CONSTRUCTIVE MATHEMATICAL ANALYSIS
8 (2025), No. 2, pp. 81-92
http://dergipark.org.tr/en/pub/cma
ISSN 2651 - 2939

Research Article

Duffin–Schaeffer inequality revisited

GENO NIKOLOV*

ABSTRACT. The classical Markov inequality asserts that the n-th Chebyshev polynomial $T_n(x) = \cos n \arccos x$, $x \in [-1,1]$, has the largest $C[-1,1]$-norm of its derivatives within the set of algebraic polynomials of degree at most n whose absolute value in $[-1,1]$ does not exceed one. In 1941 R.J. Duffin and A.C. Schaeffer found a remarkable refinement of Markov inequality, showing that this extremal property of T_n persists in the wider class of polynomials whose modulus is bounded by one at the extreme points of T_n in $[-1,1]$. Their result gives rise to the definition of DS-type inequalities, which are comparison-type theorems of the following nature: inequalities between the absolute values of two polynomials of degree not exceeding n on a given set of $n+1$ points in $[-1,1]$ induce inequalities between the $C[-1,1]$-norms of their derivatives. Here we apply the approach from a 1992 paper of A. Shadrin to prove some DS-type inequalities where Jacobi polynomials are extremal. In particular, we obtain an extension of the result of Duffin and Schaeffer.

Keywords: Markov inequality, Duffin–Schaeffer inequality, Chebyshev polynomials, Jacobi polynomials, interlacing of zeros.

2020 Mathematics Subject Classification: 41A17.

1. INTRODUCTION

Throughout this paper, π_n stands for the set of real-valued algebraic polynomials of degree not exceeding n, and $\|\cdot\|$ is the uniform norm in $[-1,1]$,

$$\|g\| := \max_{x\in[-1,1]} |g(x)|.$$

The classical inequality of the brothers Markov reads as follows:

Theorem 1.1. *If $f \in \pi_n$ satisfies*

$$\|f\| \le 1, \tag{1.1}$$

then

$$\|f^{(k)}\| \le \|T_n^{(k)}\|, \quad k = 1, \dots, n, \tag{1.2}$$

and the equality in (1.2) occurs if and only if $f = \pm T_n$.

Here and henceforth, T_n is the n-th Chebyshev polynomial of the first kind, defined by

$$T_n(x) = \cos(n \arccos x), \quad x \in [-1,1]\,.$$

The case $k = 1$ is due to Andrei Markov [7], and his brother Vladimir Markov [8] proved the general case, $1 \le k \le n$. For the intriguing history of Markov inequality and some of its proofs the reader is referred to the survey paper [21] .

Received: 03.02.2025; Accepted: 16.04.2025; Published Online: 05.06.2025
*Corresponding author: Geno Nikolov; geno@fmi.uni-sofia.bg
DOI: 10.33205/cma.1632536

In 1941 Duffin and Schaeffer [5] found the following remarkable extension of Theorem 1.1 (for a proof, see also [18, Theorem 2.24] or [19, Section 5.6]):

Theorem 1.2 ([5]). *Inequality (1.2) remains true if assumption (1.1) is replaced with*

$$\Big|f\Big(\cos\frac{\nu\pi}{n}\Big)\Big|\le 1,\quad \nu=0,\dots,n. \tag{1.3}$$

Theorem 1.2 may be viewed as a comparison type result: the inequality $|f|\le|T_n|$ at the $n+1$ points in $[-1,1]$ where $|T_n|=1$ implies inequalities between the uniform norms of the derivatives of f and T_n. This observation motivated the author to formulate in [9] the following:

Definition 1.1. *Let Q be a polynomial of degree n, and $\Delta=\{t_\nu\}_{\nu=0}^n$, where $1\ge t_0>\dots>t_n\ge-1$. The pair $\{Q,\Delta\}$ is said to admit Duffin–Schaeffer–type inequality (in short, DS–inequality), if for any $f\in\pi_n$, the assumption*

$$|f|\le|Q|\quad \text{at the points from }\ \Delta$$

implies

$$\|f^{(k)}\|\le\|Q^{(k)}\|,\quad k=1,\dots,n.$$

In this definition Q (called henceforth as *majorant*) is mutually assumed to be an oscillating polynomial in $[-1,1]$ (i.e., having n distinct zeros in $(-1,1)$), however, Δ is not necessarily the set of its critical points.

The shortest ever given proof of Markov's inequality, which moreover captures the refinement of Duffin and Schaeffer, is due to Alexei Shadrin [20]. Its main ingredient is the following:

Theorem 1.3 ([20]). *Let $Q\in\pi_n$ have n distinct zeros, all located in $(-1,1)$. If $f\in\pi_n$ satisfies*

$$|f|\le|Q|\ \text{ at the zeros of }\ (x^2-1)Q'(x),$$

then for each $k\in\{1,\dots,n\}$ and for every $x\in[-1,1]$ there holds

$$|f^{(k)}(x)|\le\max\Big\{|Q^{(k)}(x)|,\Big|\frac{x^2-1}{k}Q^{(k+1)}(x)+xQ^{(k)}(x)\Big|\Big\}.$$

Theorem 1.3 was applied in [3] for the proof of DS-inequalities where Q is an ultraspherical polynomial $P_n^{(\lambda)}$, $\lambda\ge0$ and Δ is the set of its extreme points in $[-1,1]$. As a matter of fact, Theorem 1.3 implies DS-inequality whenever Q is oscillating polynomial with positive expansion in Chebyshev polynomials of the first kind and Δ is the set of its extreme points. Using Shadrin's idea to the proof of Theorem 1.3, we established various DS-type inequalities in [9], where, typically, Q is an ultraspherical polynomial and Δ is formed by the zeros of another ultraspherical polynomial.

