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Quasi Bi-Slant Submanifolds of Bronze Riemannian Manifolds

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Abstract— In this paper, we aim to study quasi-bi-slant (qbs) submanifolds of bronze Riemannian manifolds. We present necessary and sufficient conditions for the distributions included in the definition of such lightlike submanifolds to be integrable. Moreover, we obtain totally geodesic conditions for the distribution involved in the qbs submanifold of a bronze Riemannian manifold. Afterward, we construct an example of this type of submanifold. Finally, we discuss the need for further research.

Keywords — *Bronze mean, slant submanifold, qbs submanifold*

Mathematics Subject Classification (2020) 53C15, 53C40

1. Introduction

In differential geometry, the theory of slant submanifolds is a critical area as it generalizes the cases of both complex and totally real submanifolds. In this manner, Chen initiated slant submanifolds of an almost Hermitian manifold [1]. Inspired by this paper, many geometry have introduced this notion in the different kinds of structures [2]. Then, as a generalization of semi-slant submanifolds given by Papagiuc [3], the notion of bi-slant submanifolds was introduced in [4]. However, these submanifolds were called hemi-slant submanifolds in [5]. In different ambient spaces, slant, semi-slant, and pseudo-slant submanifolds have been investigated by many geometers [6–8]. The present paper aims to study quasi-bi-slant (qbs) submanifolds, including the classes which include the classes of slant, semi-slant, hemi-slant, bi-slant, and quasi-hemi-slant submanifolds as their particular cases.

Metallic structure was introduced Spinadel [9]. Let p and q be positive integers. Therefore, members of the metallic means family are a positive solution

$$x^2 - px - q = 0$$

and this number, called $(p,q)-$metallic numbers [10], denoted by

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}$$

In view of this family, in [10], the authors introduced the metallic structure which is given by J of type $(1,1)-$tensor field satisfying

$$J^2 = pJ + qI \tag{1.1}$$

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After that, some especially remarks on this structure have been studied by many papers [11–15]. Taking $p = 3$ and $q = 1$ in (1.1), then $\tilde{\Phi}$ is named bronze structure which satisfies [16]

$$\tilde{\Phi}^2 = 3\tilde{\Phi} + I$$

In [16], the authors studied the notion of bronze structure on manifolds, using the bronze mean defined by

$$\tilde{\eta}_{br} = \frac{3+\sqrt{13}}{2}$$

which is the positive solution

$$x^2 - 3x - 1 = 0$$

Recently, in [17], the author has studied the twin bronze Riemannian metric. After, some types of slant submanifolds of bronze Riemannian manifolds have been investigated in [18].

Section 2 of this study presents some basic notions needed for the next section. Section 3 introduces and exemplifies qbs submanifolds of bronze Riemannian manifolds. Section 4 concludes the paper by discussing future studies.

2. Preliminaries

This section provides some basic notions that are required in the following section. Assume that $\tilde{G}$ is a differentiable manifold with $\tilde{\Phi}$, $(1,1)-$ tensor field. If the equation

$$\tilde{\Phi}^2 = 3\tilde{\Phi} + I \tag{2.1}$$

satisfies, then $(\tilde{G}, \tilde{\Phi})$ is called a bronze manifold. If $(\tilde{G}, \tilde{\Phi})$ is a Riemannian manifold with $\tilde{\Phi}$ bronze structure, such that $\tilde{g}$ is $\tilde{\Phi}-$compatible

$$\tilde{g}(\tilde{\Phi}\partial_1, \partial_2) = \tilde{g}(\partial_1, \tilde{\Phi}\partial_2) \tag{2.2}$$

then $(\tilde{G}, \tilde{\Phi}, \tilde{g})$ is a bronze Riemannian manifold where $\partial_1, \partial_2 \in \Gamma(T\tilde{G})$. From (2.2),

$$\tilde{g}(\tilde{\Phi}\partial_1, \tilde{\Phi}\partial_2) = 3\tilde{g}(\tilde{\Phi}\partial_1, \partial_2) + \tilde{g}(\partial_1, \partial_2) \tag{2.3}$$

Example 2.1. [18] Suppose that $\mathbb{R}^4$ is a real space and gives a map by

$$\begin{array}{cccc} \tilde{\Phi}: & \mathbb{R}^4 & \longrightarrow & \mathbb{R}^4 \\ & (\omega_1, \omega_2, \omega_3, \omega_4) & \longrightarrow & (\tilde{\eta}_{br}\omega_1, \eta_{br}\omega_2, \tilde{\eta}_{br}\omega_3, \eta_{br}\omega_4) \end{array}$$

where $\tilde{\eta}_{br} = \frac{3+\sqrt{13}}{2}$ and $\eta_{br} = \frac{3-\sqrt{13}}{2}$. In this case, $\tilde{\Phi}$ satisfies the equation (2.1). Therefore, $(\mathbb{R}^4, \tilde{\Phi})$ is an example of bronze structure.

