c}\mathfrak{D}_{\mathbf{t}}^{\theta}\nu(\varkappa,\mathbf{t}) = \nu_{\varkappa\varkappa}(\varkappa,\mathbf{t}) + \int_0^{\mathbf{t}} -\varkappa(\mathbf{t}-s)\nu(\varkappa,s)ds + \mathfrak{h}(\varkappa,\mathbf{t}), \ \ 0 \leq \varkappa \leq 1, 0 \leq \mathbf{t} \leq 0.5, \\ \nu(\varkappa,0) = 0, \end{cases} \tag{5.2}$$

where

$$\mathfrak{h}(\varkappa,\mathbf{t}) = B_1(1-\varkappa^2)\left[\frac{\mathbf{t}^{1-\theta}}{\Gamma(2-\theta)}+\Gamma(\theta+1)\right]+B_1(\mathbf{t}+\mathbf{t}^{\theta})+B_2\left[\frac{\mathbf{t}^3}{6}+\frac{\mathbf{t}^{2+\theta}}{(1+\theta)(2+\theta)}\right]$$

and $B_1=[\varsigma,2-\varsigma]$, $B_2=[2-\varsigma,\varsigma]$.

The exact solution of Equation (5.2) is $\nu(\varkappa,\mathbf{t}) = B_1(1-\varkappa^2)(\mathbf{t}+\mathbf{t}^{\theta})$. Figure 5.3 represent (a) the exact solutions and its ${}^{c}\mathfrak{D}_{\mathbf{t}}^{\theta}\nu(\varkappa,\mathbf{t})$ plotted in (b) with $\theta=\frac{1}{2}$ and different values of uncertainty $\varsigma\in[0,1]$. From Figure 5.3, $\nu(\varkappa,t)$ is $(i)-g\mathtt{H}$-differentiable. By applying the FADM for $k=2$, we get the numerical results shown in Figure 5.4.

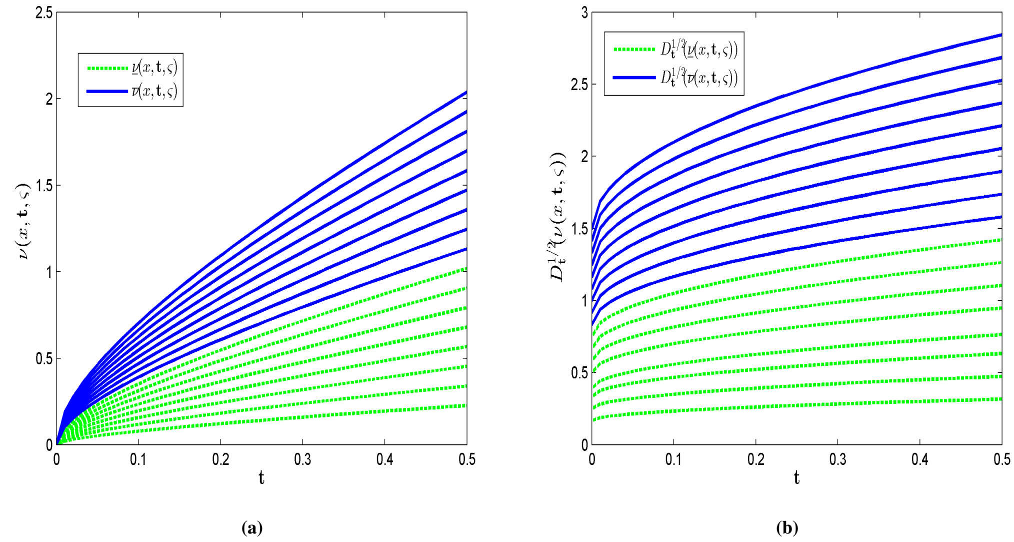

Figure 5.3: 2D plots of $\nu(\varkappa,\mathbf{t})$ and ${}^{c}\mathfrak{D}_{\mathbf{t}}^{\frac{1}{2}}\nu(\varkappa,\mathbf{t})$ of ς-level of Example 5.2 at $\varkappa=\frac{1}{4}$.

