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RESEARCH ARTICLE

Generalizations of third-order recurrence relation

K. L. Verma[1*]

[1]Career Point University, Department of Mathematics, Hamirpur (HP), 176041, India.

ABSTRACT

This paper presents a generalization of the sequence defined by the third-order recurrence relation $V_n(a_j, p_j) = \sum_{j=1}^{3} p_j V_{n-j},\ n \geq 4,$, $p_3 \neq 0$ with initial terms $V_j = a_j$, where a_j and p_j $j = 1, 2, 3$, are any non-zero real numbers. The generating function and Binet's formula are derived for this generalized tribonacci sequence. Classical second-order generalized Fibonacci sequences and other existing sequences based on second-order recurrence relations are implicitly included in this analysis. These derived sequences are discussed as special cases of the generalization. A pictorial representation is provided, illustrating the growth and variation of tribonacci numbers for different initial terms a_j and coefficients p_j. Additionally, the tribonacci constant is examined and visually represented. It is observed that the constant is influenced solely by the coefficients p_j of the recurrence relation and is unaffected by the initial terms a_j.

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Keywords: Generalized Tribonacci sequence, generating function, Binet's formula, Tribonacci constant.

1. INTRODUCTION

Tribonacci sequences are the generalization of the classical Fibonacci sequence, defined by a recurrence relation involving the sum of the three preceding terms, where each term is the sum of the three preceding terms. The enigmatic tribonacci number sequences with its captivating properties, has piqued the curiosity of mathematicians and researchers, opening doors to a world of intriguing applications as it has attracted attention in various branches of physical sciences and its applications. Sequence terms in a recursive relations are generated sequentially, the process of calculating any specific term is computationally intensive, as it necessitates the calculation of all its predecessors. Alternatively, using the index form of a generating function or Binet's formula provides efficient methods for directly computing any term of a recursive sequence. Although extensive research has been conducted on second-order Fibonacci sequences and their generalizations, the exploration of third-order recurrence relations, particularly in the context of third-order Fibonacci-like sequences, has received comparatively less attention. A generalized tribonacci sequence, $\{V_n\}$, is result of the recurrence relations with coefficients p_j and arbitrary first three initial terms a_j. The concept of tribonacci sequence mentioned and studied, first time by Feinberg M. Feinberg (1963), then number of generalizations of the Fibonacci sequence have been considered and examined by many authors W. R. Spickerman (1982); T. Komatsu (2018); R. Frontczak (2018); A. G. Shannon (1972); A. C. F. Bueno (2015); T. Komatsu and R. Li (2017); T. Koshy (2001); P. Y. Lin (1988); S. Pethe (1988); Y. Soykan (2019); C. C. Yalavigi (1972). F. T. Howard (2001) extended and generalize the main result obtained by F. T. Howard (1999) for tribonacci sequences. Generalization of Tribonacci sequences for quaternions studied by G. Cerda-Morales (2017). In the literature. Generalized Tribonacci sequence has also been considered and studied by A. G. Shannon and A. F. Horadam (1972); M. E. Waddill and L. Sacks (1967); T. Komatsu and R. Li (2017) and Y. Soykan, I. et al. (2020); A. Scott, T. et al. (1997). This research aims to address by considering and exploring the properties, patterns, and potential applications of generalized third-order Fibonacci sequences. In this article, generalized third-order recurrence relations with variable coefficients p_j and initial terms a_j are taken to derive the generalized form of generating function and the Binet's formula. Classical second-order generalized Fibonacci sequences and other existing sequences based on second-order recurrence relations are implicitly included in this analysis. These derived sequences are discussed as special cases of the generalization. A pictorial representation is provided, illustrating the growth and variation of tribonacci numbers for different initial terms a_j and coefficients p_j. Additionally, the

Corresponding Author: K. L. Verma **E-mail:** klverma@netscape.net, klverma@cpuh.edu.in
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Tribonacci constant is examined and visually represented. It is observed that the constant is influenced solely by the coefficients p_j of the recurrence relation and is unaffected by the initial terms a_j.

