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Research Article

On isolated subsemigroups of order-decreasing transformation semigroups

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Abstract

For $n \in \mathbb{N}$, let $\mathcal{D}_n$ be the semigroup of all order-decrasing transformations on $X_n = \{1, \dots, n\}$, under its natural order. In this paper, we determine isolated, completely isolated, and (left/right) convex subsemigroups of $\mathcal{D}_n$. Furthermore, for $\{1\} \neq U \subset X_n$ which contains 1, we find the rank of $\mathcal{D}_n[U] = \{\alpha \in \mathcal{D}_n : U \subsetneq X_n\}$ which is a convex subsemigroup of $\mathcal{D}_n$.

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1. Introduction

For $n \in \mathbb{N}$, let $X_n = \{1, \dots, n\}$ be the finite chain with the standard order. $\mathcal{T}_n$ denotes the (full) transformations semigroup (under composition) on X_n. For $\alpha \in \mathcal{T}_n$, if $x\alpha \leq x$ (for all $x \in X_n$), then α is called *order-decreasing transformation*. $\mathcal{D}_n$ denotes the semigroup of all order-decreasing transformations in $\mathcal{T}_n$. The *fix*, *shift* and *kernel* of any transformation $\alpha \in \mathcal{T}_n$ are defined as follows, respectively:

$$\operatorname{fix}(\alpha) = \{x \in X_n : x\alpha = x\}, \quad \operatorname{shift}(\alpha) = \{x \in X_n : x\alpha \neq x\} \quad \text{and}$$
$$\ker(\alpha) = \{(x, y) \in X_n \times X_n : x\alpha = y\alpha\}.$$

Let S be a semigroup and let $e \in S$. If $e^2 = e$ then e is called *idempotent element* of S, and $E(S)$ denotes the set of all idempotent elements of S. It is clear that $\alpha \in \mathcal{T}_n$ is an idempotent element if and only if $\operatorname{fix}(\alpha) = \operatorname{im}(\alpha)$. A commutative semigroup S is called a *semilattice* if $S = E(S)$. The set of all subsets of X_n is denoted by $\mathcal{SL}_n$. For all $U, V \in \mathcal{SL}_n$, let the multiplication on $\mathcal{SL}_n$ be defined as $U \cdot V = U \cap V$. Then $\mathcal{SL}_n$ is called the *free semilattice* on X_n. Notice that for all $U, V \in \mathcal{SL}_n$, the usual multiplication defined on $\mathcal{SL}_n$ is defined by $U \cdot V = U \cup V$. Since the first multiplication will play an essential role for this research and the map $\varphi : (\mathcal{SL}_n, \cap) \to (\mathcal{SL}_n, \cup)$ defined by $U\varphi = X_n \setminus U$, is an isomorphism, we prefer the first definition of the free semilattice. Let S be a semigroup with a zero, denoted by 0. For some $m \in \mathbb{N}$, if $a^m = 0$ then a is called *nilpotent element* of S, and $N(S)$ denotes the set of all nilpotent elements of S. Let S be a semigroup

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with 0. If $S^m = 0$ for some $m \in \mathbb{N}$, then S is called *nilpotent semigroup*. It is clear that for a finite semigroup S with 0, S is nilpotent semigroup if and only if for some $m \in \mathbb{N}$, $a^m = 0$ (for all $a \in S$). Moreover, the only idempotent element of S is 0 (see, also [3, Proposition 8.1.2]). Since $1\alpha = 1$ for all $\alpha \in \mathcal{D}_n$, it is clear that the zero element of $\mathcal{D}_n$ is $0_n = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 1 & \cdots & 1 \end{pmatrix}$. Moreover, from [11, Lemma 1.5], $\alpha \in \mathcal{D}_n$ is nilpotent if and only if $\mathrm{fix}(\alpha) = \{1\}$, and so $N(\mathcal{D}_n) = \{\alpha \in \mathcal{D}_n : \mathrm{fix}(\alpha) = \{1\}\}$. For any $\alpha \in \mathcal{T}_n$, recall that $\ker(\alpha)$ is an equivalence relation on X_n and the equivalence classes of $\ker(\alpha)$ are all of the pre-image sets of elements in $\mathrm{im}(\alpha)$, i.e. $\{a\alpha^{-1} : a \in \mathrm{im}(\alpha)\}$ is the set of all the equivalence classes of $\ker(\alpha)$. For $\alpha \in \mathcal{D}_n$, if $\mathrm{im}(\alpha) = \{1 = a_1 < a_2 < \cdots < a_r\}$ and $A_i = a_i\alpha^{-1}$ for each $1 \leq i \leq r$, then we can write α in the following tabular form: $\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ 1 & a_2 & \cdots & a_r \end{pmatrix}$. Then it is clear that $\alpha \in \mathcal{D}_n$ is an idempotent if and only if $\min(A_i) = a_i$ for each $1 \leq i \leq r$.