Assume that G is a submanifold of bronze Riemannian manifold $(\tilde{G}, \tilde{\Phi}, \tilde{g})$. Thus, the tangent space of $\tilde{G}$ can be decomposed by

$$T_x\tilde{G} = T_xG \perp T_x^{\perp}G$$

where x is a point on $\tilde{G}$. For all $\partial_1 \in \Gamma(TG)$,

$$\tilde{\Phi}\partial_1 = f_{br}\partial_1 + t_{br}\partial_1 \tag{2.4}$$

where $f_{br}\partial_1$ and $t_{br}\partial_1$, the tangential and normal parts of $\tilde{\Phi}\partial_1$, respectively. Similarly, for $N \in \Gamma(T^{\perp}G)$,

$$\tilde{\Phi}N = B_{br}N + C_{br}N \tag{2.5}$$

where $B_{br}N$ and $C_{br}N$ are the tangential and normal parts of $\tilde{\Phi}N$, respectively.

Moreover, Gauss and Weingarten equations are given by

$$\tilde{\#}_{\partial_1}\partial_2 = \#_{\partial_1}\partial_2 + h(\partial_1, \partial_2) \tag{2.6}$$

and

$$\tilde{\#}_{\partial_1}N = -A_N\partial_1 + \#^t_{\partial_1}N \tag{2.7}$$

where $\tilde{\#}$ and $\#$ are Levi-Civita connection on $(\tilde{G}, \tilde{g})$ and (G, g), respectively. In this article, we assume that

$$\tilde{\#}\tilde{\Phi} = 0 \tag{2.8}$$

i.e., $\tilde{G}$ is a locally bronze Riemannian manifold.

3. Main Results

In this section, we give some geometric characterizations and an example for qbs submanifold of $(\tilde{G}, \tilde{\Phi}, \tilde{g})$. Firstly, we give the following definition:

Definition 3.1. Suppose that (G, g) is a submanifold of $(\tilde{G}, \tilde{\Phi}, \tilde{g})$. Then G is called a qbs submanifold if the following conditions are satisfied:

i. TG admits the orthogonal direct decomposition as

$$TG = D_\gamma \perp D_\alpha \perp D_\beta$$

ii. The distributions D_γ is invariant, $\tilde{\Phi}(D_\gamma) = D_\gamma$

iii. $\tilde{\Phi}(D_\alpha)\perp D_\beta$ and $\tilde{\Phi}(D_\beta)\perp D_\alpha$

iv. The distributions D_α and D_β are slant with slant angle θ_α and θ_β.

If $\dim(D_\gamma) \neq 0$, $\dim(D_\alpha) \neq 0$, $\dim(D_\beta) \neq 0$ and $\theta\alpha, \theta_\beta \notin \{0, \frac{\pi}{2}\}$, then G is called a proper qbs submanifold.

Example 3.2. Suppose that $\mathbb{R}^6$ is the Euclidean space with the usual Euclidean metric. We define the bronze structure

$$\begin{array}{cccc} \tilde{\Phi}: & \mathbb{R}^6 & \longrightarrow & \mathbb{R}^6 \\ & (\omega_1, \omega_2, ..., \omega_6) & \longrightarrow & (\tilde{\eta}_{br}\omega_1, \tilde{\eta}_{br}\omega_2, \eta_{br}\omega_3, \tilde{\eta}_{br}\omega_4, \eta_{br}\omega_5, \tilde{\eta}_{br}\omega_6) \end{array}$$

where $\tilde{\eta}_{br} = \frac{3+\sqrt{13}}{2}$ and $\eta_{br} = \frac{3-\sqrt{13}}{2}$. Since (2.1) is achieved, then $(\mathbb{R}^6, \tilde{\Phi})$ is a bronze Riemannian manifold.

Assume that G is a submanifold of $(\mathbb{R}^6, \tilde{\Phi})$ defined by $\omega_1 = \tilde{\eta}_{br}u_1 + \eta_{br}u_2$, $\omega_2 = \eta_{br}u_1 + \tilde{\eta}_{br}u_2$, $\omega_3 = \cos su_3$, $\omega_4 = \sin su_3$, $\omega_5 = \sin su_4$, and $\omega_6 = \cos su_4$. The following are the vector fields that span the tangent space TG:

$$\tilde{\Psi}_1 = \tilde{\eta}_{br}\partial\omega_1 + \eta_{br}\partial\omega_2$$

$$\tilde{\Psi}_2 = \eta_{br}\partial\omega_1 + \tilde{\eta}_{br}\partial\omega_2$$

$$\tilde{\Psi}_3 = \cos s\partial\omega_3 + \sin s\partial\omega_4$$

and

$$\tilde{\Psi}_4 = \sin s\partial\omega_5 + \cos s\partial\omega_6$$

Putting $D_\gamma = Sp\{\tilde{\Psi}_1, \tilde{\Psi}_2\}$, $D_\alpha = Sp\{\tilde{\Psi}_3\}$, and $D_\beta = Sp\{\tilde{\Psi}_4\}$, then D_γ, D_α, and D_β satisfy the definition of qbs submanifold with θ_α and θ_β as its slant angle.