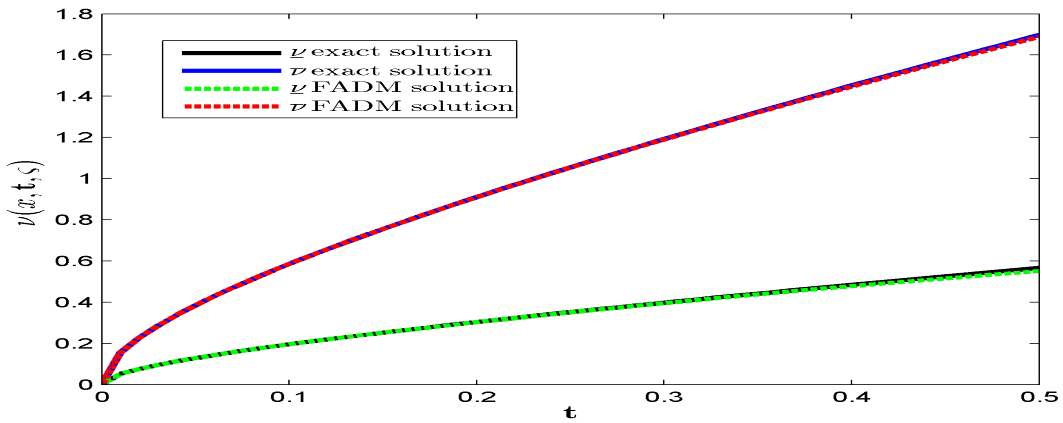

Figure 5.4: 2D graph of ς-level representations of exact and FADM solution of Example 5.2 at $\varkappa=\frac{1}{4}$ and $\varsigma=\frac{1}{2}$.

Example 5.3. *Now we consider another fuzzy fractional parabolic IDEs under the initialcondition as:*

$$\begin{cases} {}^{c}\mathfrak{D}_{\mathbf{t}}^{\theta}\nu(\varkappa,\mathbf{t}) = \nu_{\varkappa\varkappa}(\varkappa,\mathbf{t})+\int_0^{\mathbf{t}}\mathbf{t}\nu(\varkappa,s)ds+\mathfrak{h}(\varkappa,\mathbf{t}), \quad 0\le\varkappa,\mathbf{t}\le 1, \\ \nu(\varkappa,0)=0, \end{cases} \tag{5.3}$$

where

$$\mathfrak{h}(\varkappa,\mathbf{t}) = [2+3\varsigma,8-3\varsigma]\sin\varkappa\left[\frac{\Gamma(1-\theta)}{\Gamma(2-\theta)}\mathbf{t}^{-2\theta}+\mathbf{t}^{-\theta}-\frac{\mathbf{t}^{2-\theta}}{1-\theta}\right].$$

The exact solution of Equation (5.3) is $\nu(\varkappa,\mathbf{t}) = [2+3\varsigma,8-3\varsigma]\sin\varkappa\mathbf{t}^{-\theta}$. Figure 5.5 represent (a) the exact solutions and its ${}^{c}\mathfrak{D}_{\mathbf{t}}^{\theta}\nu(\varkappa,\mathbf{t})$ plotted in (b) with $\theta=\frac{3}{4}$ and different values of uncertainty $\varsigma\in[0,1]$. As it is seen, $\nu(\varkappa,\mathbf{t})$ is $(ii)-g\mathtt{H}$-differentiable, so by applying the FADM discussed in detail in Subsection 4.2, with $k=3$, we have the numerical results shown in Figure 5.6.