For a semigroup S, let $\emptyset \neq A \subseteq S$. $\langle A \rangle$ denotes the smallest subsemigroup of S containing A. For $A \subseteq S$, if $S = \langle A \rangle$ then A is called a *generating set* of S. The *rank* of S is defined by $\mathrm{rank}(S) = \min\{\,|A| : \langle A \rangle = S\}$, and a *minimal generating set* of S is defined as a generating set with $\mathrm{rank}(S)$ elements. For some $A \subseteq E(S)$, if $S = \langle A \rangle$ then S is called an *idempotent generated semigroup*, and the *idempotent rank* of S is defined by $\mathrm{idrank}(S) = \min\{\,|A| : A \subseteq E(S)$ and $\langle A \rangle = S\}$. For $s \in S$, if $s = ab$ means $s = a$ or $s = b$ (for $a, b \in S$), then s is called *indecomposable element* of S. It is clear that all indecomposable elements must be contained by every generating set of S. Moreover, for a finite nilpotent semigroup S, $S \setminus S^2$ is the unique minimal generating set of S. In addition, $\mathrm{rank}(S) = |S| - |S^2|$.

A proper subsemigroup of a semigroup S that is not contained any other proper subsemigroup of S is called *maximal subsemigroup*, and if it is unique, it is called the *maximum subsemigroup*. A subsemigroup T of S is called

- *isolated* provided that, for all $x \in S$, the condition $x^m \in T$ for some $m \in \mathbb{N}$ implies $x \in T$;
- *completely isolated* provided that, for all $x, y \in S$, $xy \in T$ implies $x \in T$ or $y \in T$;
- *left convex* provided that, for all $x, y \in S$, $xy \in T$ implies $x \in T$;
- *right convex* provided that, for all $x, y \in S$, $xy \in T$ implies $y \in T$;
- *convex* provided that, for all $x, y \in S$, $xy \in T$ implies $x \in T$ and $y \in T$.

From the definitions, it is clear that every convex subsemigroup is both left and right convex, every left/right convex subsemigroup is completely isolated, every completely isolated subsemigroup is isolated, and that any every left/right convex subsemigroup of a monoid must contain the identity. For the other terms in semigroup theory, which are not explained here, and for more details and properties on these subsemigroups, we refer to [1–4]).

The (completely) isolated and (left/right) convex subsemigroups of some special semigroups have been studied by several authors. For example, all (completely) isolated and (left/right) convex subsemigroups of $\mathcal{T}_n$ are classified in [7]. Moreover, all (completely) isolated and (left/right) convex subsemigroups of $\mathcal{I}_n$, all injective partial transformations on X_n, are classified in [10], and all (completely) isolated and (left/right) convex subsemigroups of $\mathcal{C}_n$, all order-preserving and decreasing transformations on X_n, are classified in [5]. Since the problem of finding certain algebraic and combinatorial properties of $\mathcal{D}_n$ has been an important research area in Semigroup Theory (for examples, see [6,8,9,11,13–15]), in this paper, we focus on the classification of (completely) isolated and (left/right) convex subsemigroups of $\mathcal{D}_n$. Finally, we determine a minimal generating set of $\mathcal{D}_n[U] = \{\,\alpha \in \mathcal{D}_n : U \subseteq \mathrm{fix}(\alpha)\,\}$, and so we find the rank of $\mathcal{D}_n[U]$ where U is a proper subset of X_n such that $\{1\} \subsetneq U$.

2. Isolated subsemigroups of $\mathcal{D}_n$

For an idempotent element ξ of a semigroup S, let

$$S(\xi) = \{\alpha \in S : \alpha^m = \xi \text{ for some } m \in \mathbb{N}\}.$$

Then we state the following lemma which is essential for classifying isolated subsemigroups of $\mathcal{D}_n$.

Lemma 2.1 ([3], Lemma 5.3.4)**.** *Let S be a semigroup, and let T be a subsemigroup of S.*

(i) *If T is isolated, then $S(\xi) \subseteq T$, for all $\xi \in E(T)$.*

(ii) *If T is isolated, and if S is finite, then $T = \bigcup_{\xi \in E(T)} S(\xi)$.*

For any $\xi \in E(\mathcal{D}_n)\backslash\{0_n, 1_n\}$, it is shown in [14, Theorem 2.3] that $\mathcal{D}_n(\xi)$ is the maximum nilpotent subsemigroup of $\mathcal{D}_n$ with zero element ξ. Observe that $\mathcal{D}_n(0_n) = N(\mathcal{D}_n)$ and $\mathcal{D}_n(1_n) = \{1_n\}$ where 1_n is the identity transformation on X_n. From this observation, we suppose that $\xi \in E(\mathcal{D}_n) \setminus \{0_n, 1_n\}$, unless otherwise stated. Then we state the following results from [6, 14] which also play important roles in this study.

Theorem 2.2 ([14], Theorem 2.2)**.** *For any $\xi \in E(\mathcal{D}_n)$ and any $\alpha \in \mathcal{D}_n$, the followings are equivalent:*

(*i*) $\alpha \in \mathcal{D}_n(\xi)$,

(*ii*) $\text{fix}\,(\alpha) = \text{fix}\,(\xi)$ *and* $\alpha\xi = \xi$,

(*iii*) $\text{fix}\,(\alpha) = \text{fix}\,(\xi)$ *and* $\alpha\xi = \xi = \xi\alpha$.

Moreover, we have the following.