Taking Q_γ, Q_α, and Q_β orthogonal projections on D_γ, D_α, and D_β, respectively. Thus, for all $\partial_1 \in \Gamma(TG)$,

$$\partial_1 = Q_\gamma \partial_1 + Q_\alpha \partial_1 + Q_\beta \partial_1 \tag{3.1}$$

In view of (3.1) with (2.4),

$$\begin{aligned}\tilde{\Phi}\partial_1 &= \tilde{\Phi}Q_\gamma\partial_1 + \tilde{\Phi}Q_\alpha\partial_1 + \tilde{\Phi}Q_\beta\partial_1 \\ &= f_{br}Q_\gamma\partial_1 + t_{br}Q_\gamma\partial_1 + f_{br}Q_\alpha\partial_1 + t_{br}Q_\alpha\partial_1 + f_{br}Q_\beta\partial_1 + t_{br}Q_\beta\partial_1\end{aligned}$$

Since $\tilde{\Phi}(D_\gamma) = D_\gamma$, then

$$\tilde{\Phi}\partial_1 = f_{br}Q_\gamma\partial_1 + f_{br}Q_\alpha\partial_1 + t_{br}Q_\alpha\partial_1 + f_{br}Q_\beta\partial_1 + t_{br}Q_\beta\partial_1$$

We investigate the integrability conditions for the distributions involved in defining qbs submanifolds.

Theorem 3.3. Assume that G is a qbs submanifold of $(\tilde{G}, \tilde{\Phi}, \tilde{g})$. Then, D_γ is integrable if and only if

$$g(\#_{\partial_1} f_{br}\partial_2 - \#_{\partial_2} f_{br}\partial_1, f_{br}Q_\alpha\partial_3 + f_{br}Q_\beta\partial_3) = g(h(\partial_2, f_{br}\partial_1) - h(\partial_1, f_{br}\partial_2), t_{br}Q_\alpha\partial_3 + t_{br}Q_\beta\partial_3)$$

and

$$g(\#_{\partial_1}\partial_2 - \#_{\partial_2}\partial_1, f_{br}Q_\alpha\partial_3 + f_{br}Q_\beta\partial_3) = g(h(\partial_2, \partial_1) - h(\partial_1, \partial_2), t_{br}Q_\alpha\partial_3 + t_{br}Q_\beta\partial_3)$$

for $\partial_1, \partial_2 \in \Gamma(D_\gamma)$ and $\partial_3 \in \Gamma(D_\alpha \perp D_\beta)$.

Proof. In view of (2.3), (2.4), and (2.6) with (2.8), for all $\partial_1, \partial_2 \in \Gamma(D_\gamma)$ and $\partial_3 = Q_\alpha\partial_1 + Q_\beta\partial_1 \in \Gamma(D_\alpha \perp D_\beta)$,

$$\begin{aligned}\tilde{g}([\partial_1, \partial_2], \partial_3) &= \tilde{g}(\tilde{\Phi}[\partial_1, \partial_2], \tilde{\Phi}\partial_3) - 3\tilde{g}(\tilde{\Phi}[\partial_1, \partial_2], \partial_3) \\ &= \tilde{g}(\tilde{\#}_{\partial_1}\tilde{\Phi}\partial_2, \tilde{\Phi}\partial_3) - \tilde{g}(\tilde{\#}_{\partial_2}\tilde{\Phi}\partial_1, \tilde{\Phi}\partial_3) - 3\tilde{g}(\tilde{\#}_{\partial_1}\partial_2, \tilde{\Phi}\partial_3) + 3\tilde{g}(\tilde{\#}_{\partial_2}\partial_1, \tilde{\Phi}\partial_3) \\ &= g(\#_{\partial_1}\tilde{\Phi}\partial_2 + h(\partial_1, \tilde{\Phi}\partial_2), \tilde{\Phi}\partial_3) - g(\#_{\partial_2}\tilde{\Phi}\partial_1 + h(\partial_2, \tilde{\Phi}\partial_1), \tilde{\Phi}\partial_3) \\ &\quad -3g(\#_{\partial_1}\partial_2 + h(\partial_1, \partial_2), \tilde{\Phi}\partial_3) + 3g(\#_{\partial_2}\partial_1 + h(\partial_2, \partial_1), \tilde{\Phi}\partial_3) \\ &= g(\#_{\partial_1} f_{br}\partial_2, f_{br}Q_\alpha\partial_3 + f_{br}Q_\beta\partial_3) - g(\#_{\partial_2} f_{br}\partial_1, f_{br}Q_\alpha\partial_3 + f_{br}Q_\beta\partial_3) \\ &\quad +g(h(\partial_1, f_{br}\partial_2), t_{br}Q_\alpha\partial_3 + t_{br}Q_\beta\partial_3) - g(h(\partial_2, f_{br}\partial_1), t_{br}Q_\alpha\partial_3 + t_{br}Q_\beta\partial_3) \\ &\quad -3g(\#_{\partial_1}\partial_2, f_{br}Q_\alpha\partial_3 + f_{br}Q_\beta\partial_3) + 3g(\#_{\partial_2}\partial_1, f_{br}Q_\alpha\partial_3 + f_{br}Q_\beta\partial_3) \\ &\quad -3g(h(\partial_1, \partial_2), t_{br}Q_\alpha\partial_3 + t_{br}Q_\beta\partial_3) + 3g(h(\partial_2, \partial_1), t_{br}Q_\alpha\partial_3 + t_{br}Q_\beta\partial_3)\end{aligned}$$