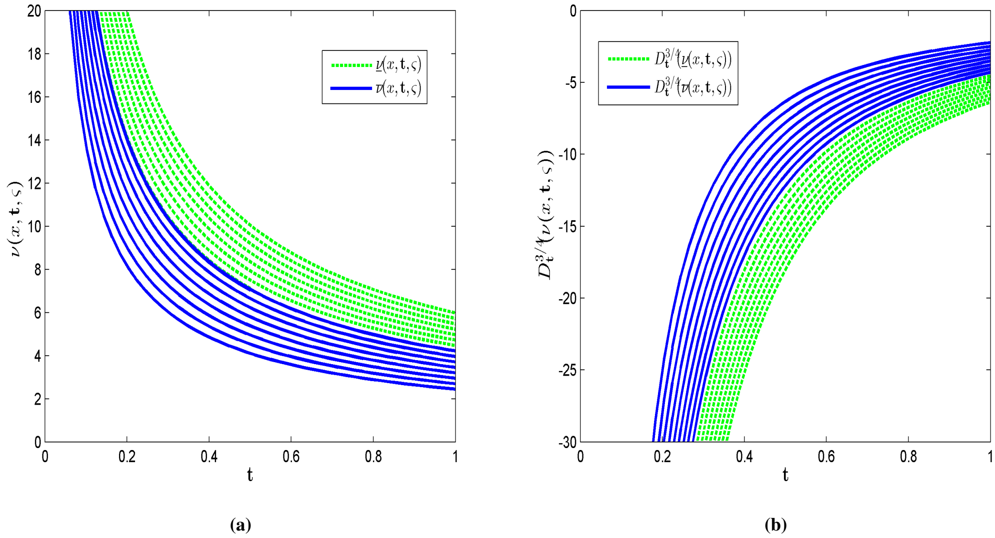

Figure 5.5: 2D plots of $\nu(\varkappa,\mathbf{t})$ and ${}^c\mathfrak{D}_{\mathbf{t}}^{\frac{3}{4}}\nu(\varkappa,\mathbf{t})$ of ς-level of Example 5.3 at $\varkappa=1$.

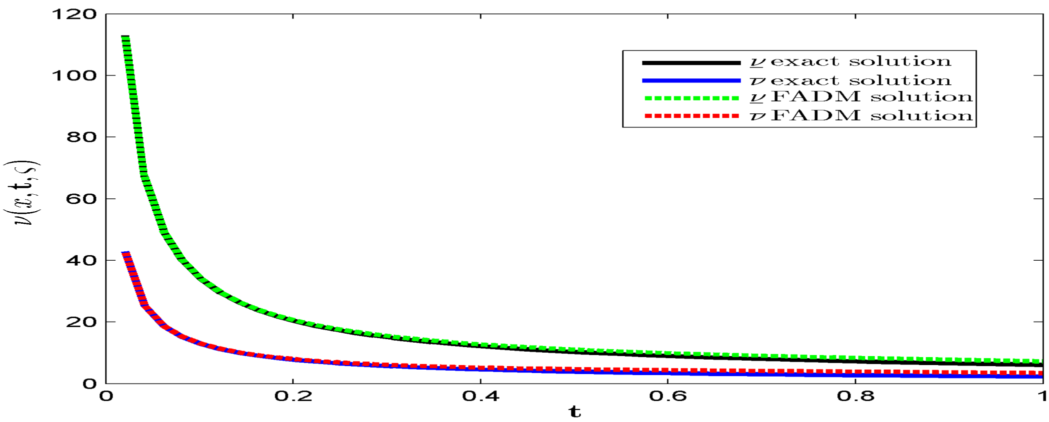

Figure 5.6: 2D graph of ς-level representations of exact and FADM solution of Example 5.3 at $\varkappa=1$ and $\varsigma=\frac{1}{4}$.