Definition 1.1. We define the Generalized Tribonacci sequence $\{V_n\}$ by the following linear recurrence relation:

$$V_n(a_1, a_2, a_3, p_1, p_2, p_3) = p_1 V_{n-1} + p_2 V_{n-2} + p_3 V_{n-3}, \; n \geq 4, \tag{1}$$

with the initial conditions,$a_j = V_j$, a_j, p_j, $j = 1, 2, 3$ are any non–zero real numbers.

The expression for $\{V_n\}$ in (1) is holds true T. Koshy (2001) for every integer $n \geq 4$.

Terms of the Generalized Tribonacci Sequence The first few terms in the generalized form of the sequence defined in (1) are:

$$\{V_n\} = \left\{ \begin{array}{l} a_1, a_2, a_3, p_1 a_3 + p_2 a_2 + p_3 a_1, \left(p_1^2 + p_2\right) a_3 + \left(p_1 p_2 + p_3\right) a_2 + p_1 p_3 a_1, \\ \left(p_1^3 + p_3 + 2 p_1 p_2\right) a_3 + \left(p_1^2 p_2 + p_2^2 + p_1 p_3\right) a_2 + \left(p_1^2 p_3 + p_2 p_3\right) a_1 + \cdots \end{array} \right\}.$$

Tribonacci Sequences pictorial representations A few values Y. Soykan, I. et al. (2020) of Tribonacci sequences represented in the following figure.

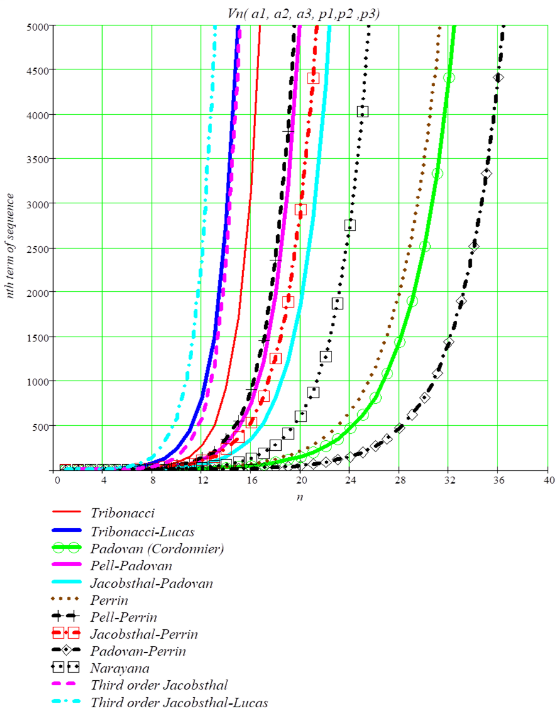

Figure 1. Tribonacci sequences progression and comparison

Special Cases

Remark 1.2. With initial conditions $V_0 = 0, V_1 = 1$, $V_2 = 1$, and $p_1 = p_2 = p_3 = 1$, recurrence relation (1) is known as the generalized Lucas tribonacci sequence and is denoted by T_n in F. T. Howard (1999). The first few terms of the sequence deduced from the above generalization:

$$\{V_n\}_{n\geq 0} = \{T_n\} = \{0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, \cdots\}.$$

This tribonacci number sequence is A000073 on the OEIS, N. J. A. Sloane (1973).

Remark 1.3. If we substitute the initial conditions $V_0 = 3, V_1 = 1$, $V_2 = 3$, and $p_1 = p_2 = p_3 = 1$ in (1), it reduces to K_n sequence which is explained in **?**. The first few terms of the sequence K_n are:

$$\{V_n\}_{n\geq 0} = \{K_n\} = \{3, 1, 3, 7, 11, 21, 39, 71, 131, 241, 443, 815, 1499, 2757, 5071, 9327, \cdots\}.$$

This tribonacci number sequence is A001644 on the OEIS, N. J. A. Sloane (1973).

2. GENERALIZED GENERATING FUNCTIONS

A generating function W. Watkins (1987) is a representation of a sequence as the coefficients of a power series in mathematics. By analyzing the generating function, we can derive various properties of the generalized Tribonacci sequence, such as closed-form expressions, asymptotic behavior, and generating function identities.

Theorem 2.1. (Generalized Generating Functions) *The generalized generating function of the sequence defined in (1) is*

$$V(x) = \frac{f(x)}{1 - p_1x - p_2x^2 - p_3x^3}, \tag{2}$$

where $f(x) = V_0 + (V_1 - p_1V_0)\,x + (V_2 - p_1V_1 - p_2V_0)\,x^2$.