□

Theorem 3.4. Assume that G is a qbs submanifold of $(\tilde{G}, \tilde{\Phi}, \tilde{g})$. Then, D_α is integrable if and only if

$$g(A_{t_{br}\partial_1}\partial_2 - A_{t_{br}\partial_2}\partial_1, f_{br}Q_\gamma\partial_3 + f_{br}Q_\beta\partial_3) = g(A_{t_{br}f_{br}\partial_2}\partial_1 - A_{t_{br}f_{br}\partial_1}\partial_2, \partial_3) + g(\#^\perp_{\partial_2} t_{br}\partial_1 - \#^\perp_{\partial_1} t_{br}\partial_2, t_{br}Q_\beta\partial_3)$$

and

$$g(\#_{\partial_1} f_{br}\partial_2 - \#_{\partial_2} f_{br}\partial_1, \partial_3) = g(A_{t_{br}\partial_2}\partial_1 - A_{t_{br}\partial_1}\partial_2, \partial_3)$$

for $\partial_1, \partial_2 \in \Gamma(D_\alpha)$ and $\partial_3 \in \Gamma(D_\gamma \perp D_\beta)$.

Proof. Using (2.3), (2.4), and (2.6) with (2.7), for all $\partial_1, \partial_2 \in \Gamma(D_\alpha)$, and $\partial_3 = Q_\gamma\partial_1 + Q_\beta\partial_1 \in \Gamma(D_\gamma \perp D_\beta)$,

$$\begin{aligned}\tilde{g}([\partial_1, \partial_2], \partial_3) &= \tilde{g}(\tilde{\Phi}[\partial_1, \partial_2], \tilde{\Phi}\partial_3) - 3\tilde{g}(\tilde{\Phi}[\partial_1, \partial_2], \partial_3) \\ &= \tilde{g}(\tilde{\#}_{\partial_1}\tilde{\Phi}\partial_2, \tilde{\Phi}\partial_3) - \tilde{g}(\tilde{\#}_{\partial_2}\tilde{\Phi}\partial_1, \tilde{\Phi}\partial_3) - 3\tilde{g}(\tilde{\Phi}\tilde{\#}_{\partial_1}\partial_2, \partial_3) + 3\tilde{g}(\tilde{\#}_{\partial_2}\partial_1, \partial_3) \\ &= \tilde{g}(\tilde{\#}_{\partial_1} t_{br}\partial_2, \tilde{\Phi}\partial_3) - \tilde{g}(\tilde{\#}_{\partial_2} t_{br}\partial_1, \tilde{\Phi}\partial_3) + \tilde{g}(\tilde{\#}_{\partial_1}\tilde{\Phi} f_{br}\partial_2, \partial_3) - \tilde{g}(\tilde{\#}_{\partial_2}\tilde{\Phi} f_{br}\partial_1, \partial_3) \\ &\quad -3\tilde{g}(\tilde{\#}_{\partial_1}\tilde{\Phi}\partial_2, \partial_3) + 3\tilde{g}(\tilde{\#}_{\partial_2}\tilde{\Phi}\partial_1, \partial_3)\end{aligned}$$

$$= g(A_{t_{br}\partial_1}\partial_2, \tilde{\Phi}\partial_3) - g(A_{t_{br}\partial_2}\partial_1, \tilde{\Phi}\partial_3) + g(\#^{\perp}_{\partial_1} t_{br}\partial_2, \tilde{\Phi}\partial_3) - g(\#^{\perp}_{\partial_2} t_{br}\partial_1, \tilde{\Phi}\partial_3)$$
$$+\tilde{g}(\tilde{\#}_{\partial_1} f^2_{br}\partial_2, \partial_3) - \tilde{g}(\tilde{\#}_{\partial_2} f^2_{br}\partial_1, \partial_3) - g(A_{t_{br}f_{br}\partial_2}\partial_1, \partial_3) + g(A_{t_{br}f_{br}\partial_1}\partial_2, \partial_3)$$
$$-3\tilde{g}(\tilde{\#}_{\partial_1} f_{br}\partial_2, \partial_3) + 3\tilde{g}(\tilde{\#}_{\partial_2} f_{br}\partial_1, \partial_3) - 3\tilde{g}(\tilde{\#}_{\partial_1} t_{br}\partial_2, \partial_3) + 3\tilde{g}(\tilde{\#}_{\partial_2} t_{br}\partial_1, \partial_3)$$