In the present paper, we apply the approach from [20] to obtain DS-inequalities, where some Jacobi polynomials are the extremisers. As a particular case, we prove the following extension of the inequality of Duffin and Schaeffer, given by Theorem 1.2:

Theorem 1.4. *Let $f\in\pi_n$ satisfy $|f(1)|\le1+2nc$ for some $c\in[0,1]$ and*

$$\Big|f\Big(\cos\frac{\nu\pi}{n}\Big)\Big|\le1,\qquad \nu=1,\dots,n\,.$$

Then

$$\|f^{(k)}\|\le\|Q_n^{(k)}\|\,,\qquad k=1,\dots,n, \tag{1.4}$$

where $Q_n(x)=(1-c)\,T_n(x)+c\,W_n(x)$, with T_n and W_n being the n-th Chebyshev polynomials of the first and the fourth kind,

$$T_n(x)=\cos(n\theta),\qquad W_n(x)=\frac{\sin\big(n+\frac12\big)\theta}{\sin\big(\frac12\theta\big)},\qquad x=\cos\theta\,.$$

The equality in (1.4) *is attained only for* $f = \pm Q_n$.

The rest of the paper is organized as follows. In Section 2 we prove Theorem 2.5, which provides pointwise estimates for the derivatives of a polynomial $f \in \pi_n$ whose modulus is bounded at a set of $n+1$ distinct points in $[-1,1]$. In Section 3 we apply Theorem 2.5 to obtain some DS-inequalities where the majorants are Jacobi polynomials, Theorem 1.4 being a particular case of them. In Section 4 we discuss applications of DS-inequalities and the interlink between DS-inequalities and Markov-type inequalities for polynomials with a curved majorant.

2. Pointwise estimates for derivatives of a polynomial

If p and q are algebraic polynomials with only real and simple zeros, we say that the zeros of p and q *interlace*, if one can trace all the zeros of both polynomials, switching alternatively from a zero of p to zero of q and vice versa and moving only in one direction. If, in addition, no zero of p coincides with a zero of q, then the zeros of p and q are said to *interlace strictly*.

Clearly, interlacing is only possible if p and q are polynomials of the same degree or of degrees which differ by one. In the latter case, if p is of degree $n+1$, q is of degree n and the zeros of p and q interlace strictly, we say shortly that *the zeros of q separate the zeros of p*. The following lemma, due to V. Markov [8], asserts that the interlacing property is inherited by the zeros of the derivatives:

Lemma 2.1. *If the zeros of polynomials p and q interlace, then the zeros of p' and q' interlace strictly.*

Proofs of Lemma 2.1 can be found, e.g., in [11, Lemma 4], [18, Lemma 2.7.1] and [20]. For the sake of brevity, we write in this section

$$p \prec q$$

to say that p and q are polynomials of the same degree with interlacing zeros, with relation "$\le$" between the corresponding zeros of p and q. The notation

$$p \prec q \prec p$$

means that the zeros of p and q interlace and p is of higher degree than q.

The following theorem provides pointwise bounds for derivatives of polynomials $f \in \pi_n$ satisfying $|f| \le |Q_n|$ on a set of $n+1$ points related in a specific way to the majorant Q_n.

Theorem 2.5. *Let Q_n be a polynomial of degree n with only real and distinct zeros, all located in $(-1,1)$, and let ω be a polynomial of degree $n-1$ whose zeros separate the zeros of Q_n. Assume that for some $k \in \{1,\dots,n\}$ and constants $a \ge 1$ and $b \le -1$,*

$$Q_n^{(k)}(x) = \big[(a-b-2)x + a + b\big]\omega^{(k)}(x) + k(a-b)\omega^{(k-1)}(x)\,. \tag{2.5}$$

If $f \in \pi_n$ satisfies

$$|f| \le |Q_n| \quad \textit{at the zeros of} \;\; (x^2-1)\omega(x), \tag{2.6}$$

then

$$|f^{(k)}(x)| \le \max\Big\{\,\Big|Q_n^{(k)}(x)\Big|\,, |Z_{n,k}(x)|\,\Big\} \quad \textit{for all} \;\; x \in \mathbb{R},$$

where

$$Z_{n,k}(x) = \big[2x^2 - (a+b)x + b - a\big]\omega^{(k)}(x) + k(2x-a-b)\omega^{(k-1)}(x)\,. \tag{2.7}$$

Proof. We consider first the case $1 \le k \le n-1$. With the notation introduced above, the assumption for the zeros of Q_n and ω can be written shortly as

$$Q_n \prec \omega \prec Q_n\,. \tag{2.8}$$

If

$$\omega_0(x) = (x+1)\omega(x)\,, \quad \omega_n(x) = (x-1)\omega(x),$$

then obviously

$$\omega_0 \prec Q_n \prec \omega_n$$

and, by Lemma 2.1,

$$\omega_0^{(k)} \prec Q_n^{(k)} \prec \omega_n^{(k)}\,. \tag{2.9}$$

Denote by $\{\alpha_i^k\}_{i=1}^{n-k}$ and $\{\beta_i^k\}_{i=1}^{n-k}$ the zeros of $\omega_0^{(k)}$ and $\omega_n^{(k)}$, respectively, labeled in increasing order, then it follows from (2.9) that each interval (α_i^k, β_i^k), $i = 1, \dots, n-k$, contains exactly one zero of $Q_n^{(k)}$, hence the zeros of $Q_n^{(k)}$ belong to the set

$$J_{n,k} = J_{n,k}(\omega) = \bigcup_{i=1}^{n-k} (\alpha_i^k, \beta_i^k)$$

and each interval (α_i^k, β_i^k) contains one zero of $Q_n^{(k)}$. Consequently,

$$Q_n^{(k)}(\beta_i^k)\, Q_n^{(k)}(\alpha_{i+1}^k) > 0, \quad i = 1, \dots, n-k-1 \tag{2.10}$$

(this statement is void if $k = n-1$). Denote by $I_{n,k} = I_{n,k}(\omega)$ the complementary set $\mathbb{R} \setminus J_{n,k}$,

$$I_{n,k} = I_{n,k}(\omega) = (-\infty, \alpha_1^k] \cup \bigcup_{i=1}^{n-k-1} [\beta_i^k, \alpha_{i+1}^k] \cup [\beta_{n-k}^k, \infty)\,.$$

The sets $I_{n,k}$ and $J_{n,k}$ are referred to as *Chebyshev set* and *Zolotarev set*, respectively.