Proposition 2.3 ([6], Lemma 1.1)**.** *For all $\alpha, \beta \in \mathcal{D}_n$, $\text{fix}\,(\alpha\beta) = \text{fix}\,(\alpha) \cap \text{fix}\,(\beta)$, and so $\text{fix}\,(\alpha^k) = \text{fix}\,(\alpha)$ for every $k \in \mathbb{N}$.*

Therefore, for any non-identity $\xi \in E(\mathcal{D}_n)$, $\mathcal{D}_n(\xi)$ is the maximum nilpotent subsemigroup of $\mathcal{D}_n$ with zero element ξ. Now we start to give our new results.

Theorem 2.4. (*i*) *For any non-identity $\xi \in E(\mathcal{D}_n)$, $\mathcal{D}_n(\xi)$ is the unique isolated nilpotent subsemigroup of $\mathcal{D}_n$ with zero element ξ.*

(*ii*) *For any subsemigroup T of $\mathcal{D}_n$, T is isolated if and only if $T = \bigcup_{\xi \in E(T)} \mathcal{D}_n(\xi)$.*

Proof. (*i*) It is enough to show that for any $\xi \in E(\mathcal{D}_n)$, $\mathcal{D}_n(\xi)$ is the unique isolated subsemigroup of $\mathcal{D}_n$. For any $\xi \in E(\mathcal{D}_n)$ and $\alpha \in \mathcal{D}_n$, suppose $\alpha^m \in \mathcal{D}_n(\xi)$ for some $m \in \mathbb{N}$. Then it follows from Proposition 2.3 and Theorem 2.2 that $\text{fix}\,(\alpha) = \text{fix}\,(\alpha^m) = \text{fix}\,(\xi)$. Moreover, from the definition of $\mathcal{D}_n(\xi)$, there exists $k \in \mathbb{N}$ such that $(\alpha^m)^k = \xi$, and so we have the equality $\alpha\xi = \xi\alpha$. For any $b \in X_n$, since $b\xi \in \text{im}\,(\xi) = \text{fix}\,(\xi) = \text{fix}\,(\alpha)$, it follows that $b(\alpha\xi) = (b\xi)\alpha = b\xi$, and so $\alpha\xi = \xi$. From Theorem 2.2, we conclude that $\alpha \in \mathcal{D}_n(\xi)$, and so it follows from Lemma 2.1 (*ii*) that $\mathcal{D}_n(\xi)$ is a unique isolated nilpotent subsemigroup of $\mathcal{D}_n$ with zero element ξ.

(*ii*) ($\Rightarrow$) It is clear from Lemma 2.1 (*ii*).

($\Leftarrow$) Suppose $T = \bigcup_{\xi \in E(T)} \mathcal{D}_n(\xi)$. For any $\alpha \in \mathcal{D}_n$, if $\alpha^m \in T$ for some $m \in \mathbb{N}$, then there exists an idempotent ξ of T such that $\alpha^m \in \mathcal{D}_n(\xi)$. Thus, it follows from the proof of the first item that $\alpha \in \mathcal{D}_n(\xi)$, and so $\alpha \in T$, as required. $\square$

From Theorem 2.4, we have the following immediate results.

Corollary 2.5. *Let T be an isolated subsemigroup of $\mathcal{D}_n$.*

(*i*) *For each idempotent ξ of T, $\mathcal{D}_n(\xi)$ is a subsemigroup of T.*

(*ii*) *If ξ_1 and ξ_2 are two idempotents of T, then there exists an idempotent ξ_3 of T such that $\xi_1\xi_2 \in \mathcal{D}_n(\xi_3)$.*

Notice that for $\xi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 1 \end{pmatrix}$ and $\xi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 2 \end{pmatrix} \in E(\mathcal{D}_4)$, it is clear that $\xi_1\xi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 1 \end{pmatrix} \notin \mathcal{D}_4(\xi_1) \cup \mathcal{D}_4(\xi_2)$, and so the union of isolated subsemigroups may not be even a subsemigroup. For any subsemigroup T of $\mathcal{D}_n$, let

$$\mathrm{sl}\,(T) = \{\mathrm{fix}\,(\xi) : \xi \in E(T)\} \quad \text{and} \quad \mathcal{SL}_n(1) = \{Y \subseteq X_n : 1 \in Y\}.$$

Then it is clear that $\mathrm{sl}\,(T) \subseteq \mathcal{SL}_n(1)$ and $\mathrm{sl}\,(\mathcal{D}_n) = \mathcal{SL}_n(1)$. Moreover, $\mathcal{SL}_n(1)$ is a subsemigroup of $\mathcal{SL}_n$ which is isomorphic to $\mathcal{SL}_{n-1}$. More generally, since $\mathcal{SL}_n(1)$ is a semilattice, from Proposition 2.3 and Corollary 2.5, we have the following immediate result.