From the above equation,

$$\begin{aligned}(1 - \cos^2\theta_\alpha)\tilde{g}([\partial_1, \partial_2], \partial_3) = {} & g(A_{t_{br}\partial_1}\partial_2, f_{br}Q_\gamma\partial_3 + f_{br}Q_\beta\partial_3) - g(A_{t_{br}\partial_2}\partial_1, f_{br}Q_\gamma\partial_3 + f_{br}Q_\beta\partial_3) \\ & + g(\#^{\perp}_{\partial_1} t_{br}\partial_2, t_{br}Q_\beta\partial_3) - g(\#^{\perp}_{\partial_2} t_{br}\partial_1, t_{br}Q_\beta\partial_3) - g(A_{t_{br}f_{br}\partial_2}\partial_1, \partial_3) \\ & + g(A_{t_{br}f_{br}\partial_1}\partial_2, \partial_3) - 3g(\#_{\partial_1} f_{br}\partial_2, \partial_3) + 3g(\#_{\partial_2} f_{br}\partial_1, \partial_3) \\ & + 3g(A_{t_{br}\partial_2}\partial_1, \partial_3) - 3g(A_{t_{br}\partial_1}\partial_2, \partial_3)\end{aligned}$$

□

Theorem 3.5. Assume that G is a qbs submanifold of $(\tilde{G}, \tilde{\Phi}, \tilde{g})$. Then, D_β is integrable if and only if

$$g(A_{t_{br}\partial_1}\partial_2 - A_{t_{br}\partial_2}\partial_1, f_{br}Q_\gamma\partial_3 + f_{br}Q_\alpha\partial_3) = g(A_{t_{br}f_{br}\partial_2}\partial_1 - A_{t_{br}f_{br}\partial_1}\partial_2, \partial_3) + g(\#^{\perp}_{\partial_2} t_{br}\partial_1 - \#^{\perp}_{\partial_1} t_{br}\partial_2, t_{br}Q_\alpha\partial_3)$$

and

$$g(\#_{\partial_1} f_{br}\partial_2 - \#_{\partial_2} f_{br}\partial_1, \partial_3) = g(A_{t_{br}\partial_2}\partial_1 - A_{t_{br}\partial_1}\partial_2, \partial_3)$$

for $\partial_1, \partial_2 \in \Gamma(D_\beta)$ and $\partial_3 \in \Gamma(D_\gamma \perp D_\alpha)$.

PROOF. From (2.3), (2.4), and (2.6) with (2.7), for all $\partial_1, \partial_2 \in \Gamma(D_\beta)$ and $\partial_3 = Q_\gamma\partial_1 + Q_\alpha\partial_1 \in \Gamma(D_\gamma \perp D_\alpha)$,

$$\begin{aligned}\tilde{g}([\partial_1, \partial_2], \partial_3) = {} & \tilde{g}(\tilde{\Phi}[\partial_1, \partial_2], \tilde{\Phi}\partial_3) - 3\tilde{g}(\tilde{\Phi}[\partial_1, \partial_2], \partial_3) \\ = {} & \tilde{g}(\tilde{\#}_{\partial_1}\tilde{\Phi}\partial_2, \tilde{\Phi}\partial_3) - \tilde{g}(\tilde{\#}_{\partial_2}\tilde{\Phi}\partial_1, \tilde{\Phi}\partial_3) - 3\tilde{g}(\tilde{\Phi}\tilde{\#}_{\partial_1}\partial_2, \partial_3) + 3\tilde{g}(\tilde{\#}_{\partial_2}\partial_1, \partial_3) \\ = {} & \tilde{g}(\tilde{\#}_{\partial_1} t_{br}\partial_2, \tilde{\Phi}\partial_3) - \tilde{g}(\tilde{\#}_{\partial_2} t_{br}\partial_1, \tilde{\Phi}\partial_3) + \tilde{g}(\tilde{\#}_{\partial_1}\tilde{\Phi} f_{br}\partial_2, \partial_3) - \tilde{g}(\tilde{\#}_{\partial_2}\tilde{\Phi} f_{br}\partial_1, \partial_3) \\ & -3\tilde{g}(\tilde{\#}_{\partial_1}\tilde{\Phi}\partial_2, \partial_3) + 3\tilde{g}(\tilde{\#}_{\partial_2}\tilde{\Phi}\partial_1, \partial_3) \\ = {} & g(A_{t_{br}\partial_1}\partial_2, \tilde{\Phi}\partial_3) - g(A_{t_{br}\partial_2}\partial_1, \tilde{\Phi}\partial_3) + g(\#^{\perp}_{\partial_1} t_{br}\partial_2, \tilde{\Phi}\partial_3) - g(\#^{\perp}_{\partial_2} t_{br}\partial_1, \tilde{\Phi}\partial_3) \\ & +\tilde{g}(\tilde{\#}_{\partial_1} f^2_{br}\partial_2, \partial_3) - \tilde{g}(\tilde{\#}_{\partial_2} f^2_{br}\partial_1, \partial_3) - g(A_{t_{br}f_{br}\partial_2}\partial_1, \partial_3) + g(A_{t_{br}f_{br}\partial_1}\partial_2, \partial_3) \\ & -3\tilde{g}(\tilde{\#}_{\partial_1} f_{br}\partial_2, \partial_3) + 3\tilde{g}(\tilde{\#}_{\partial_2} f_{br}\partial_1, \partial_3) - 3\tilde{g}(\tilde{\#}_{\partial_1} t_{br}\partial_2, \partial_3) + 3\tilde{g}(\tilde{\#}_{\partial_2} t_{br}\partial_1, \partial_3)\end{aligned}$$