Example 5.4. *Consider the following fuzzy fractional parabolic IDEs:*

$$\begin{cases} {}^c\mathfrak{D}_{\mathbf{t}}^{\theta}\nu(\varkappa,\mathbf{t}) = \nu_{\varkappa\varkappa}(\varkappa,\mathbf{t}) + \int_0^{\mathbf{t}} -\nu(\varkappa,s)ds, \quad 0\le\varkappa,\mathbf{t}\le 1, \\ \qquad\qquad \nu(\varkappa,0) = [0.5\varsigma, 1-0.5\varsigma]\varkappa. \end{cases} \tag{5.4}$$

The exact solution of Equation (5.4) is given by $\nu(\varkappa,\mathbf{t}) = [0.5\varsigma, 1-0.5\varsigma]\varkappa E_{\theta+1}(-\mathbf{t}^{\theta+1})$. Figure 5.7 represent (a) the exact solutions and its ${}^c\mathfrak{D}_{\mathbf{t}}^{\theta}\nu(\varkappa,\mathbf{t})$ plotted in (b) with $\theta=\frac{1}{2}$ and different values of uncertainty $\varsigma\in[0,1]$. Figure 5.7 shows that $\nu(\varkappa,\mathbf{t})$ is $(ii)-g\mathrm{H}$-differentiable. Thus, by applying the FADM discussed in detail in case 4, for $k=4$. Figure 5.8 shows the exact and approximate results.
The results in Figure 5.8 show that the numerical solution converges to the exact solution. This confirms that the proposed method is highly efficient for obtaining numerical solutions to such problems.

6. Conclusion

In this article, we have established sufficient conditions for the existence and uniqueness of solutions to FFPIDEs. Additionally, we applied the FADM to obtain approximate solutions for the problem taking into account the type of differentiability. Also, the convergence of FADM to the exact solution is proved. Four illustrative examples of fuzzy fractional parabolic IDEs are provided to validate the effectiveness and performance of our method. The proposed method provides reliable series solutions with continuity depending on the fuzzy fractional derivative. As the number of decomposed terms increases, the numerical solution converges. As a future extension, this method could be applied to two-dimensional fuzzy fractional parabolic IDEs with both constant and variable coefficients and could also be expanded to address nonlinear problems.

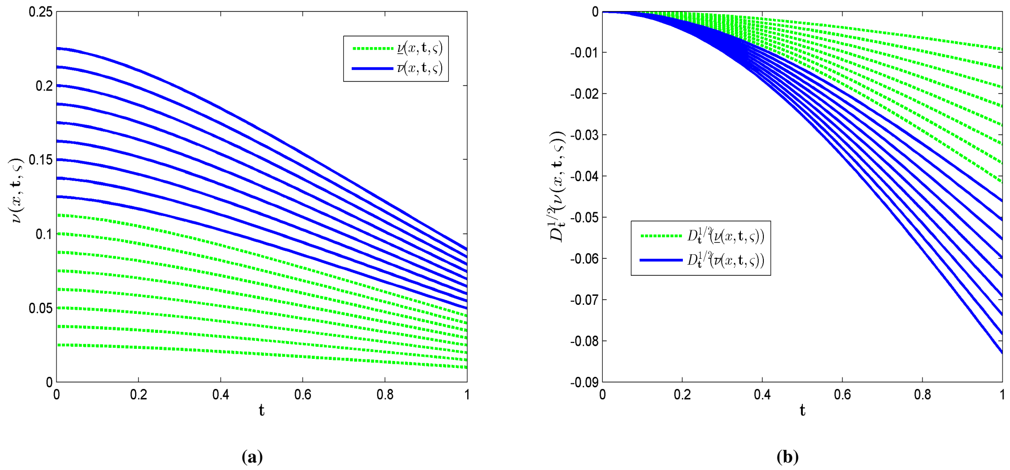

Figure 5.7: 2D plots of $\nu(\varkappa,\mathbf{t})$ and ${}^c\mathfrak{D}_{\mathbf{t}}^{\frac{1}{2}}\nu(\varkappa,\mathbf{t})$ of ς-level of Example 5.4 at $\varkappa=\frac{1}{4}$.