Let $t_1 < \cdots < t_{n-1}$ be the zeros of ω, and $t_0 = -1$, $t_n = 1$. If $f \in \pi_n$ satisfies $|f(t_i)| \le |Q_n(t_i)|$ for $i = 0, \dots, n$, then

$$\left|f^{(k)}(x)\right| \le \left|Q_n^{(k)}(x)\right|, \qquad x \in I_{n,k}\,. \tag{2.11}$$

Indeed, if $\{\ell_i\}_{i=0}^n$ are the fundamental polynomials for interpolation at $\{t_i\}_{i=0}^n$, then

$$\ell_n \prec \ell_{n-1} \prec \cdots \prec \ell_0$$

and, by Lemma 2.1,

$$\ell_n^{(k)} \prec \ell_{n-1}^{(k)} \prec \cdots \prec \ell_0^{(k)}\,.$$

This observation, combined with the fact that the sign of the leading coefficient of $\ell_i(x)$ is $(-1)^i$, $i = 0, 1, \dots, n$, implies that if x is an interior point of $I_{n,k}$, then the signs of $\{\ell_i^{(k)}(x)\}_{i=0}^n$ alternate. In view of (2.8), so do the signs of $\{Q_n(t_i)\}_{i=0}^n$, and using (2.6) we conclude that

$$\begin{aligned} \left|f^{(k)}(x)\right| &= \left|\sum_{i=0}^n \ell_i^{(k)}(x) f(t_i)\right| \le \sum_{i=0}^n \left|\ell_i^{(k)}(x) f(t_i)\right| \\ &\le \sum_{i=0}^n \left|\ell_i^{(k)}(x) Q_n(t_i)\right| = \left|\sum_{i=0}^n \ell_i^{(k)}(x) Q_n(t_i)\right| = \left|Q_n^{(k)}(x)\right|\,. \end{aligned} \tag{2.12}$$

Obviously, (2.12) remains true also when x is a boundary point of $I_{n,k}$, and hence (2.11) is proved. It is readily seen from (2.12) that in the case when x is an interior point of $I_{n,k}$, the inequality (2.11) is strict unless $f = \pm Q_n$.

Next, we show that on the Zolotarev set $J_{n,k}$, $\left|f^{(k)}\right|$ is bounded by $|Z_{n,k}|$, i.e.,

$$\left|f^{(k)}(x)\right| \le |Z_{n,k}(x)|\,, \qquad x \in J_{n,k}. \tag{2.13}$$

Using the representation of $Q_n^{(k)}$ and $Z_{n,k}$, given in (2.5) and (2.7), we find that

$$\begin{aligned} Z_{n,k}(x) - Q_n^{(k)}(x) &= 2(x-a)\omega_0^{(k)}(x), \\ Z_{n,k}(x) + Q_n^{(k)}(x) &= 2(x-b)\omega_n^{(k)}(x). \end{aligned}$$

Hence,

$$Z_{n,k}(x) = \begin{cases} Q_n^{(k)}(x), & x \in \{\alpha_i^k\}_{i=1}^{n-k} \cup \{a\}, \\[2ex] -Q_n^{(k)}(x), & x \in \{b\} \cup \{\beta_i^k\}_{i=1}^{n-k}. \end{cases} \tag{2.14}$$

In view of (2.11), for an arbitrary constant $c \in (-1,1)$, we have

$$\left|c\, f^{(k)}(x)\right| < \left|Q_n^{(k)}(x)\right| = |Z_{n,k}(x)|\ , \quad x \in \{b\} \cup \{\alpha_i^k\}_{i=1}^{n-k} \cup \{\beta_i^k\}_{i=1}^{n-k} \cup \{a\}. \tag{2.15}$$

It follows from (2.10), (2.11), (2.14) and (2.15) that $Z_{n,k} - c\, f^{(k)}$ has a zero in each interval $(\beta_i^k, \alpha_{i+1}^k)$, $1 \le i \le n-k-1$ (again, this statement is void if $k = n-1$). Moreover, $Z_{n,k} - c\, f^{(k)}$ has a zero in each of intervals (b, α_1^k) and (β_{n-k}^k, a). To see this, we observe from (2.5) and (2.7) that the leading coefficients of $Q_n^{(k)}(x)$ and $Z_{n,k}(x)$ have the same sign. Since $Q_n^{(k)}$ has no zeros outside the interval $(\alpha_1^k, \beta_{n-k}^k)$, we find from (2.14) and (2.15) that

$$\operatorname{sign}\{Z_{n,k}(x) - c\, f^{(k)}(x)\}_{|x=\alpha_1^k} = \operatorname{sign}\{Q_n^{(k)}(\alpha_1^k)\} = -\operatorname{sign}\{Z_{n,k}(x) - c\, f^{(k)}(x)\}_{|x=b},$$

$$\operatorname{sign}\{Z_{n,k}(x) - c\, f^{(k)}(x)\}_{|x=\beta_{n-k}^k} = -\operatorname{sign}\{Q_n^{(k)}(\beta_{n-k}^k)\} = -\operatorname{sign}\{Z_{n,k}(x) - c\, f^{(k)}(x)\}_{|x=a}\,,$$

thus concluding that $Z_{n,k} - c\, f^{(k)}$ has a zero in each of the intervals (b, α_1^k) and (β_{n-k}^k, a).