which implies

$$\begin{aligned}(1 - \cos^2\theta_\beta)\tilde{g}([\partial_1, \partial_2], \partial_3) = {} & g(A_{t_{br}\partial_1}\partial_2, f_{br}Q_\gamma\partial_3 + f_{br}Q_\alpha\partial_3) - g(A_{t_{br}\partial_2}\partial_1, f_{br}Q_\gamma\partial_3 + f_{br}Q_\alpha\partial_3) \\ & + g(\#^{\perp}_{\partial_1} t_{br}\partial_2, t_{br}Q_\alpha\partial_3) - g(\#^{\perp}_{\partial_2} t_{br}\partial_1, t_{br}Q_\alpha\partial_3) - g(A_{t_{br}f_{br}\partial_2}\partial_1, \partial_3) \\ & + g(A_{t_{br}f_{br}\partial_1}\partial_2, \partial_3) - 3g(\#_{\partial_1} f_{br}\partial_2, \partial_3) + 3g(\#_{\partial_2} f_{br}\partial_1, \partial_3) \\ & + 3g(A_{t_{br}\partial_2}\partial_1, \partial_3) - 3g(A_{t_{br}\partial_1}\partial_2, \partial_3)\end{aligned}$$

□

Theorem 3.6. Assume that G is a qbs submanifold of $(\tilde{G}, \tilde{\Phi}, \tilde{g})$. Then, D_γ defines totally geodesic foliation on G if and only if

$$g(\#_{\partial_1} f_{br}\partial_2, f_{br}\partial_3) = -g(h(\partial_1, f_{br}\partial_2), t_{br}\partial_3), g(\#_{\partial_1}\partial_2, f_{br}\partial_3) = -g(h(\partial_1, \partial_2), t_{br}\partial_3)$$

and

$$g(\#_{\partial_1} f_{br}\partial_2, B_{br}N) = -g(h(\partial_1, f_{br}\partial_2), C_{br}N), g(\#_{\partial_1}\partial_2, B_{br}N) = -g(h(\partial_1, \partial_2), C_{br}N)$$

for $\partial_1, \partial_2 \in \Gamma(D_\gamma)$, $\partial_3 \in \Gamma(D_\alpha \perp D_\beta)$, and $N \in \Gamma(T^{\perp}G)$.

PROOF. Using (2.3), (2.4), and (2.8), for all $\partial_1, \partial_2 \in \Gamma(D_\gamma)$ and $\partial_3 = Q_\alpha\partial_1 + Q_\beta\partial_1 \in \Gamma(D_\alpha \perp D_\beta)$

$$\begin{aligned}\tilde{g}(\tilde{\#}_{\partial_1}\partial_2, \partial_3) &= \tilde{g}(\tilde{\Phi}\tilde{\#}_{\partial_1}\partial_2, \tilde{\Phi}\partial_3) - 3\tilde{g}(\tilde{\#}_{\partial_1}\partial_2, \tilde{\Phi}\partial_3)\\ &= \tilde{g}(\tilde{\#}_{\partial_1} f_{br}\partial_2, f_{br}\partial_3 + t_{br}\partial_3) - 3\tilde{g}(\tilde{\#}_{\partial_1}\partial_2, f_{br}\partial_3 + t_{br}\partial_3)\\ &= g(\#_{\partial_1} f_{br}\partial_2, f_{br}\partial_3) + g(h(\partial_1, f_{br}\partial_2), t_{br}\partial_3) - 3g(\#_{\partial_1}\partial_2, f_{br}\partial_3)\\ &\quad -3g(h(\partial_1, \partial_2), t_{br}\partial_3)\end{aligned}$$

In view of (2.8), for $N \in \Gamma(T^\perp G)$,

$$\begin{aligned}\tilde{g}(\tilde{\#}_{\partial_1}\partial_2, N) &= \tilde{g}(\tilde{\#}_{\partial_1}\tilde{\Phi}\partial_2, \tilde{\Phi}N) - 3\tilde{g}(\tilde{\#}_{\partial_1}\partial_2, \tilde{\Phi}N)\\ &= \tilde{g}(\tilde{\#}_{\partial_1} f_{br}\partial_2, B_{br}N + C_{br}N) - 3\tilde{g}(\tilde{\#}_{\partial_1}\partial_2, B_{br}N + C_{br}N)\\ &= g(\#_{\partial_1} f_{br}\partial_2, B_{br}N) + g(h(\partial_1, f_{br}\partial_2), C_{br}N) - 3g(\#_{\partial_1}\partial_2, B_{br}N) - 3g(h(\partial_1, \partial_2), C_{br}N)\end{aligned}$$