Lemma 2.6. *If T is an isolated subsemigroup of $\mathcal{D}_n$, then $\mathrm{sl}\,(T)$ is an isolated subsemigroup of $\mathcal{SL}_n(1)$.*

Although, for any partition $\{A_1, \ldots, A_r\}$ of X_n, there exists a unique idempotent in $\mathcal{D}_n$ whose kernel classes are $A_1, \ldots, A_r$, namely $\xi = \begin{pmatrix} A_1 & \cdots & A_r \\ a_1 & \cdots & a_r \end{pmatrix}$ where $a_i = \min(A_i)$ for each $1 \leq i \leq r$, for any $U \subseteq X_n$ containing 1, there might be more than one idempotent in $\mathcal{D}_n$ whose fix set is U. For example, there exist exactly four idempotents in $\mathcal{D}_5$ whose fix sets are equal to $\{1, 3\}$ and there exist exactly nine idempotents in $\mathcal{D}_5$ whose fix sets are equal to $\{1, 2, 3\}$. Moreover, let $\xi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 3 & 1 & 3 \end{pmatrix}$ and $\xi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 3 & 3 \end{pmatrix}$, whose fix sets are $\{1, 3\}$ and $\{1, 2, 3\}$, respectively. For $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 3 & 2 & 3 \end{pmatrix} \in \mathcal{D}_5(\xi_1)$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 3 & 4 \end{pmatrix} \in \mathcal{D}_5(\xi_2)$, it is clear that $\alpha\beta = \alpha \in \mathcal{D}_5(\xi_1) \cup \mathcal{D}_5(\xi_2)$, but $\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 3 & 3 & 2 \end{pmatrix} \notin \mathcal{D}_5(\xi_1) \cup \mathcal{D}_5(\xi_2)$. Thus, we observe that if $T = \mathcal{D}_5(\xi_1) \cup \mathcal{D}_5(\xi_2)$, then T is not even a subsemigroup of $\mathcal{D}_5$ although $\mathrm{sl}\,(T) = \{\{1, 3\}, \{1, 2, 3\}\}$ is an isolated subsemigroup of $\mathcal{SL}_5(1)$. That is, we do not have a similar result as in [5, Theorem 2.7]. However, if $\xi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 3 & 3 & 1 \end{pmatrix}$ and $\xi_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 3 & 3 & 3 \end{pmatrix}$ in $E(\mathcal{D}_5)$, then we observe that $\beta\alpha \in \mathcal{D}_5(\xi_3)$ and $\xi_1 = \xi_1\xi_2 \neq \xi_2\xi_1 = \xi_4$. After these observations, we give a description of some isolated proper subsemigroups of $\mathcal{D}_n$.

Theorem 2.7. *Let T be a proper subset of $\mathcal{D}_n$ such that $E(T)$ is not empty. If $T = \bigcup_{\xi \in E(T)} \mathcal{D}_n(\xi)$, and if for all $\xi_1, \xi_2 \in E(T)$, there exists $\xi_3 \in E(T)$ such that $\xi_1\xi_2, \xi_2\xi_1 \in \mathcal{D}_n(\xi_3)$, then T is an isolated subsemigroup of $\mathcal{D}_n$.*

Proof. It is enough to only show that T is a subsemigroup. For any $\alpha, \beta \in T$, there exist three idempotents $\xi_1, \xi_2, \xi_3 \in E(T)$ such that $\alpha \in \mathcal{D}_n(\xi_1)$, $\beta \in \mathcal{D}_n(\xi_2)$ and $\xi_1\xi_2, \xi_2\xi_1 \in \mathcal{D}_n(\xi_3)$. Similarly, it follows from Proposition 2.3 and Theorem 2.2 that

$$\mathrm{fix}\,(\alpha\beta) = \mathrm{fix}\,(\alpha) \cap \mathrm{fix}\,(\beta) = \mathrm{fix}\,(\xi_1) \cap \mathrm{fix}\,(\xi_2) = \mathrm{fix}\,(\xi_1\xi_2) = \mathrm{fix}\,(\xi_3).$$

Moreover, from Theorem 2.2, since $\alpha\xi_1 = \xi_1$, $\beta\xi_2 = \xi_2$, $(\xi_1\xi_2)\xi_3 = \xi_3$ and $(\xi_2\xi_1)\xi_3 = \xi_3$, it follows that

$$\begin{aligned}(\alpha\beta)\xi_3 &= \alpha\beta((\xi_2\xi_1)\xi_3) = \alpha(\beta\xi_2)\xi_1\xi_3 = \alpha((\xi_2\xi_1)\xi_3) = \alpha\xi_3 = \alpha((\xi_1\xi_2)\xi_3) \\ &= (\alpha\xi_1)\xi_2\xi_3 = (\xi_1\xi_2)\xi_3 = \xi_3,\end{aligned}$$

and so $\alpha\beta \in \mathcal{D}_n(\xi_3) \subseteq T$. Therefore, from Theorem 2.4 (*ii*), we conclude that T is an isolated subsemigroup of $\mathcal{D}_n$. □

3. Convex subsemigroups of $\mathcal{D}_n$

First we concentrate on completely isolated subsemigroups. It is clear from the concerning definitions that a proper subsemigroup T of a semigroup S is completely isolated if and only if its complement $\overline{T} = S \setminus T$ is also a subsemigroup of S. Thus, if T is completely isolated, then $\overline{T}$ is also completely isolated.