Proof. If $V(x)$ is the generating function of $V_n = p_1V_{n-1} + p_2V_{n-2} + p_3V_{n-3}$, then we have

$$V(x) = \sum_{n=0}^{\infty} V_nx^n = V_0 + V_1x + V_2x^2 + V_3x^3 + \cdots \tag{3}$$

Multiplying $V(x)$ by p_1x, p_2x^2 and p_3x^3, we have

$$\begin{aligned} p_1xV(x) &= p_1V_0x + p_1V_1x^2 + p_1V_2x^3 + p_1V_3x^4 + \cdots \\ p_2x^2V(x) &= p_2V_0x^2 + p_2V_1x^3 + p_2V_2x^4 + p_2V_3x^5 + \cdots. \\ p_3x^3V(x) &= p_3V_0x^3 + p_3V_1x^4 + p_3V_2x^5 + p_3V_3x^6 + \cdots \end{aligned} \tag{4}$$

Subtracting (3)- (4) and rearranging the above equations, we have

$$V(x)\left[1 - p_1x - p_2x^2 - p_3x^3\right] = f(x).$$

Solving for $V(x)$, we obtain

$$V(x) = \frac{f(x)}{1 - p_1x - p_2x^2 - p_3x^3}, \tag{5}$$

where $f(x) = V_0 + (V_1 - p_1V_0)\,x + (V_2 - p_1V_1 - p_2V_0)\,x^2$ is a polynomial.
Hence $V(x)$ is the generating function of the sequence $\{V_n\}$.

Remark 2.2. If we substitute $V_0 = 3, V_1 = 1,\ V_2 = 3$, and $p_1 = p_2 = p_3 = 1$ in the result obtained in (5), it reduces to the generating function

$$V(x) = \frac{3 - 2x - x^2}{1 - x - x^2 - x^3} = T(x),$$

which is the same as result, which is explained in M. Elia (2001); M. Catalani (2002).

Remark 2.3. If we substitute $V_0 = 0, V_1 = 1,\ V_2 = 1$, and $p_1 = p_2 = p_3 = 1$ in result of (5), it reduces to the generating function

$$V(x) = \frac{x}{1 - x - x^2 - x^3} = K(x),$$

which is the same as result, which is explained in M. Elia (2001); M. Catalani (2002).

2.1. Even and odd terms Generating Functions of the Generalized Tribonacci Sequence

Theorem 2.4. *[Even and odd terms Generating Functions] The generating functions of even* $V_{2n}(x)$ *and odd* $V_{2n+1}(x)$ *terms of the Generalized Tribonacci Sequence (1) are:*

$$V_{even}(x) = \frac{V_0 - \left[(2p_2 + p_1^2)V_0 - V_2\right]x + \left[(p_2^2 - p_1p_3)V_0 + (p_1p_2 + p_3)V_1 - p_2V_2\right]x^2}{1 - (p_1^2 + 2p_2)x - (2p_1p_3 - p_2^2)x^2 - p_3^2x^3},$$

and

$$V_{odd}(x) = \frac{V_1 + \left[V_0p_3 - (p_1^2 + p_2)V_1 + p_1V_2\right]x + \left[p_3V_2 - p_1p_3V_1 - p_2p_3V_0\right]x^2}{1 - (p_1^2 + 2p_2)x - (2p_1p_3 - p_2^2)x^2 - p_3^2x^3}.$$

Proof. From the definition of the even $V_{2n}(x) = \frac{V_n(\sqrt{x}) + V_n(-\sqrt{x})}{2}$ and odd $V_{2n+1}(x) = \frac{V_n(\sqrt{x}) - V_n(-\sqrt{x})}{2\sqrt{x}}$ functions and employing the Generalized generating function of the Tribonacci sequence (1) obtained in the Theorem (2.1) we have

$$V_n(x) = \frac{f(x)}{1 - p_1x - p_2x^2 - p_3x^3},$$

where $f(x) = V_0 + (V_1 - p_1V_0)\,x + (V_2 - p_1V_1 - p_2V_0)\,x^2$.
On simplification we obtained the Generalized Generating function of even and odd terms of Tribonacci sequence