Hence, all $n-k+1$ zeros of $Z_{n,k} - c\, f^{(k)}$ belong to the Chebyshev set $I_{n,k}$, and consequently $Z_{n,k} - c\, f^{(k)} \ne 0$ on $J_{n,k}$. Since $c \in (-1,1)$ is arbitrary, it follows that $\left|c\, f^{(k)}(x)\right| \ne |Z_{n,k}(x)|$, $x \in J_{n,k}$. On the boundary points of $J_{n,k} = \mathbb{R} \setminus I_{n,k}$ we have $\left|c\, f^{(k)}(x)\right| < |Z_{n,k}(x)|$, hence the same inequality holds true on $J_{n,k}$. Therefore, $\left|f^{(k)}(x)\right| \le |Z_{n,k}(x)|$ on the Zolotarev set, i.e., (2.13) holds true. The proof of Theorem 2.5 in the case $1 \le k \le n-1$ follows from (2.11) and (2.13). The remaining case $k = n$ is readily verified: since $\{\ell_i^{(n)}(x)\}_{i=0}^n$ is a sequence of sign alternating constants, (2.12) holds true in this case, too. □

Remark 2.1. *When $1 \le k \le n-1$ and x is an interior point of $I_{n,k}$, (2.12) implies the strict inequality $\left|f^{(n)}(x)\right| < \left|Q_n^{(n)}(x)\right|$ unless $f = \pm Q_n$. The same conclusion follows from (2.12) in the case $k = n$, i.e., $\left|f^{(n)}\right| < \left|Q_n^{(n)}\right|$ unless $f = \pm Q_n$.*

Remark 2.2. *Theorem 1.3 can be obtained as a special case of Theorem 2.5 with $a = 1$, $b = -1$ and $\omega = \frac{1}{2k}\, Q_n'$.*

3. DS-inequalities with Jacobi polynomials as majorants

Theorem 2.5 is applicable when the majorant Q_n is a Jacobi polynomial. Recall that Jacobi polynomials $\{P_m^{(\alpha,\beta)}\}_{m\in\mathbb{N}_0}$ are the orthogonal polynomials in $[-1,1]$ with respect to the weight function $w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$, see e.g, [22, Chapt. 4].

Jacobi polynomials $P_n^{(\alpha,\beta)}$, $P_n^{(\alpha,\beta+1)}$ and $P_{n-1}^{(\alpha+1,\beta+1)}$ are connected with the identity

$$P_n^{(\alpha,\beta+1)}(x) = P_n^{(\alpha,\beta)}(x) + \frac{1}{2}(x-1)\,P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{3.16}$$

which is a consequence of Gauss' contiguous relations (see, e.g. [1, Section 2.5]) and the representation of Jacobi polynomials as hypergeometric functions. It follows from (3.16) and

$$\frac{d}{dx}\big\{P_n^{(\alpha,\beta)}(x)\big\} = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x) \tag{3.17}$$

(see [22, eqn. (4.21.7)]) that the zeros of $P_n^{(\alpha,\beta+1)}(x)$ and $\frac{d}{dx}\big\{P_n^{(\alpha,\beta)}(x)\big\}$ interlace.

Setting $Q_n := P_n^{(\alpha,\beta+1)}$ and $q := P_n^{(\alpha,\beta)}$, by k-fold differentiation of (3.16) we get

$$Q_n^{(k)}(x) = \frac{1}{n+\alpha+\beta+1}\big[(x-1)q^{(k+1)}(x) + (n+\alpha+\beta+k+1)\,q^{(k)}(x)\big]. \tag{3.18}$$

This representation of $Q_n^{(k)}$ provides relation (2.5) between Q_n and ω in Theorem 2.5 with

$$\omega(x) = \frac{1}{2k}\,q'(x), \quad a=1, \quad b = -1 - \frac{2k}{n+\alpha+\beta+1}.$$

Replacing these quantities in (2.7), we find that in this particular case Theorem 2.5 reads as:

Theorem 3.6. *Let $Q_n = P_n^{(\alpha,\beta+1)}$ and $q = P_n^{(\alpha,\beta)}$. If $f \in \pi_n$ satisfies*

$$|f| \le |Q_n| \quad \text{at the zeros of } (x^2-1)q'(x),$$

then for $k = 1,\dots,n$ and for every $x \in \mathbb{R}$,

$$\left|f^{(k)}(x)\right| \le \max\left\{\left|Q_n^{(k)}(x)\right|, |Z_{n,k}(x)|\right\},$$

where

$$Z_{n,k}(x) = \frac{x^2-1}{k}q^{(k+1)}(x) + xq^{(k)}(x) + \frac{\big((x-1)q'(x)\big)^{(k)}}{n+\alpha+\beta+1}. \tag{3.19}$$

Theorem 3.6 enables us to prove the following DS-inequality:

Theorem 3.7. *Let $Q_n = P_n^{(\alpha,\beta+1)}$ and $q = P_n^{(\alpha,\beta)}$, where $-1/2 < \alpha \le \beta$. If $f \in \pi_n$ satisfies*

$$|f| \le |Q_n| \quad \text{at the zeros of } (x^2-1)q'(x),$$

then

$$\|f^{(k)}\| \le \|Q_n^{(k)}\|, \qquad k=1,\dots,n\,. \tag{3.20}$$

The equality occurs in (3.20) if and only if $f = \pm Q_n$.

Proof of Theorem 3.7. We shall show that for $Z_{n,k}$ defined in (3.19) there holds

$$\|Z_{n,k}\| < \|Q_n^{(k)}\|, \qquad k=1,\dots,n. \tag{3.21}$$

We need the following property of Jacobi polynomials (cf. [22, Theorem 5.32.1] or [1, p. 350, Problem 40]):

Lemma 3.2. *Let $\max\{\alpha,\beta\} \ge -1/2$. Then for every $m \in \mathbb{N}$,*

$$\|P_m^{(\alpha,\beta)}\| = \begin{cases} P_m^{(\alpha,\beta)}(1) & = \binom{m+\alpha}{m}, \ \text{if } \ \alpha \ge \beta, \\ \left|P_m^{(\alpha,\beta)}(-1)\right| & = \binom{m+\beta}{m}, \ \text{if } \ \beta \ge \alpha. \end{cases}$$

Unless $\alpha = \beta = -1/2$, the norm of $P_m^{(\alpha,\beta)}$ is attained only at end point of the interval $[-1,1]$.