If we consider the idempotents ξ_1, ξ_2 and ξ_4 which are defined just before Theorem 2.7, then we observe that $\xi_2\xi_1 \in \mathcal{D}_5(\xi_4)$ but neither ξ_1 nor ξ_2 in $\mathcal{D}_5(\xi_4)$. Thus, $\mathcal{D}_n(\xi)$ is not a completely isolated, and so (right/left) convex subsemigroup of $\mathcal{D}_n$ in general. Since 1_n is indecomposable in $\mathcal{D}_n$, if $1_n \in T \neq \{1_n\}$ is a subsemigroup of $\mathcal{D}_n$, then it is clear that T is a completely isolated subsemigroup of $\mathcal{D}_n$ if and only if $T \setminus \{1_n\}$ is a completely isolated subsemigroup of $\mathcal{D}_n$. Thus we conclude that both $\mathcal{D}_n \setminus \{1_n\}$ and $\{1_n\}$ are completely isolated (in fact, they are convex) subsemigroups of $\mathcal{D}_n$. For any $U \subseteq X_n$ containing 1, we define two sets as follows:

$$\mathcal{C}_n(U) = \{\alpha \in \mathcal{C}_n : \operatorname{fix}(\alpha) = U\} \quad \text{and} \quad \mathcal{D}_n(U) = \{\alpha \in \mathcal{D}_n : \operatorname{fix}(\alpha) = U\}.$$

For any $\xi \in E(\mathcal{D}_n)$ such that $\operatorname{fix}(\xi) = U$, if $\xi \in E(\mathcal{C}_n)$, then we have $\mathcal{C}_n(\xi) = \mathcal{C}_n(U)$, however we have just observed that $\mathcal{D}_n(\xi) \neq \mathcal{D}_n(U)$ in general. It is clear that $\mathcal{D}_n(\{1\}) = \mathcal{D}_n(0_n) = N(\mathcal{D}_n)$ and $\mathcal{D}_n(X_n) = \{1_n\}$ are both isolated subsemigroups of $\mathcal{D}_n$. Moreover, $\mathcal{D}_n(X_n) = \{1_n\}$ is completely isolated, but $\mathcal{D}_n(\{1\})$ is not completely isolated. In fact, we have the following more general result.

Lemma 3.1. *For any $U \subseteq X_n$ containing 1, $\mathcal{D}_n(U)$ is an isolated subsemigroup of $\mathcal{D}_n$. Moreover, $\mathcal{D}_n(U)$ is convex, and so completely isolated if and only if $|U| \geq n-1$.*

Proof. For any $U \subseteq X_n$ containing 1, it is clear from Proposition 2.3 that $\mathcal{D}_n(U)$ is an isolated subsemigroup of $\mathcal{D}_n$.

Suppose $|U| \leq n-2$. For two distinct elements u and v in $X_n \setminus U$, if we consider two (idempotent) transformations $\alpha, \beta : X_n \to X_n$ defined by

$$x\alpha = \begin{cases} x & \text{if } x \in U \cup \{u\} \\ 1 & \text{if } x \in X_n \setminus (U \cup \{u\}) \end{cases} \quad \text{and} \quad x\beta = \begin{cases} x & \text{if } x \in U \cup \{v\} \\ 1 & \text{if } x \in X_n \setminus (U \cup \{v\}) \end{cases}$$

for all $x \in X_n$, then we immediately see that $\alpha, \beta \in \mathcal{D}_n$. Moreover, since $\alpha\beta : X_n \to X_n$ can be defined by $x\alpha\beta = \begin{cases} x & \text{if } x \in U \\ 1 & \text{if } x \in X_n \setminus U \end{cases}$ for all $x \in X_n$, it follows that $\alpha\beta \in \mathcal{D}_n(U)$. However, neither α nor β is an element of $\mathcal{D}_n(U)$.

On the other hand, since $\mathcal{D}_n(X_n) = \{1_n\}$ is convex, we need to consider only the case $|U| = n-1$. For any two $\alpha, \beta \in \mathcal{D}_n \setminus \{1_n\}$, if $\alpha\beta \in \mathcal{D}_n(U)$, then it is follows from Proposition 2.3 that $U \subseteq \operatorname{fix}(\alpha), \operatorname{fix}(\beta)$, and so since $|\operatorname{fix}(\alpha)|, |\operatorname{fix}(\beta)| \leq n-1$, it follows that $\operatorname{fix}(\alpha) = U = \operatorname{fix}(\beta)$. Therefore, $\alpha, \beta \in \mathcal{D}_n(U)$, and so $\mathcal{D}_n(U)$ is convex, as required. □

For a proper subset $U = \{1 = u_1 < u_2 < \cdots < u_r\}$ of X_n with $r \geq 2$, let $E(\mathcal{D}_n(U)) = \{\xi_1, \ldots, \xi_t\}$. For each $1 \leq i \leq t$ and each $1 \leq j \leq r$, if we let $|u_j\xi_i^{-1}| = m_{ij}$, then it follows from [14, Theorem 3.6] that

$$|\mathcal{D}_n(U)| = \sum_{i=1}^{t} \left(\prod_{j=1}^{r} (m_{ij} - 1)! \right)$$

since $\mathcal{D}_n(U)$ is the union of its disjoint subsemigroups $\mathcal{D}_n(\xi_i)$ for $1 \leq i \leq t$. Furthermore, let $m_k = u_{k+1} - u_k - 1$ for each $1 \leq k \leq r$ where $u_{r+1} = n+1$. If $\xi \in E(\mathcal{D}_n(U))$, then it is clear that for each $x \in \operatorname{shift}\xi$, there exists $1 \leq k \leq r$ such that $u_k < x < u_{k+1}$, and so $x\xi \in \{u_1, \ldots, u_k\}$. Thus, for each $1 \leq i \leq r$ and each $u_i < x < u_{i+1}$ (if exists), there are k

choices for the value of $x\xi$. Since there are m_k many x such that $u_k < x < u_{k+1}$, we have

$$|E(\mathcal{D}_n(U))| = \prod_{k=1}^{r} k^{m_k}.$$

It is also clear that $\xi\zeta = \xi$ for all $\xi, \zeta \in E(\mathcal{D}_n(U))$, i.e. $E(\mathcal{D}_n(U))$ is a left zero semigroup.