$$V_{even}(x) = \frac{V_0 - \left[(2p_2 + p_1^2)V_0 - V_2\right]x + \left[(p_2^2 - p_1p_3)V_0 + (p_1p_2 + p_3)V_1 - p_2V_2\right]x^2}{1 - (p_1^2 + 2p_2)x - (2p_1p_3 - p_2^2)x^2 - p_3^2x^3}, \tag{6}$$

and

$$V_{odd}(x) = \frac{V_1 + \left[V_0p_3 - (p_1^2 + p_2)V_1 + p_1V_2\right]x + \left[p_3V_2 - p_1p_3V_1 - p_2p_3V_0\right]x^2}{1 - (p_1^2 + 2p_2)x - (2p_1p_3 - p_2^2)x^2 - p_3^2x^3}. \tag{7}$$

2.2. Special cases of Even and odd terms Generating Functions

Remark 2.5. With initial conditions $V_0 = 0, V_1 = 1,\ V_2 = 1$, and $p_1 = p_2 = p_3 = 1$, the even and odd terms Generating Functions of the generalized Lucas sequence T_n T. Koshy (2001)) are deduced from the (6) and (7) generalized even and odd terms generating functions as:

$$V_{even}(x) = T_{even}(x) = \frac{x + x^2}{1 - 3x - x^2 - x^3},$$

and

$$V_{odd}(x) = T_{odd}(x) = \frac{1 - x}{1 - 3x - x^2 - x^3}.$$

Similarly with initial conditions $V_0 = 3, V_1 = 1,\ V_2 = 3$, and $p_1 = p_2 = p_3 = 1$ in (1), the even and odd terms Generating Functions of the generalized Lucas sequence K_n. T. Koshy (2001)) are deduced from the (6) and (7) generalized even and odd terms Generating Functions are:

$$V_{2n}(x) = K_{even}(x) = \frac{3 - 6x - x^2}{1 - 3x - x^2 - x^3},$$

and

$$V_{odd}(x) = K_{odd}(x) = \frac{3 + 4x - x^2}{1 - 3x - x^2 - x^3}.$$

These even and odd terms of the Generating Functions of T_n and K_n are same as obtained by T. Komatsu (2018).

Theorem 2.6. (Generalized Binet's formula for Tribonacci sequence) *Generalized form of the Binet's formula for the sequence defined in (1) is*

$$V_n(x) = \sum_{j=1}^{3}\left[\frac{A_1\alpha_j^2 + A_2\alpha_j + A_3}{\prod\limits_{\substack{1\le i\le 3\\ i\ne j}}(\alpha_j - \alpha_i)}\right]\alpha_j^n = \sum_{j=1}^{3}\frac{\left(\alpha_j^2A_1 + \alpha_jA_2 + A_3\right)p_3\alpha_j^{n+1}}{\alpha_j^3p_3 - \alpha_jp_1 + 2}.$$

Proof. Since $V(x)$ is the generating function of the sequence $\{V_n\}$

$$V(x) = \frac{f(x)}{1 - p_1x - p_2x^2 - p_3x^3},$$

where $f(x) = V_0 + (V_1 - p_1V_0)\,x + (V_2 - p_1V_1 - p_2V_0)\,x^2$.
Consider the partial fraction decomposition of the right-hand side of the generating function , we have

$$V(x) = \frac{A_1 + A_2x + A_3x^2}{1 - p_1x - p_2x^2 - p_3x^3} = \frac{A_1 + A_2x + A_3x^2}{(1 - \alpha_1x)(1 - \alpha_2x)(1 - \alpha_3x)},$$

where $A_1 = V_0, A_2 = V_1 - p_1V_0, A_3 = V_2 - p_1V_1 - p_2V_0$ and $\alpha_i,\ i = 1, 2, 3$, are roots of the equation $1 - p_1x - p_2x^2 - p_3x^3 = 0$.