In view of (3.17), apart from constant factors, the polynomials $Q_n^{(k)}$, $q^{(k)}$ and $q^{(k+1)}$ are equal respectively to $P_{n-k}^{(\alpha+k,\beta+k+1)}$, $P_{n-k}^{(\alpha+k,\beta+k)}$ and $P_{n-k-1}^{(\alpha+k+1,\beta+k+1)}$. Since $\beta \ge \alpha \ge -1/2$, we have $\beta+k+1 \ge \alpha+k+1 > \alpha+k \ge 1/2$, then (3.18) and Lemma 3.2 imply

$$\|Q_n^{(k)}\| = |Q_n^{(k)}(-1)| = \frac{2}{n+\alpha+\beta+1}\left|q^{(k+1)}(-1)\right| + \left(1+\frac{k}{n+\alpha+\beta+1}\right)\left|q^{(k)}(-1)\right|. \tag{3.22}$$

We represent the polynomial $Z_{n,k}$ defined in (3.19) in the form

$$Z_{n,k}(x) = \varphi(x) + \frac{1}{n+\alpha+\beta+1}\,\psi(x), \tag{3.23}$$

where

$$\begin{aligned}\varphi(x) =& \frac{x^2-1}{k}q^{(k+1)}(x) + xq^{(k)}(x),\\ \psi(x) =& (x-1)\,q^{(k+1)}(x) + kq^{(k)}(x).\end{aligned}$$

Lemma 3.3. *Let $q = P_n^{(\alpha,\beta)}$, where $\beta \ge \alpha \ge -1/2$. Then*

$$\|\psi\| = \left\| \left(x-1)q^{(k+1)}(x) + kq^{(k)}(x)\right) \right\| = 2\left|q^{(k+1)}(-1)\right| + k\left|q^{(k)}(-1)\right| = |\psi(-1)|$$

and, in addition, $\|\psi\|$ is attained only at $x = -1$.

Proof of Lemma 3.3. By triangle inequality and Lemma 3.2,

$$\begin{aligned}\|\psi\| &\le \left\|(x-1)q^{(k+1)}(x)\right\| + k\left\|q^{(k)}(x)\right\| \le 2\left\|q^{(k+1)}\right\| + k\left\|q^{(k)}\right\|\\ &= 2\left|q^{(k+1)}(-1)\right| + k\left|q^{(k)}(-1)\right| = \left|-2\,q^{(k+1)}(-1) + k\,q^{(k)}(-1)\right|\\ &= |\psi(-1)|.\end{aligned}$$

On account of the last claim of Lemma 3.2, one can readily see that $x = -1$ is the unique point $[-1,1]$ where the norm of ψ is attained. □

Next, we estimate the norm of φ.

Lemma 3.4. *Let $q = P_n^{(\alpha,\beta)}$, where $\beta \ge \alpha \ge -1/2$. Then*

$$\|\varphi\| = \left\|\frac{x^2-1}{k}q^{(k+1)}(x) + xq^{(k)}(x)\right\| = \left|q^{(k)}(-1)\right| = |\varphi(-1)|, \qquad k = 1,\dots,n.$$

Proof of Lemma 3.4. We consider separately three cases.

Case 1: $\alpha = \beta = -1/2$. This case, corresponding to $q = T_n$, has been proven by Shadrin in [20, Lemma 3], it reads as

$$\left\|\frac{x^2-1}{k}T_n^{(k+1)}(x) + x\,T_n^{(k)}(x)\right\| = \left|T_n^{(k)}(-1)\right| = T_n^{(k)}(1).$$

Case 2: $\alpha = \beta > -1/2$. We make use of the fact that $q = P_n^{(\alpha,\alpha)}$ admits non-negative expansion in the Chebyshev polynomials of the first kind (cf. [2, eq. (7.34)]):

$$q(x) = \sum_{\nu=0}^{n} c_\nu\,T_\nu(x), \qquad c_\nu \ge 0.$$

Using the result from *Case 1*, we find

$$\begin{aligned}\Big\|\frac{x^2-1}{k}q^{(k+1)}(x)+x\,q^{(k)}(x)\Big\| &= \Big\|\sum_{\nu=0}^{n}c_\nu\left(\frac{x^2-1}{k}T_\nu^{(k+1)}(x)+x\,T_\nu^{(k)}(x)\right)\Big\|\\ &\le \sum_{\nu=0}^{n}c_\nu\Big\|\frac{x^2-1}{k}T_\nu^{(k+1)}(x)+x\,T_\nu^{(k)}(x)\Big\|\\ &= \sum_{\nu=0}^{n}c_\nu T_\nu^{(k)}(1)=q^{(k)}(1)=\left|q^{(k)}(-1)\right|.\end{aligned}$$

Case 3: $\beta>\alpha\ge -1/2$. Set $r=P_n^{(\beta,\alpha)}$, then $r(x)$ admits representation in the basis of $\{P_\nu^{(\alpha,\alpha)}(x)\}_{\nu=0}^{n}=:\{P_\nu(x)\}_{\nu=0}^{n}$ with non-negative coefficients (cf. [2, eq. (7.33)]):

$$r(x)=\sum_{\nu=0}^{n}c_\nu\,P_\nu(x),\qquad c_\nu=c_\nu(n,\alpha,\beta)\ge 0.$$

This representation and the result from *Case 2* imply

$$\begin{aligned}\Big\|\frac{x^2-1}{k}r^{(k+1)}(x)+x\,r^{(k)}(x)\Big\| &= \Big\|\sum_{\nu=0}^{n}c_\nu\left(\frac{x^2-1}{k}P_\nu^{(k+1)}(x)+xP_\nu^{(k)}(x)\right)\Big\|\\ &\le \sum_{\nu=0}^{n}c_\nu\Big\|\frac{x^2-1}{k}P_\nu^{(k+1)}(x)+xP_\nu^{(k)}(x)\Big\|\\ &= \sum_{\nu=0}^{n}c_\nu P_\nu^{(k)}(1)=r^{(k)}(1).\end{aligned}$$