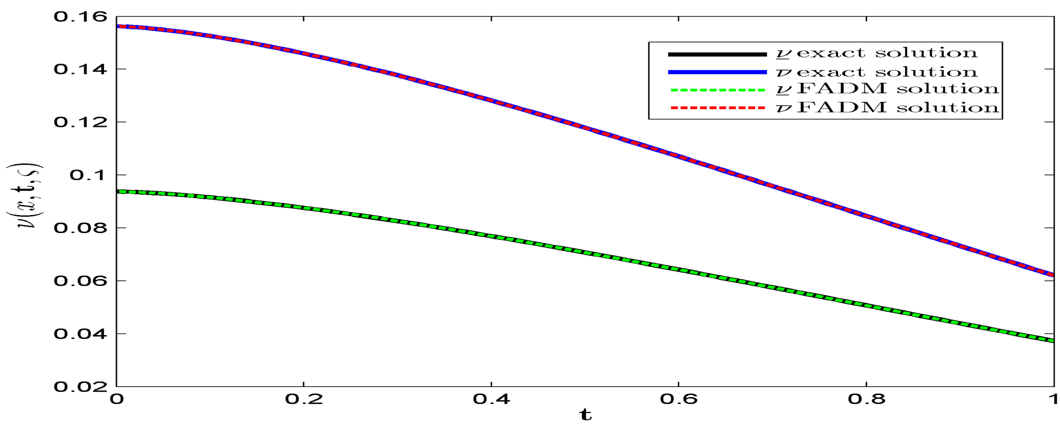

Figure 5.8: 2D graph of ς-level representations of exact and FADM solution of Example 5.4 at $\varkappa=\frac{1}{4}$ and $\varsigma=\frac{3}{4}$.

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References

[1] H. R. Marasi, H. Afshari, C. B. Zhai, *Some existence and uniqueness results for nonlinear fractional partial differential equations*, Rocky Mountain J. Math., **47**(2) (2017), 571–585. https://doi.org/10.1216/RMJ-2017-47-2-571

[2] D.B. Pachpatte, *Existence and stability of some nonlinear ψ-Hilfer partial fractional differential equation*, Partial Differential Equations in Applied Mathematics, **3** (2021), Article ID 100032. https://doi.org/10.1016/j.padiff.2021.100032

[3] L. Deng, A. Ducrot, *Existence of multi-dimensional pulsating fronts for KPP equations: a new formulation approach*, Calc. Var. Partial Differential Equations, **62** (2023), Article ID 134. https://doi.org/10.1007/s00526-023-02473-y

[4] R. Shah, H. Khan, P. Kumam, M. Arif, *An analytical technique to solve the system of nonlinear fractional partial differential equations*, Mathematics, **7**(6) (2019), Article ID 505. https://doi.org/10.3390/math7060505

[5] P. Jena, S. M. S. Mishra, A. A. Al-Moneef, A. A. Hindi, K. S. Nisar, *The solution of nonlinear time-fractional differential equations: an approximate analytical approach*, Progress in Fractional Differentiation and Applications, **8**(1) (2022), 191–204. http://dx.doi.org/10.18576/pfda/080112

[6] M. N. Koleva, L. G. Vulkov, *Numerical analysis of direct and inverse problems for a fractional parabolic integro-differential equation*, Fractal Fract., **7**(8) (2023), Article ID 601. https://doi.org/10.3390/fractalfract7080601

[7] R. P. Agarwal, V. Lakshmikantham, J. J. Nieto, *On the concept of solution for fractional differential equations with uncertainty*, Nonlinear Anal., **72**(6) (2010), 2859–2862. https://doi.org/10.1016/j.na.2009.11.029

[8] S. Arshad, V. Lupulescu, *On the fractional differential equations with uncertainty*, Nonlinear Anal., **74**(11) (2011), 3685–3693. https://doi.org/10.1016/j.na.2011.02.048