On simplification we have

$$V_n(x) = \sum_{j=1}^{3}\left[\frac{A_1\alpha_j^2 + A_2\alpha_j + A_{3k}}{\prod\limits_{\substack{1\le i\le 3\\ i\ne j}}(\alpha_j-\alpha_i)}\right]\alpha_j^n = \sum_{j=1}^{3}\frac{\left(\alpha_j^2A_1+\alpha_jA_2+A_3\right)p_3\alpha_j^{n+1}}{p_3\alpha_j^3-\alpha_jp_1+2}. \tag{8}$$

The above relation is the Generalized Binet's formula for Tribonacci sequence

2.3. Special cases: Generalized Binet's formula for Tribonacci sequence

Remark 2.7. Generalized form of the Binet's formula (8) for the generalized sequence(1) can also be written as

$$V_n = \left[\begin{array}{l}\left(\frac{\alpha_3^{n+2}-\alpha_2^{n+2}}{\alpha_3-\alpha_2}\right)\frac{\alpha_1}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_1)} + \left(\frac{\alpha_1^{n+2}-\alpha_3^{n+2}}{\alpha_1-\alpha_3}\right)\frac{\alpha_2}{(\alpha_3-\alpha_2)(\alpha_2-\alpha_1)}\\ + \left(\frac{\alpha_2^{n+2}-\alpha_1^{n+2}}{\alpha_2-\alpha_1}\right)\frac{\alpha_3}{(\alpha_3-\alpha_2)(\alpha_1-\alpha_3)}\end{array}\right].$$

$$= \sum_{j=0}^{n}\left(\sum_{k=0}^{n-j}\alpha_1^j\alpha k\alpha_3^{n-j-k}\right)$$

Remark 2.8. If we put $V_3 = 0, p_3 = 0$, in equation (1) then tribonacci sequences becomes the generalized classical Fibonacci sequence, and the Binet's formula (8) in this case reduces to

$$\begin{aligned} V_n(x) &= \frac{A_1 + A_2x}{1 - p_1x - p_2x^2} \\ V_n(x) &= \frac{(\alpha_1A_1 + A_2)\left(\alpha_1^n\right) - (\alpha_2A_1 + A_2)\left(\alpha_2^n\right)}{\alpha_1 - \alpha_2} \\ V_n(x) &= \frac{A_1\left(\alpha_1^{n+1} - \alpha_2^{n+1}\right) + A_2\left(\alpha_1^n - \alpha_2^n\right)}{\alpha_1 - \alpha_2} = A_1\left(\frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2}\right) + A_2\left(\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}\right) \end{aligned},$$

where $A_1 = V_0, A_2 = (V_1 - p_1V_0)$ and $\alpha_i,\ i = 1, 2$ are roots of the equation $1 - p_1x - p_2x^2 = 0$.

Remark 2.9. If we take $V_0 = 0, V_1 = 1,\ V_2 = 1$, and $p_1 = p_2 = p_3 = 1$, in the expression (8) this reduces to

$$V_n(x) = \frac{\alpha_1^{n+1}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} + \frac{\alpha_2^{n+1}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)} + \frac{\alpha_3^{n+1}}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)} = T_n$$

. which is same, as obtained by R. Frontczak (2018).

When$V_0 = 3, V_1 = 1,\ V_2 = 3$, and $p_1 = p_2 = p_3 = 1$, in the expression (8) this reduces to

$$V(x) = \alpha_1^n + \alpha_2^n + \alpha_3^n = K_n$$

where $\alpha_i,\ i = 1, 2, 3$ are roots of the equation $1 - x - x^2 - x^3 = 0$. This is in agreement with W. R. Spickerman (1982); R. Frontczak (2018). .

Theorem 2.10. *If*

$$V_n = \begin{cases} a_i \text{ if } 1 \le n \le 3 \\ p_1V_{n-1} + p_2V_{n-2} + \cdots + p_3V_{n-3} \text{ if } n > 3 \end{cases}$$

then for $n \ge 4$, we have

$$V_n = 2p_1V_{n-1} + \left(p_2 - {p_1}^2\right)V_{n-2} + (p_3 - p_1p_2)V_{n-3} - p_1p_3V_{n-4}.$$