Now using the symmetry property $P_n^{(\beta,\alpha)}(-x)=(-1)^nP_n^{(\alpha,\beta)}(x)$ (cf. [4, p. 144, eq. (2.8)], for $q(x)=P_n^{(\alpha,\beta)}(x)=(-1)^n r(-x)$ we obtain

$$\Big\|\frac{x^2-1}{k}q^{(k+1)}(x)+x\,q^{(k)}(x)\Big\|=\Big\|\frac{x^2-1}{k}r^{(k+1)}(x)+x\,r^{(k)}(x)\Big\|=r^{(k)}(1)=\left|q^{(k)}(-1)\right|.$$

Lemma 3.4 is proved. □

Lemma 3.3 and Lemma 3.4 imply

$$\begin{aligned}\|Z_{n,k}\| &= \Big\|\varphi+\frac{1}{n+\alpha+\beta+1}\,\psi\Big\|\\ &\le \|\varphi\|+\frac{1}{n+\alpha+\beta+1}\|\psi\|\\ &= |\varphi(-1)|+\frac{1}{n+\alpha+\beta+1}\,|\psi(-1)|\\ &= \left|q^{(k)}(-1)\right|+\frac{1}{n+\alpha+\beta+1}\left(2\left|q^{(k+1)}(-1)\right|+k\left|q^{(k)}(-1)\right|\right)\\ &= \frac{2}{n+\alpha+\beta+1}\left|q^{(k+1)}(-1)\right|+\left(1+\frac{k}{n+\alpha+\beta+1}\right)\left|q^{(k)}(-1)\right|.\end{aligned}\tag{3.24}$$

According to (3.22), the last expression is equal to $\|Q_n^{(k)}\|$, so we have proved the inequality $\|Z_{n,k}\|\le\|Q_n^{(k)}\|$, and now inequality (3.20) in Theorem 3.7 follows from Theorem 3.6.

For the last statement of Theorem 3.7, we need to prove the strict inequality (3.21). We observe that $\varphi(-1)$ and $\psi(-1)$ have opposite signs, namely,

$$\operatorname{sign}\varphi(-1) = -\operatorname{sign} q^{(k)}(-1) = (-1)^{n-1-k}, \qquad \operatorname{sign}\psi(-1) = \operatorname{sign} q^{(k)}(-1) = (-1)^{n-k}.$$

Therefore, the inequality in the second line of (3.24) is strict, and hence (3.21) holds true. We are ready now to prove the last claim of Theorem 3.7. The case $k = n$ is a direct consequence from Remark 2.1. The case $1 \le k \le n-1$ is also justified with Remark 2.1 as follows. We recall that if $f \in \pi_n$ satisfies the assumptions of Theorem 3.6, then $|Z_{n,k}(x)|$ (resp. $\left|Q_n^{(k)}(x)\right|$) furnishes upper bound for $\left|f^{(k)}(x)\right|$ when x belongs to the Zolotarev set $J_{n,k}$ (resp. Chebyshev set $I_{n,k}$). In view of (3.21), the equality $\|f^{(k)}\| = \|Q_n^{(k)}\|$ can happen only when the norm of $f^{(k)}$ is attained at a point x from the set $I_{n,k}\bigcap[-1,1]$. Since $\left|f^{(k)}(x)\right| \le \left|Q_n^{(k)}(x)\right|$ for $x \in I_{n,k}\bigcap[-1,1]$ and, by Lemma 3.2, $\|Q_n^{(k)}\| = \left|Q_n^{(k)}(-1)\right|$ with $x = -1$ being the unique point where the norm of $Q_n^{(k)}$ is attained, it follows that $\|f^{(k)}\| = \|Q_n^{(k)}\|$ is possible only when $\left|f^{(k)}(-1)\right| = \left|Q_n^{(k)}(-1)\right|$. Since $x = -1$ is an interior point for $I_{n,k}$, the last equality holds only if $f = \pm Q_n$, by virtue of Remark 2.1. □

Let us consider the special case $\alpha = \beta = -1/2$. According to (3.17),

$$q'(x) = \frac{1}{2} n\, P_{n-1}^{(1/2,1/2)}(x)$$

and, apart from a constant factor, q' is equal to the $(n-1)^{\text{th}}$ Chebyshev polynomial of the second kind U_{n-1}, which is defined for $x \in [-1,1]$ by

$$U_{n-1}(x) = \frac{\sin n\theta}{\sin\theta}, \quad x = \cos\theta,$$

and whose zeros are

$$t_\nu = \cos\frac{\nu\pi}{n}, \qquad \nu = 1,\dots,n-1.$$

On the other hand, apart from a constant multiplier, $Q_n = P_n^{(-1/2,1/2)}$ is equal to the Chebyshev polynomial of the third kind $V_n(x)$, which is defined for $x \in [-1,1]$ by

$$V_n(x) = \frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\left(\frac{1}{2}\theta\right)}, \qquad x = \cos\theta\,. \tag{3.25}$$

Clearly,

$$\begin{aligned} V_n\left(\cos\frac{\nu\pi}{n}\right) &= (-1)^\nu, \quad \nu = 0,1,\dots,n-1,\\ V_n(-1) &= (-1)^n(2n+1)\,. \end{aligned}$$

Thus, in the case $\alpha = \beta = -1/2$, Theorem 3.7 comes down to the following:

Corollary 3.1. *Let $f \in \pi_n$ satisfy $|f(-1)| \le 2n+1$ and*

$$\left|f\left(\cos\frac{\nu\pi}{n}\right)\right| \le 1, \qquad \nu = 0,\dots,n-1\,.$$

Then

$$\|f^{(k)}\| \le \|V_n^{(k)}\|, \qquad k = 1,\dots,n, \tag{3.26}$$

where V_n is the n-th Chebyshev polynomial of the third kind (3.25). The equality in (3.26) occurs only if $f = \pm V_n$.