[9] M. Hukuhara, *Integration des applications measurables dont la valuer est un compact convexe*, Funkcialaj Ekavacioy, **10**(3) (1967), 205–223.
[10] B. Bede, L. Stefanini, *Generalized differentiability of fuzzy-valued functions*, Fuzzy Sets and systems, **230** (2013), 119-141. https://doi.org/10.1016/j.fss.2012.10.003
[11] B. Shiri, *A unified generalization for Hukuhara types differences and derivatives: Solid analysis and comparisons*, AIMS Math., **8**(1) (2023), 2168–2190.https://doi.org/10.3934/math.2023112
[12] F. Longo, B. Laiate, M. C. Gadotti, J. F. da CA Meyer, *Characterization results of generalized differentiabilities of fuzzy functions*, Fuzzy Set Syst., **490** (2024), Article ID 109038. https://doi.org/10.1016/j.fss.2024.109038
[13] S. Salahshour, T. Allahviranloo, S. Abbasbandy, *Solving fuzzy fractional differential equations by fuzzy Laplace transforms*, Commun. Nonlinear Sci. Numer. Simul., **17**(3) (2012), 1372–1381. https://doi.org/10.1016/j.cnsns.2011.07.005
[14] H. V. Long, N. T. Son, H. T. Tam, *The solvability of fuzzy fractional partial differential equations under Caputo gH-differentiability*, Fuzzy Sets Syst., **309** (2017), 35–63. https://doi.org/10.1016/j.fss.2016.06.018
[15] M. A. Alqudah, R. Ashraf, S. Rashid, J. Singh, Z. Hammouch, T. Abdeljawad, *Novel numerical investigations of fuzzy Cauchy reaction–diffusion models via generalized fuzzy fractional derivative operators*, Fractal Fract., **5**(4) (2021), Article ID 151. https://doi.org/10.3390/fractalfract5040151
[16] N. A. Saeed, D. B. Pachpatte, *A modified fuzzy Adomian decomposition method for solving time-fuzzy fractional partial differential equations with initial and boundary conditions*, Bound. Value Probl., **2024**(1) (2024), Article ID 82. https://doi.org/10.1186/s13661-024-01885-9
[17] R. Alikhani, F. Bahrami, *Global solutions for nonlinear fuzzy fractional integral and integrodifferential equations*, Commun. Nonlinear Sci., **18** (2013), 2007–2017.
[18] A. Armand, Z. Gouyandeh, *Fuzzy fractional integro-differential equations under generalized Caputo differentiability*, Ann. Fuzzy Math. Inform., **10** (2015), 789–798.
[19] O. H. Mohammed, O. I. Khaleel, *Fractional differential transform method for solving fuzzy integro-differential equations of fractional order*, Basrah J. Sci., **34** (2016), 31–40.
[20] V. Padmapriya, M. Kaliyappan, V. Parthiban, *Solution of fuzzy fractional Integro-Differential equations using a domain decomposition method*, J. Inform. Math. Sci., **9** (2017), 501-507.
[21] S. Rashid, M. K. Kaabar, A. Althobaiti, M. S. Alqurashi, *Constructing analytical estimates of the fuzzy fractional-order Boussinesq model and their application in oceanography*, J. Ocean Eng. Sci., **8**(2) (2023), 196–215. https://doi.org/10.1016/j.joes.2022.01.003
[22] M. R. Nourizadeh, T. Allahviranloo, N. Mikaeilvand, *Positive solutions of fuzzy fractional Volterra integro-differential equations with the fuzzy Caputo fractional derivative using the Jacobi polynomials operational matrix*, Int. J. Comput. Sci. Netw. Secur., **18** (2018), 241–252.
[23] M. R. M. Shabestari, R. Ezzati, T. Allahviranloo, *Numerical solution of fuzzy fractional integro-differential equation via two-dimensional Legendre wavelet method*, J. Intell. Fuzzy Syst., **34**(4) (2018), 2453–2465. https://doi.org/10.3233/JIFS-171707
[24] B. Shiri, Z. Alijani, Y. Karaca, *A power series method for the fuzzy fractional logistic differential equation*, Fractals, **31**(10) (2023), Article ID 2340086. https://doi.org/10.1142/S0218348X23400868