Proof. Rewrite the recurrence relation (1) as

$$\begin{aligned} V_n &= p_1V_{n-1} + p_2V_{n-2} + p_3V_{n-3} + 0 \\ &= p_1V_{n-1} + p_2V_{n-2} + p_3V_{n-3} + \left(p_1V_{n-1} - {p_1}^2V_{n-2} - p_1p_2V_{n-3} - p_1p_3V_{n-4}\right) \\ &= 2p_1V_{n-1} + \left(p_2 - {p_1}^2\right)V_{n-2} + (p_3 - p_1p_2)V_{n-3} - p_1p_3V_{n-4} \end{aligned}$$

$$\left[\begin{array}{l} \because\ V_n = p_1V_{n-1} + p_2V_{n-2} + p_3V_{n-3} \text{ by multipling } p_1 \text{ and replacing } n \text{ by } (n-1)\text{, we have} \\ p_1V_{n-1} - p_1{}^2V_{n-2} - p_1p_2V_{n-3} - p_1p_3V_{n-4} \end{array}\right]$$

$$V_n = 2p_1V_{n-1} + \left(p_2 - p_1{}^2\right)V_{n-2} + (p_3 - p_1p_2)V_{n-3} - p_1p_3V_{n-4}. \tag{9}$$

Remark 2.11. On substituting $p_1 = p_2 = p_3 = 1$,in the result of above Theorem (2.10) we have

$$V_n = 2V_{n-1} + (0)\,V_{n-2} + (0)\,V_{n-3} - V_{n-4}$$

, this implies that

$$V_n = 2V_{n-1} - V_{n-4}$$

. which is in agreement with F. T. Howard and C. Cooper (1970); M. E. Waddill and L. Sacks (1967).

Theorem 2.12. *If* $V_n = p_1V_{n-1} + p_2V_{n-2} + p_3V_{n-3}$, $n > 3$, $f(x) = x^3 - p_1x^2 - p_2x - p_3 = 0$,, *then*

$$\lim_{n\to\infty} \frac{V_{n+1}(a_1, a_2, a_3, p_1, p_2, p_3)}{V_n(a_1, a_2, a_3, p_1, p_2, p_3)} = \begin{cases} \alpha, \text{ real root of } f(x) = 0, p_i > 0, \text{others roots are complex,} \\ \alpha \text{ (largest root), if all roots of } f(x) = 0 \text{ are real,} \\ p_i > 0, \text{others roots are complex,} \\ 1.839, \text{ if } p_j = 1 \text{ and } a_j\ (j = 1, 2, 3) \text{ are any real numbers,} \\ 1.618, \text{ if } p_3 = 0,\ a_3 = 0, \text{ and } p_j, a_j, (j = 1, 2) \text{ are any real numbers.} \end{cases}$$

Remark 2.13. Graphical representation of the theorem (2.12)for the polynomials
$f(x) = x^3 - x^2 - x = 0$, $f(x) = x^3 - x^2 - x - 1 = 0$, and $f(x) = x^3 - x^2 - x - 3 = 0$.

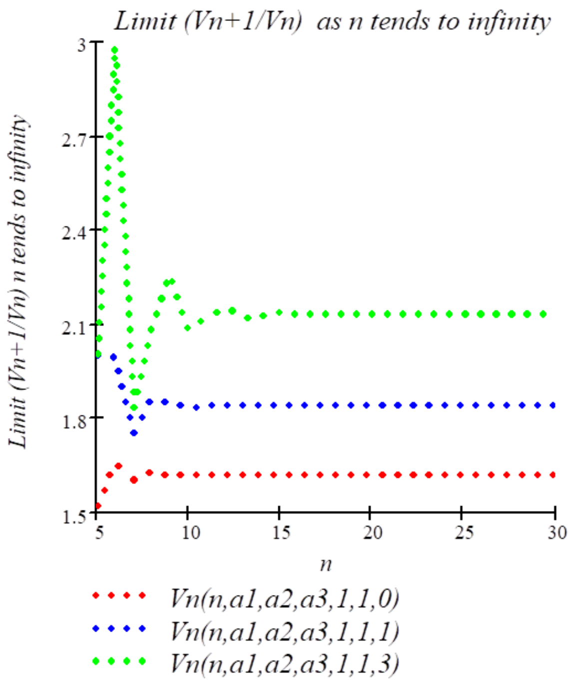

Figure 2. Tribonacci sequences progression and comparison

3. IDENTITIES

Theorem 3.1. *If $n \geq m$, then on employing the result of theorem (2.6)*

$$V_n V_{n+m} = \left(\sum_{j=1}^{3}\left[\frac{A_1\alpha_j^2 + A_2\alpha_j + A_3}{\prod\limits_{\substack{1\leq i\leq 3\\ i\neq j}}(\alpha_j - \alpha_i)}\right]\right)\alpha_j^n . \left(\sum_{j=1}^{3}\left[\frac{A_1\alpha_j^2 + A_2\alpha_j + A_3}{\prod\limits_{\substack{1\leq i\leq 3\\ i\neq j}}(\alpha_j - \alpha_i)}\right]\right)\alpha_j^{n+m}.$$