□

Theorem 3.7. Assume that G is a qbs submanifold of $(\tilde{G}, \tilde{\Phi}, \tilde{g})$. Then, D_α defines totally geodesic foliation on G if and only if

$$g(A_{t_{br}f_{br}\partial_2}\partial_1, \partial_3) + g(A_{t_{br}\partial_2}\partial_1, f_{br}\partial_3) = g(\#^\perp_{\partial_1} t_{br}\partial_2, t_{br}Q_\beta\partial_3), g(A_{t_{br}\partial_2}\partial_1, \partial_3) = g(\#^\perp_{\partial_1} f_{br}\partial_2, \partial_3)$$

and

$$g(A_{t_{br}\partial_2}\partial_1, B_{br}N) - g(\#^\perp_{\partial_1} t_{br}f_{br}\partial_2, N) = g(\#^\perp_{\partial_1} t_{br}\partial_2, C_{br}N), g(\#_{\partial_1}\partial_2, B_{br}N) = -g(h(\partial_1, \partial_2), C_{br}N)$$

for $\partial_1, \partial_2 \in \Gamma(D_\alpha)$, $\partial_3 \in \Gamma(D_\gamma \perp D_\beta)$, and $N \in \Gamma(T^\perp G)$.

PROOF. In view of (2.3) and (2.4) with (2.8), for all $\partial_1, \partial_2 \in \Gamma(D_\alpha)$ and $\partial_3 = Q_\gamma\partial_1 + Q_\beta\partial_1 \in \Gamma(D_\gamma \perp D_\beta)$,

$$\begin{aligned}\tilde{g}(\tilde{\#}_{\partial_1}\partial_2, \partial_3) &= \tilde{g}(\tilde{\#}_{\partial_1}\tilde{\Phi}\partial_2, \tilde{\Phi}\partial_3) - 3\tilde{g}(\tilde{\#}_{\partial_1}\tilde{\Phi}\partial_2, \partial_3)\\ &= \tilde{g}(\tilde{\#}_{\partial_1} f_{br}\partial_2, \tilde{\Phi}\partial_3) + \tilde{g}(\tilde{\#}_{\partial_1} t_{br}\partial_2, \tilde{\Phi}\partial_3) - 3\tilde{g}(\tilde{\#}_{\partial_1} f_{br}\partial_2, \partial_3) - 3\tilde{g}(\tilde{\#}_{\partial_1} t_{br}\partial_2, \partial_3)\\ &= \cos\theta_\alpha \tilde{g}(\tilde{\#}_{\partial_1}\partial_2, \partial_3) - g(A_{t_{br}f_{br}\partial_2}\partial_1, \partial_3) - g(A_{t_{br}\partial_2}\partial_1, f_{br}Q_\gamma\partial_3 + f_{br}Q_\beta\partial_3)\\ &\quad +g(\#^\perp_{\partial_1} t_{br}\partial_2, t_{br}Q_\beta\partial_3) - 3g(\#_{\partial_1} f_{br}\partial_2, \partial_3) + 3g(A_{t_{br}\partial_2}\partial_1, \partial_3)\end{aligned}$$

Similarly, by (2.5), for $N \in \Gamma(T^\perp G)$,

$$\begin{aligned}\tilde{g}(\tilde{\#}_{\partial_1}\partial_2, N) &= \tilde{g}(\tilde{\#}_{\partial_1}\tilde{\Phi}\partial_2, \tilde{\Phi}N) - 3\tilde{g}(\tilde{\#}_{\partial_1}\partial_2, \tilde{\Phi}N)\\ &= \tilde{g}(\tilde{\#}_{\partial_1} f_{br}\partial_2, \tilde{\Phi}N) + \tilde{g}(\tilde{\#}_{\partial_1} t_{br}\partial_2, \tilde{\Phi}N) - 3\tilde{g}(\tilde{\#}_{\partial_1}\partial_2, B_{br}N + C_{br}N)\\ &= \cos\theta_\alpha \tilde{g}(\tilde{\#}_{\partial_1}\partial_2, N) + g(\#^\perp_{\partial_1} t_{br}f_{br}\partial_2, N) - g(A_{t_{br}\partial_2}\partial_1, B_{br}N)\\ &\quad +g(\#^\perp_{\partial_1} t_{br}\partial_2, C_{br}N) - 3g(\#_{\partial_1}\partial_2, B_{br}N) - 3g(h(\partial_1, \partial_2), C_{br}N)\end{aligned}$$