For any $U \subseteq X_n$ containing 1, we also define the following sets:

$$\mathcal{D}_n[U] = \{\alpha \in \mathcal{D}_n : U \subseteq \mathrm{fix}\,(\alpha)\} \quad \text{and} \quad \overline{\mathcal{D}}_n[U] = \{\alpha \in \mathcal{D}_n : U \setminus \mathrm{fix}\,(\alpha) \neq \emptyset\}.$$

Similarly, $\mathcal{D}_n[\{1\}] = \mathcal{D}_n$ and $\overline{\mathcal{D}}_n[\{1\}] = \emptyset$ since $1\alpha = 1$ for all $\alpha \in \mathcal{D}_n$. Moreover, $\mathcal{D}_n[X_n] = \{1_n\}$ and $\overline{\mathcal{D}}_n[X_n] = \mathcal{D}_n \setminus \{1_n\}$ are both completely isolated subsemigroups of $\mathcal{D}_n$. Moreover, from Proposition 2.3 and Lemma 2.1, we have the following immediate result.

Lemma 3.2. *(i) Let U be any subset of X_n such that $\{1\} \subsetneq U$. Then $\mathcal{D}_n[U]$ and $\overline{\mathcal{D}}_n[U]$ are completely isolated subsemigroups of $\mathcal{D}_n$. Moreover, $\mathcal{D}_n[U]$ is convex, but $\overline{\mathcal{D}}_n[U]$ is not necessarily convex.*

(ii) If T is an isolated subsemigroup of $\mathcal{D}_n$, then T is a subsemigroup of $\mathcal{D}_n[U]$ where $U = \bigcap_{\xi \in E(T)} \mathrm{fix}\,(\xi)$.

For a completely isolated subsemigroup T of $\mathcal{D}_n$, T is not necessarily equal to $\mathcal{D}_n[U]$ where $U = \bigcap_{\xi \in E(T)} \mathrm{fix}\,(\xi)$. For example, let $T = \{\xi, 1_3\}$ where $\xi = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$. Then it is easy to check that both T and $\mathcal{D}_3 \setminus T = \{0_3, \alpha, \beta, \gamma\}$, where $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}$ and $\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}$, are subsemigroups of $\mathcal{D}_3$, and so T is completely isolated. However, $\mathcal{D}_3[\{1,2\}] \neq T$ since $\gamma \in \mathcal{D}_3[\{1,2\}] \setminus T$. Moreover, since $\xi\alpha = 0_3$, $\alpha\xi = \alpha$, $\xi\beta = 0_3 = \beta\xi$, $\xi\gamma = \xi$ and $\gamma\xi = \gamma$, we conclude that T is left convex but not right convex. Similarly, we check that $\mathcal{D}_3[\{1,2\}] = \{\xi, \gamma, 1_3\}$ is convex.

Lemma 3.3. *If T is a left (right) convex subsemigroup of $\mathcal{D}_n$, then $\mathrm{sl}\,(T)$ is a convex subsemigroup of $\mathcal{SL}_n(1)$.*

Proof. If T is left (right) convex, then it is isolated, and so from Lemma 2.6, $\mathrm{sl}\,(T)$ is a subsemigroup of $\mathcal{SL}_n(1)$. For any $U, V \in \mathcal{SL}_n(1)$, suppose that $U \cap V \in \mathrm{sl}\,(T)$. Then there exists $\xi \in E(T)$ such that $\mathrm{fix}\,(\xi) = U \cap V$. If we define two maps $\alpha, \beta : X_n \to X_n$ by

$$x\alpha = \begin{cases} x & \text{if } x \in U \\ x\xi & \text{if } x \in X_n \setminus U \end{cases} \qquad \text{and} \qquad x\beta = \begin{cases} x & \text{if } x \in V \\ x\xi & \text{if } x \in U \setminus V \\ 1 & \text{if } x \in X_n \setminus (U \cup V) \end{cases}$$

for all $x \in X_n$, then it is easy to see that $\alpha \in \mathcal{D}_n(U)$ and $\beta \in \mathcal{D}_n(V)$. Moreover, for any $x \in X_n$, since $x\xi \in \mathrm{im}\,(\xi) = \mathrm{fix}\,(\xi) \subseteq V$, it is routine matter to check that $x(\alpha\beta) = x\xi$, and so $\alpha\beta = \xi \in T$. Since T is a left (right) convex subsemigroup, $\alpha \in T$ ($\beta \in T$), and so from Theorems 2.2 and 2.4, $\mathrm{fix}\,(\alpha) = U \in \mathrm{sl}\,(T)$ ($\mathrm{fix}\,(\beta) = V \in \mathrm{sl}\,(T)$). Therefore, since $\mathrm{sl}\,(T)$ is commutative, it follows that $\mathrm{sl}\,(T)$ is convex. □

In the last example above, $\mathrm{sl}\,(T) = \{\{1,2\}, \{1,2,3\}\}$ is a convex subsemigroup of $\mathcal{SL}_3(1)$, but T is not right convex. Thus, the converse of Lemma 3.3 does not hold in general.