Proof. Using

$$V_n(x) = \sum_{j=1}^{3}\left[\frac{A_1\alpha_j^2 + A_2\alpha_j + A_{3k}}{\prod\limits_{\substack{1\leq i\leq 3\\ i\neq j}}(\alpha_j - \alpha_i)}\right]\alpha_j^n .$$

On simplifying the RHS, we obtain

$$V_n V_{n+m} = V_{2n+m} + V_n V_m - \sum_{j=1}^{3}\left(\frac{A_1\alpha_j^2 + A_2\alpha_j + A_3}{\prod\limits_{\substack{1\leq i\leq 3\\ i\neq j}}(\alpha_j - \alpha_i)}\right)^2 \alpha_j^{2n+m}.$$

3.1. Special cases:Identities

If we replace n by $n-1$ and taking $m = 1$, in (3.1) then we obtain

$$V_{n-1}V_n = V_{2n-1} + V_{n-1}V_1 - \sum_{j=1}^{3}\left(\frac{A_1\alpha_j^2 + A_2\alpha_j + A_3}{\prod\limits_{\substack{1\leq i\leq 3\\ i\neq j}}(\alpha_j - \alpha_i)}\right)^2 \alpha_j^{2n-1}.$$

If we take $m = n$ then we get

$$V_n V_{2n} = V_{3n} + V_n V_n - \sum_{j=1}^{3}\left(\frac{A_1\alpha_j^2 + A_2\alpha_j + A_3}{\prod\limits_{\substack{1\leq i\leq 3\\ i\neq j}}(\alpha_j - \alpha_i)}\right)^2 \alpha_j^{3n}.$$

If we take $m = 0$ in (3.1) then we get

$$V_n{}^2 = V_{2n} + V_n V_0 - \sum_{j=1}^{3}\left(\frac{A_1\alpha_j^2 + A_2\alpha_j + A_3}{\prod\limits_{\substack{1\leq i\leq 3\\ i\neq j}}(\alpha_j - \alpha_i)}\right)^2 \alpha_j^{2n}$$

.

$$V_n{}^3 = V_n{}^2 V_n = V_{2n}V_n + V_n{}^2 V_0 - V_n\sum_{j=1}^{3}\left(\frac{A_1\alpha_j^2 + A_2\alpha_j + A_3}{\prod\limits_{\substack{1\leq i\leq 3\\ i\neq j}}(\alpha_j - \alpha_i)}\right)^2 \alpha_j^{2n} V_n$$

In general from (3.1), we have

$$V_n V_{nm} = V_{mn+n} + V_n V_{nm-n} - \sum_{j=1}^{3}\left(\frac{A_1\alpha_j^2 + A_2\alpha_j + A_3}{\prod\limits_{\substack{1\leq i\leq 3\\ i\neq j}}(\alpha_j - \alpha_i)}\right)^2 \alpha_j^{nm-n}.$$

4. DISCUSSION AND CONCLUSION

This study investigates a generalized third-order recurrence relation. After defining the initial terms in general form, we present a graphical representation in Figure 1 to illustrate the progression of Tribonacci numbers for various cases considered by previous authors. Figure 2 depicts the ratio of consecutive terms as the number of terms approaches infinity. We observe that the Tribonacci constant is solely influenced by the coefficients p_j of the recurrence relation and is unaffected by the terms a_j.We derive the generating function and Binet formula in their general forms. By applying these results, we show that many existing results from previous studies emerge as special cases. Future research could delve deeper into this generalized third-order sequence, extending the analysis to explore additional properties and applications. Employing alternative approaches, such as matrix methods, combinatorial arguments, or number theory, may lead to the discovery of new identities and theorems.

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LIST OF AUTHOR ORCIDS

K. L. Verma https://orcid.org/0000-0002-6486-8736

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