□

Theorem 3.8. Assume that G is a qbs submanifold of $(\tilde{G}, \tilde{\Phi}, \tilde{g})$. Then, D_β defines totally geodesic foliation on G if and only if

$$g(A_{t_{br}f_{br}\partial_2}\partial_1, \partial_3) + g(A_{t_{br}\partial_2}\partial_1, f_{br}\partial_3) = g(\#^\perp_{\partial_1} t_{br}\partial_2, t_{br}Q_\alpha\partial_3), g(A_{t_{br}\partial_2}\partial_1, \partial_3) = g(\#^\perp_{\partial_1} f_{br}\partial_2, \partial_3)$$

and

$$g(A_{t_{br}\partial_2}\partial_1, B_{br}N) - g(\#^\perp_{\partial_1} t_{br}f_{br}\partial_2, N) = g(\#^\perp_{\partial_1} t_{br}\partial_2, C_{br}N), g(\#_{\partial_1}\partial_2, B_{br}N) = -g(h(\partial_1, \partial_2), C_{br}N)$$

for $\partial_1, \partial_2 \in \Gamma(D_\beta)$, $\partial_3 \in \Gamma(D_\gamma \perp D_\alpha)$, and $N \in \Gamma(T^\perp G)$.

PROOF. In view of (2.3) and (2.4) with (2.8), for all $\partial_1, \partial_2 \in \Gamma(D_\alpha)$ and $\partial_3 = Q_\gamma\partial_1 + Q_\beta\partial_1 \in \Gamma(D_\gamma \perp D_\beta)$

$$\begin{aligned}
\tilde{g}(\tilde{\#}_{\partial_1}\partial_2, \partial_3) &= \tilde{g}(\tilde{\#}_{\partial_1}\tilde{\Phi}\partial_2, \tilde{\Phi}\partial_3) - 3\tilde{g}(\tilde{\#}_{\partial_1}\tilde{\Phi}\partial_2, \partial_3) \\
&= \tilde{g}(\tilde{\#}_{\partial_1} f_{br}\partial_2, \tilde{\Phi}\partial_3) + \tilde{g}(\tilde{\#}_{\partial_1} t_{br}\partial_2, \tilde{\Phi}\partial_3) - 3\tilde{g}(\tilde{\#}_{\partial_1} f_{br}\partial_2, \partial_3) - 3\tilde{g}(\tilde{\#}_{\partial_1} t_{br}\partial_2, \partial_3) \\
&= \cos\theta_\beta \tilde{g}(\tilde{\#}_{\partial_1}\partial_2, \partial_3) - g(A_{t_{br} f_{br}\partial_2}\partial_1, \partial_3) - g(A_{t_{br}\partial_2}\partial_1, f_{br}Q_\gamma\partial_3 + f_{br}Q_\alpha\partial_3) \\
&\quad + g(\#^\perp_{\partial_1} t_{br}\partial_2, t_{br}Q_\alpha\partial_3) - 3g(\#_{\partial_1} f_{br}\partial_2, \partial_3) + 3g(A_{t_{br}\partial_2}\partial_1, \partial_3)
\end{aligned}$$

Using (2.5), for $N \in \Gamma(T^\perp G)$,

$$\begin{aligned}
\tilde{g}(\tilde{\#}_{\partial_1}\partial_2, N) &= \tilde{g}(\tilde{\#}_{\partial_1}\tilde{\Phi}\partial_2, \tilde{\Phi}N) - 3\tilde{g}(\tilde{\#}_{\partial_1}\partial_2, \tilde{\Phi}N) \\
&= \tilde{g}(\tilde{\#}_{\partial_1} f_{br}\partial_2, \tilde{\Phi}N) + \tilde{g}(\tilde{\#}_{\partial_1} t_{br}\partial_2, \tilde{\Phi}N) - 3\tilde{g}(\tilde{\#}_{\partial_1}\partial_2, B_{br}N + C_{br}N) \\
&= \cos\theta_\beta \tilde{g}(\tilde{\#}_{\partial_1}\partial_2, N) + g(\#^\perp_{\partial_1} t_{br} f_{br}\partial_2, N) - g(A_{t_{br}\partial_2}\partial_1, B_{br}N) \\
&\quad + g(\#^\perp_{\partial_1} t_{br}\partial_2, C_{br}N) - 3g(\#_{\partial_1}\partial_2, B_{br}N) - 3g(h(\partial_1, \partial_2), C_{br}N)
\end{aligned}$$

□

4. Conclusion

This paper has focused on a specific class of submanifolds on bronze Riemannian manifolds known as qbs submanifolds. It has presented the integrability conditions for the distribution associated with qbs submanifolds within the context of bronze Riemannian manifolds and provided illustrative examples to elucidate these conditions. Future research can further explore additional types of slant submanifolds, such as semi-slant and bi-slant submanifolds, within this framework, aiming to identify non-trivial examples for them and analyze their geometric properties.

Author Contributions

All the authors contributed equally to this work. They also read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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