The n-th Chebyshev polynomial of the fourth kind $W_n(x) = (-1)^n V_n(-x)$ is defined for $x \in [-1,1]$ by

$$W_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\left(\frac{1}{2}\theta\right)}, \qquad x = \cos\theta\,. \tag{3.27}$$

Reflection of the variable in Corollary 3.1 (i.e., replacement of x with $-x$) yields another corollary of Theorem 3.7:

Corollary 3.2. *Let $f \in \pi_n$ satisfy $|f(1)| \le 2n+1$ and*

$$\left|f\left(\cos\frac{\nu\pi}{n}\right)\right| \le 1, \qquad \nu = 1,\dots,n\,,$$

then

$$\|f^{(k)}\| \le \|W_n^{(k)}\|\,, \qquad k = 1,\dots,n, \tag{3.28}$$

where W_n is the n-th Chebyshev polynomial of the fourth kind (3.27). The equality in (3.28) occurs only if $f = \pm W_n$.

Proof of Theorem 1.4. Theorem 1.4 is deduced as a convex combination of the result of Duffin and Schaeffer (Theorem 1.2) and Corollary 3.2. Assume that for some $c \in [0,1]$, the polynomial $f \in \pi_n$ satisfies

$$\left|f\left(\cos\frac{\nu\pi}{n}\right)\right| \le 1, \quad \nu = 1,\dots,n,$$
$$|f(1)| \le 1 + 2cn.$$

Clearly, f can be represented as $f(x) = (1-c)\,g(x) + c\,h(x)$, where $f,\ h \in \pi_n$ obey the restrictions

$$\left|g\left(\cos\frac{\nu\pi}{n}\right)\right| \le 1, \qquad \nu = 0,\dots,n,$$
$$\left|h\left(\cos\frac{\nu\pi}{n}\right)\right| \le 1, \qquad \nu = 1,\dots,n,$$
$$|h(1)| \le 2n+1.$$

Theorem 1.2 implies

$$\|g^{(k)}\| \le \|T_n^{(k)}\| = T_n^{(k)}(1)$$

while Corollary 3.2 yields

$$\|h^{(k)}\| \le \|W_n^{(k)}\| = W_n^{(k)}(1).$$

Consequently,

$$\begin{aligned}\|f^{(k)}\| &\le (1-c)\,\|g^{(k)}\| + c\,\|h^{(k)}\| \le (1-c)\,\|T_n^{(k)}\| + c\,\|W_n^{(k)}\| \\ &= (1-c)\,T_n^{(k)}(1) + c\,W_n^{(k)}(1) = \|(1-c)\,T_n^{(k)} + c\,W_n^{(k)}\| \\ &= \|Q_n^{(k)}\|\,,\end{aligned}$$

where $Q_n(x) = (1-c)\,T_n(x) + c\,W_n(x)$. Since the norm of $Q_n^{(k)}$ is attained at $x=1$, we apply Remark 2.1 to conclude that the equality $\|f^{(k)}\| = \|Q_n^{(k)}\|$ occurs only when $f = \pm Q_n$. $\square$

4. CONCLUDING REMARKS

DS-inequalities can find application to the estimation of the round-off error of interpolatory formulae for numerical differentiation (see [9, p. 174, Remark 2]). Also, DS-inequalities may serve as a useful tool for establishing Markov-type inequalities for polynomials with curved majorants. For the readers convenience, we provide below a brief information on this topic.

We call *majorant* a continuous positive (or non-negative) function $\mu(x)$ in $[-1,1]$. If there exists a polynomial $P \in \pi_n$, $P \neq 0$, such that $-\mu(x) \leq P(x) \leq \mu(x)$, $x \in [-1,1]$, then there exists a unique (up to orientation) polynomial $\omega_\mu \in \pi_n$ (*snake polynomial*) which oscillates most between $\pm\mu$. The n-th snake polynomial ω_μ, associated with the majorant μ, is uniquely determined by the following properties:

a) $|\omega_\mu(x)| \leq \mu(x) \quad x \in [-1,1]$;

b) There exists a set $\delta^* = (\tau_i^*)_{i=0}^n$, $1 \geq \tau_0^* > \cdots > \tau_n^* \geq -1$, such that

$$\omega_\mu(\tau_i^*) = (-1)^i \mu(\tau_i^*), \quad i = 0, \ldots, n.$$

The set δ^* is referred to as *the set of alternation points of* ω_μ.

Associated with a given a majorant $\mu(x)$, we have the following extremal problems (cf. [14]):

Problem 1: Markov inequality with a majorant. *Given* $n, k \in \mathbb{N}$, $1 \leq k \leq n$, *and a majorant* $\mu \geq 0$, *find*

$$M_{k,n}(\mu) := \sup\{\|p^{(k)}\| \,:\, p \in \pi_n,\ |p(x)| \leq \mu(x),\ x \in [-1,1]\}\,.$$

Problem 2: Duffin-Schaeffer inequality with a majorant. *Given* $n, k \in \mathbb{N}$, $1 \leq k \leq n$, *and a majorant* $\mu \geq 0$, *find*

$$D_{k,n}(\mu) := \sup\{\|p^{(k)}\| \,:\, p \in \pi_n,\ |p(x)| \leq \mu(x),\ x \in \delta^*\}\,.$$

Clearly, $M_{k,n}(\mu) \leq D_{k,n}(\mu)$, and the results of V. A. Markov and of R. J. Duffin and A. C. Schaeffer (Theorem 1.2) read as:

$$\mu(x) \equiv 1 \Rightarrow M_{k,n}(\mu) = D_{k,n}(\mu) = \|T_n^{(k)}\|\,, \ \ 1 \leq k \leq n\,.$$

A natural question is: for which other majorants μ the snake-polynomial ω_μ is extremal to both Problems 1 and 2, i.e., when do we have the equalities

$$M_{k,n}(\mu) \stackrel{?}{=} D_{k,n}(\mu) \stackrel{?}{=} \|\omega_\mu^{(k)}\|\,?$$

A conjecture (belonging to mathematical folklore) states that the extremal polynomial to Problem 1 is the snake polynomial ω_μ. So far, no counterexample to this conjecture is found. On the contrary, ω_μ is not always the extremal polynomial to Problem 2, the following counterexamples are known:

$$\begin{aligned} &1) \quad \mu(x) = \sqrt{1-x^2}, \quad k = 1 \quad \text{(cf. [16]);} \\ &2) \quad \mu(x) = 1 - x^2, \quad k = 1, 2 \quad \text{(cf. [17]).} \end{aligned}$$