Theorem 3.4. *For a submonoid T of $\mathcal{D}_n$, let $U = \bigcap_{\xi \in E(T)} \mathrm{fix}\,(\xi)$ and suppose that T contains $\mathcal{D}_n(U)$. Then T is convex if and only if $T = \mathcal{D}_n[U]$.*

Proof. From Lemma 3.2 (i), it is enough to show that if T is convex, then $T = \mathcal{D}_n[U]$. For any convex submonoid T of $\mathcal{D}_n$, let $E(T) = \{\xi_1, \dots, \xi_t\}$ and $U = \bigcap_{i=1}^{t} \mathrm{fix}\,(\xi_i)$. Since $\xi_1 \cdots \xi_t \in T$ and T is finite, by [4, Proposition 1.2.3], there exists $\xi_U \in E(T)$ such that $(\xi_1 \cdots \xi_t)^m = \xi_U$ for some $m \in \mathbb{N}$, and so $\xi_1 \cdots \xi_t \in \mathcal{D}_n(\xi_U)$. Then it follows from Proposition 2.3 that $\mathrm{fix}\,(\xi_U) = \mathrm{fix}\,((\xi_1 \cdots \xi_t)^m) = \mathrm{fix}\,(\xi_1 \cdots \xi_t) = \bigcap_{i=1}^{t} \mathrm{fix}\,(\xi_i) = U$. Since T is also (completely) isolated and $\xi_U \in E(T)$, it follows from Corollary 2.5 (i) that $\mathcal{D}_n(\xi_U)$ is a subsemigroup of T. Since $\xi_1 \cdots \xi_t \in \mathcal{D}_n(\xi_U)$, it follows from Theorem 2.2 that $(\xi_1 \cdots \xi_t)\xi_U = \xi_U$.

Now we show the equality. For any $\alpha \in \mathcal{D}_n[U]$, there exist $\xi \in E(\mathcal{D}_n)$ and a positive integer m such that $(\alpha\xi_U)^m = \xi$, and so $\alpha\xi_U \in \mathcal{D}_n(\xi)$. Since $U \subseteq \mathrm{fix}\,(\alpha)$, it follows from Proposition 2.3 that

$$\mathrm{fix}\,(\xi) = \mathrm{fix}\,((\alpha\xi_U)^m) = \mathrm{fix}\,(\alpha\xi_U) = \mathrm{fix}\,(\alpha) \cap \mathrm{fix}\,(\xi_U) = \mathrm{fix}\,(\xi_U) = U.$$

If $E(\mathcal{D}_n(U)) = \{\zeta_1, \dots, \zeta_r\}$, then it follows from Lemma 2.1 (ii) that $\mathcal{D}_n(U)$ is the union of its disjoint subsemigroups $\mathcal{D}_n(\zeta_j)$ for $1 \le j \le r$, and so $\alpha\xi_U \in \mathcal{D}_n(\xi) \subseteq \mathcal{D}_n(U) \subseteq T$. Since $\alpha\xi_U \in T$, it follows from the convexity of T that $\alpha \in T$, and so $\mathcal{D}_n[U] \subseteq T$.

Conversely, for any $\alpha \in T$, since $\alpha \in \mathcal{D}_n(\xi_i)$ for some $1 \le i \le t$, it similarly follows that $U \subseteq \mathrm{fix}\,(\xi_i) = \mathrm{fix}\,(\alpha)$, and so $\alpha \in \mathcal{D}_n[U]$. Therefore, we obtain the required equality $T = \mathcal{D}_n[U]$. □

4. Generating sets and ranks of $\mathcal{D}_n[U]$

For every $1 \le j < i \le n$, let

$$\xi_{ij} = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & n \\ 1 & \cdots & i-1 & j & i+1 & \cdots & n \end{pmatrix}.$$

It is shown in [11, Theorem 3.4] that $\{\xi_{ij} : 1 \le j < i \le n\} \cup \{1_n\}$ is a minimal generating set of $\mathcal{D}_n$. Since all elements of this generating set are idempotents, it follows that $\mathrm{rank}\,(\mathcal{D}_n) = \mathrm{idrank}\,(\mathcal{D}_n) = \frac{n(n-1)}{2} + 1$.

Although the following result is likely to be known, we state and prove it since our proof is very short.