[25] M. Alaroud, M. Al-Smadi, R. Ahmad, U. K. Din, *An analytical numerical method for solving fuzzy fractional Volterra integro-differential equations*, Symmetry, **11**(2) (2019), Article ID 205. https://doi.org/10.3390/sym11020205
[26] H. Hashemi, R. Ezzati, N. Mikaeilvand, M. Nazari, *Study on fuzzy fractional European option pricing model with Mittag-Leffler kernel*, J. Intell. Fuzzy Syst., **45**(5) (2023), 8567-8582.
[27] S. Ahsan, R. Nawaz, M. Akbar, S. Abdullah, K. S. Nisar, V. Vijayakumar, *Numerical solution of system of fuzzy fractional order Volterra integro-differential equation using optimal homotopy asymptotic method*, AIMS Math., **7**(7) (2022), 13169–13191. https://doi.org/10.3934/math.2022726
[28] T. Allahviranloo, *Fuzzy Fractional Differential Operators and Equations: Fuzzy Fractional Differential Equations*, Springer Nature, 2020. https://doi.org/10.1007/978-3-030-51272-9
[29] B. Shiri, D. Baleanu, C. Y. Ma, *Pathological study on uncertain numbers and proposed solutions for discrete fuzzy fractional order calculus*, Open Phys., **21**(1) (2023), Article ID 20230135. http://dx.doi.org/10.1515/phys-2023-0135
[30] B. Shiri, I. Perfieliva, Z. Alijani, *Classical approximation for fuzzy Fredholm integral equation*, Fuzzy Sets Syst., **404** (2021), 159–177. https://doi.org/10.1016/j.fss.2020.03.023
[31] S. Hartman, J. Mikusinski, *The Theory of Lebesgue Measure and Integration*, Elsevier, 2014.
[32] G. Adomian, *A review of the decomposition method in applied mathematics*, J. Math. Anal. Appl., **135**(2) (1988), 501–544. https://doi.org/10.1016/0022-247X(88)90170-9
[33] P. Pandit, P. Mistry, P. Singh, *Population dynamic model of two species solved by fuzzy Adomian decomposition method*, Math. Model., Comput. Intell. Tech. Renew. Energy, Proceedings of the First International Conference (MMCITRE 2020), (2021), 493–507. https://doi.org/10.1007/978-981-15-9953-8_42
[34] N. A. Saeed, D. B. Pachpatte, *Usage of the fuzzy Adomian decomposition method for solving some fuzzy fractional partial differential equations*, Adv. Fuzzy Syst., **2024**(1) (2024), Article ID 8794634. https://doi.org/10.1155/2024/8794634
[35] A. Ullah, A. Ullah, S. Ahmad, I. Ahmad, A. Akgül, *On solutions of fuzzy fractional order complex population dynamical model*, Numer. Methods Partial Differential Equations, **39**(6) (2020), 4595–4615. https://doi.org/10.1002/num.22654
[36] A. Georgieva, A. Pavlova, *Fuzzy Sawi decomposition method for solving nonlinear partial fuzzy differential equations*, Symmetry, **13**(9) (2021), Article ID 1580. https://doi.org/10.3390/sym13091580
[37] K. Shah, A. R. Seadawy, M. Arfan, *Evaluation of one dimensional fuzzy fractional partial differential equations*, Alex. Eng. J., **59**(5) (2020), 3347–3353. https://doi.org/10.1016/j.aej.2020.05.003
[38] S. Askari, T. Allahviranloo, S. Abbasbandy, *Solving fuzzy fractional differential equations by Adomian decomposition method used in optimal control theory*, Int. Transact. J. Eng. Manage. Appl. Sci. Technol., **10**(12) (2019), 1–10.
[39] M. Osman, Y. Xia, M. Marwan, O. A. Omer, *Novel approaches for solving fuzzy fractional partial differential equations*, Fractal Fract., **6**(11) (2022), Article ID 656. https://doi.org/10.3390/fractalfract6110656
[40] Z. Alijani, B. Shiri, I. Perfilieva, D. Baleanu, *Numerical solution of a new mathematical model for intravenous drug administration*, Evol. Intell., **17** (2024), 559–575. https://doi.org/10.1007/s12065-023-00840-4