The difficulty with the proof of the above conjecture comes from the fact that only in some exceptional cases the snake polynomials are known explicitly (and the same applies to the associated sets of alternation points). Assuming the snake polynomial ω_μ is known, a possible approach to showing that ω_μ is the extreme polynomial to Problem 2 is to show that DS-inequality holds for any pair (ω_μ, Δ) such that the points from Δ are separated by the zeros of ω_μ. For $\mu \equiv 1$ (and $\omega_\mu = T_n$) this plan was realized by the author in [10] (for $k = 1$) and [12] (the general case $1 \leq k \leq n$), thus showing that whenever T_n is a snake polynomial associated with some majorant μ, then T_n is the extreme polynomial to Problem 2. Even more is true: in [14, 15] we proved that whenever the snake polynomial associated with a majorant μ possessses positive or sign-alternating expansion in the Chebyshev polynomials of the first kind, it is the extreme polynomial to Problem 2.

The DS-inequalities in this paper were announced without proof in [13]. Although they can be derived from the results in [14, 15], we decided to propose here a direct self-contained proof, emphasizing to the important particular case presented by Theorem 1.4.

5. Acknowledgment

This study is financed by the European Union-NextGenerationEU, through the National Recovery and Resilience Plan of the Republic of Bulgaria, project No BG-RRP-2.004-0008.

References

[1] G. E. Andrews, R. Askey and R. Roy: *Special Functions*, Cambridge University Press (1999).
[2] R. Askey: *Orthogonal Polynomials and Special Functions*, SIAM, Philadelphia (1975).
[3] B. Bojanov, G. Nikolov: *Duffin and Schaeffer type inequalities for ultraspherical polynomials*, J. Approx. Theory, **84** (1996), 129–138.
[4] T. S. Chihara: *An Introduction to Orthogonal Polynomials*, Gordon and Breach Sci. Pub., New York (1978).
[5] R. J. Duffin, A. C. Schaeffer: *A refinement of an inequality of brothers Markoff*, Trans. Amer. Math. Soc., **50** (1941), 517–528.
[6] M. E. H. Ismail: *Classical and Quantum Orthogonal Polynomials in One Variable*, Encyclopedia in Mathematics and its Applications, vol. **98**, Cambridge University Press (2005).
[7] A. A. Markov: *On a question of D. I. Mendeleev*, Zap. Petersburg Akad. Nauk, **62** (1889), 1–24, [In Russian].
[8] V. A. Markov: *On functions least deviated from zero in a given interval*, St. Petersburg (1892), [In Russian].
[9] G. Nikolov: *On certain Duffin and Schaeffer type inequalities*, J. Approx. Theory, **93** (1998), 157–176.
[10] G. Nikolov: *Inequalities of Duffin–Schaeffer type*, SIAM J. Math. Anal., **33** (3) (2001), 686–698.
[11] G. Nikolov: *An extremal porperty of Hermite polynomials*, J. Math. Anal. Appl., **290** (2004), 405–413.
[12] G. Nikolov: *Inequalities of Duffin–Schaeffer type, II*, East J. Approx., **11** (2005), 147–168.
[13] G. Nikolov: *Jacobi polynomials and inequalities of Duffin–Schaeffer type*, In Proceedings Book of MICOPAM 2024 (Edited by Y. Şimşek, M. Alcan, I. Kücükoglu, and O. Öneş), 141–145.
[14] G. Nikolov, A. Shadrin: *On Markov-Duffin-Schaeffer Inequalities with a Majorant*, In Constructive Theory of Functions. Sozopol, June 3–10, 2010. In Memory of Borislav Bojanov, (Edited by G. Nikolov and R. Uluchev), pp. 227–264, Prof. Marin Drinov Academic Publishing House, Sofia, (2012).
[15] G. Nikolov, A. Shadrin: *On Markov-Duffin-Schaeffer Inequalities with a Majorant, II*, In Constructive Theory of Functions. Sozopol, June 9–14, 2013. Dedicated to Blagovest Sendov and to the memory of Vasil Popov, (Edited by K. Ivanov et al.), pp. 175–197, Prof. Marin Drinov Academic Publishing House, Sofia, (2014).
[16] Q. I. Rahman, G. Schmeisser: *Markov–Duffin–Schaeffer inequality for polynomials with a circular majorant*, Trans. Amer. Math. Soc., **310** (1988), 693–702.
[17] Q. I. Rahman, A. Q. Watt: *Polynomials with a parabolic majorant and the Duffin–Schaeffer inequality*, J. Approx. Theory, **69** (1992), 338–354.
[18] T. Rivlin: *Chebyshev Polynomials: From Approximation Theory to Algebra & Number Theory*, Second Edition, Dover Publications, New York (2020).
[19] A. Schönhage: *Approximationstheorie*, Walter de Gruyter & Co, Berlin, New York (1971).
[20] A. Yu. Shadrin: *Interpolation with Lagrange polynomials: A simple proof of Markov inequality and some of its generalization*, Approx. Theory Appl. (N.S.), **8** (1992), 51–61.
[21] A. Yu. Shadrin: *Twelve Proofs of the Markov Inequality*, In Approximation Theory: A Volume Dedicated to Borislav Bojanov, (Edited by D.K. Dimitrov, G. Nikolov and R. Uluchev), pp. 233-298, Prof. Marin Drinov Academic Publishing House, Sofia (2004).
[22] G. Szegő: *Orthogonal Polynomials*, 4th edition, AMS Colloquium Publications, Providence, RI (1975).

Geno Nikolov
Faculty of Mathematics and Informatics
Sofia University St. Kliment Ohridski
5, J. Bourchier Blvd, 1164 Sofia, Bulgaria
ORCID: 0000-0001-5608-2488
Email address: `geno@fmi.uni-sofia.bg`