Proposition 4.1. *For every $1 \le j < i \le n$, ξ_{ij} is indecomposable in $\mathcal{D}_n$.*

Proof. Assume that $\xi_{ij} = \alpha\beta$ for some $\alpha, \beta \in \mathcal{D}_n$, and also $\alpha \ne 1_n$. It follows from Proposition 2.3 that $\mathrm{fix}\,(\xi_{ij}) = X_n \setminus \{i\}$ is a subset of both $\mathrm{fix}\,(\alpha)$ and $\mathrm{fix}\,(\beta)$, and so $\mathrm{fix}\,(\alpha) = \mathrm{fix}\,(\xi_{ij})$ since $|\mathrm{fix}\,(\alpha)| \le n-1$. Moreover, since $i\alpha \le i-1$, it follows that $i\alpha \in \mathrm{fix}\,(\xi_{ij}) \subseteq \mathrm{fix}\,(\beta)$, and so $j = i\xi_{ij} = (i\alpha)\beta = i\alpha$, i.e. $i\alpha = j$. Thus, $\xi_{ij} = \alpha$. □

For a proper subset U of X_n such that $\{1\} \subsetneq U$, let

$$\mathcal{A}_U = \{\xi_{ij} : i \in X_n \setminus U \text{ and } j < i\} \cup \{1_n\}.$$

Theorem 4.2. *Let U be a proper subset of X_n such that $\{1\} \subsetneq U$. Then $\mathcal{A}_U$ is the minimum generating set of $\mathcal{D}_n[U]$, and so*

$$\mathrm{rank}\,(\mathcal{D}_n[U]) = \mathrm{idrank}\,(\mathcal{D}_n[U]) = 1 + \sum_{i=1}^{m}(k_i - 1)$$

where $X_n \setminus U = \{k_1 < \cdots < k_m\}$.

Proof. First of all, if $|U| = n-1$, or equivalently, $X_n \setminus U = \{i\}$ for some $2 \le i \le n$, then it is clear that

$$\mathcal{D}_n[U] = \{\xi_{ij} : 1 \le j \le i-1\} \cup \{1_n\} = \langle\{\xi_{ij} : 1 \le j \le i-1\} \cup \{1_n\}\rangle.$$

Suppose $|U| \leq n-2$. For any $\alpha \in \mathcal{D}_n[U] \setminus \{1_n\}$, let shift $(\alpha) = \{i_1 < \cdots < i_r\}$ and $i_k\alpha = j_k$ for each $1 \leq k \leq r$. Thus, since $i_1, \ldots, i_r \in X_n \setminus U$ and $j_k < i_k$ for each $1 \leq k \leq r$, it follows that $\xi_{i_1 j_1}, \ldots, \xi_{i_r j_r} \in \mathcal{A}_U$. Then it is clear that $\alpha = \xi_{i_1 j_1} \cdots \xi_{i_r j_r}$, and so $\mathcal{A}_U$ is a generating set of $\mathcal{D}_n[U]$. Since each element of $\mathcal{A}_U$ is indecomposable (idempotent) in $\mathcal{D}_n$, and so in $\mathcal{D}_n[U]$, it follows that $\mathcal{A}_U$ is a unique minimal (idempotent) generating set of $\mathcal{D}_n[U]$.

Suppose that $X_n \setminus U = \{k_1 < \cdots < k_m\}$ and notice that $k_1 \geq 2$. Since $\mathcal{A}_U$ is the union of the disjoint sets $\{1_n\}$ and $\{\xi_{k_i j} : 1 \leq j \leq k_i - 1\}$ for all $1 \leq i \leq m$, we obtain

$$\operatorname{rank}(\mathcal{D}_n[U]) = \operatorname{idrank}(\mathcal{D}_n[U]) = 1 + \sum_{i=1}^{m}(k_i - 1),$$

as required. □

For example, if $U = \{1,3\} \subseteq X_4$, then $\mathcal{A}_U = \{\xi_{21}, \xi_{41}, \xi_{42}, \xi_{43}, 1_4\}$, and for $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 2 \end{pmatrix} \in \mathcal{D}_4[\{1,3\}]$, we have $\alpha = \xi_{21}\xi_{42}$.

It is shown in [12, Theorem 5.2.5] that $\operatorname{rank}(\mathcal{D}_n(1)) = \operatorname{rank}(N(\mathcal{D}_n)) = (n-2)!(n-2)$. From [14, Theorem 3.7], we also know that $\operatorname{rank}(\mathcal{D}_n(\xi)) = |\mathcal{D}_n(\xi)| - |\mathcal{D}_n(\xi)^2|$ since $\mathcal{D}_n(\xi)$ is a finite nilpotent semigroup with the zero element ξ. Moreover, we have the following observations:

- If U is a subset of $n-1$ elements containing 1, i.e. $U = X_n \setminus \{k\}$ for some $2 \leq k \leq n$, then it is clear that $\mathcal{D}_n(U)$ is a left zero semigroup with $k-1$ elements. In addition, since every element of a left zero semigroup is indecomposable, $\operatorname{rank}(\mathcal{D}_n(U)) = k-1$.
- If $U = \{1, m+1, m+2, \ldots, n\}$ for some $2 \leq m \leq n-1$, then there exists unique idempotent ξ such that $\operatorname{fix}(\xi) = U$. Thus, $\mathcal{D}_n(U) = \mathcal{D}_n(\xi)$ whose rank is known.
- It follows from [14, Theorem 3.7] that if $E(\mathcal{D}_n(U)) = \{\xi_1, \ldots, \xi_r\}$, then $\mathcal{A} = \bigcup_{i=1}^{r} (\mathcal{D}_n(\xi_i) \setminus \mathcal{D}_n(\xi_i)^2)$ is a generating set of $\mathcal{D}_n(U)$. It is easy to find some examples which shows that $\mathcal{A}$ is not minimal in general.

Despite all these observations and experiences, we could not find the rank of $\mathcal{D}_n(U)$, and therefore, we have to leave the rank of $\mathcal{D}_n(U)$ as an open problem.

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