Volume 14, Issue 3, Page 10-29, 2025 https://doi.org/10.46810/tdfd.1633095 Research Article

BİNGÖL UNIVERSITY

Türk Doğa ve Fen Dergisi
Turkish Journal of Nature and Science
www.dergipark.gov.tr/tdfd

Solitary Wave Solution of the Third Order Gilson-Pickering Equation Via the Abel's Equation with Variable Coefficients

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(Received: 04.02.2025, Accepted: 18.06.2025, Online Publication: 26.09.2025)

Keywords
The Gilson Pİckering equation, Exact solution, Variable Coefficient Generalized Abel's equation (VCAEM)

Abstract: A novel analytical method, termed the Variable Coefficient Second Degree Generalized Abel Equation Method, is presented in this paper. It has been specifically developed to address the complexities of the Gilson Pickering equation—an important nonlinear partial differential equation encountered in various physical applications. This equation is recognized as a general case of several significant nonlinear partial differential equations, including the Fornberg Whitham, Rosenau Hyman, and Fuchssteiner-Fokas-Camassa-Holm equations. Unlike conventional approaches, which are typically based on constant coefficient ordinary differential equations (ODEs) or auxiliary equations, a unique framework based on variable coefficient ODEs integrated within a subequation structure is introduced by this method.. By applying the method to the Gilson Pickering equation, new exact analytical solutions are derived. The correctness of the technique is validated, and its superior efficiency and robustness are highlighted through these results. The intricate dynamics of the equation are effectively captured, demonstrating the method's suitability for modeling phenomena in fluid dynamics, nonlinear optics, and wave propagation. Furthermore, the potential application of the method to a broader class of nonlinear partial differential equations across mathematical physics is implied by its generalizability. The importance of advancing analytical tools to better understand and resolve complex physical systems is emphasized by the findings, thereby marking a significant contribution to the analytical study of nonlinear differential equations. Ultimately, an expansion of the repertoire of solution strategies available to researchers in applied mathematics and theoretical physics is achieved through this work.

Üçüncü Mertebe Gilson-Pickering Denkleminin Değişken Katsayılı Abel Denklemi Yoluyla Soliter Dalga Çözümü

Anahtar Kelimeler
Gilson Pickering denklemi, Tam çözüm, Değişken katsayılı genelleştirilmiş Abel denklemi

Öz: Bu makalede, Değişken Katsayılı İkinci Derece Genelleştirilmiş Abel Denklemi Yöntemi olarak adlandırılan yeni bir analitik yöntem sunulmaktadır. Bu yöntem, çeşitli fiziksel uygulamalarda karşılaşılan önemli bir doğrusal olmayan kısmi diferansiyel denklem olan Gilson Pickering denkleminin karmaşıklıklarını ele almak için özel olarak geliştirilmiştir. Bu denklem, Fornberg-Whitham, Rosenau-Hyman ve Fuchssteiner-Fokas-Camassa-Holm denklemleri de dahil olmak üzere, diğer bazı önemli doğrusal olmayan kısmi diferansiyel denklemlerin genel bir durumu olarak kabul edilmektedir. Genellikle sabit katsayılı adi diferansiyel denklemlere (ADD) veya yardımcı denklemlere dayanan geleneksel yaklaşımların aksine, bu yöntem alt denklem yapısına entegre edilmiş değişken katsayılı ADD'lere dayanan benzersiz bir çerçeve sunmaktadır. Yöntem, Gilson Pickering denklemine uygulanarak yeni, tam analitik çözümler elde edilmiştir. Bu sonuçlar, tekniğin doğruluğunu kanıtlamakta ve üstün verimliliğini ve sağlamlığını ortaya koymaktadır. Yöntemin denklemin karmaşık dinamiklerini etkin şekilde yakalaması, akışkanlar dinamiği, optik ve dalga yayılımı gibi alanlardaki olguları modellemede uygunluğunu göstermiştir. Genel geçerliliği sayesinde, yöntem daha geniş bir doğrusal olmayan diferansiyel denklem sınıfına uygulanabilir. Bu çalışma, karmaşık fiziksel sistemleri çözmek için analitik araçların geliştirilmesinin önemine dikkat çekerek, ilgili alanlardaki çözüm stratejilerini genişletmektedir.

1 Introduction

Nonlinear partial differential equations (PDEs) hold a profound significance in understanding and modeling various real-world phenomena across diverse fields such as physics, engineering, biology, and finance. Unlike their linear counterparts, these equations encompass intricate relationships where the effects are not proportional to the causes, leading to complex and often unexpected behaviors. From describing fluid dynamics and electromagnetism to population dynamics and wave propagation, nonlinear PDEs capture the intricate interplay of multiple factors, making them indispensable in modeling phenomena exhibiting nonlinearity, chaos, and emergent behavior. However, solving nonlinear PDEs presents formidable challenges due to their inherent complexity, often requiring sophisticated numerical methods, computational techniques, and mathematical analysis [1, 2, 3, 4, 5]. Despite these challenges, mastering nonlinear PDEs unlocks a deeper understanding of the rich tapestry of phenomena that shape our world, enabling scientists and engineers to tackle some of the most pressing challenges facing humanity.

Analytical methods for solving nonlinear PDEs play a pivotal role in understanding the fundamental behavior and properties of complex systems in various fields of science and engineering. Unlike numerical techniques which rely on approximations and computational power, analytical methods aim to derive exact solutions or obtain insightful approximations that offer profound insights into the underlying mechanisms governing nonlinear phenomena. These methods not only provide a deeper understanding of the system's behavior but also serve as the foundation for developing numerical algorithms and computational models. Furthermore, analytical solutions offer invaluable benchmarks for validating numerical simulations and experimental observations, ensuring the accuracy and reliability of predictive models. Moreover, analytical approaches often reveal hidden symmetries, conservation laws, and qualitative features of solutions, shedding light on the underlying physics and guiding the design of efficient algorithms and control strategies. In essence, the importance of analytical methods for solving nonlinear PDEs lies in their ability to unveil the intricate dynamics of complex systems, enabling scientists and engineers to make informed decisions and advance our understanding of the world around us [6, 7, 8, 9, 10, 11].

This paper examines the nonlinear Gilson Pickering (GP) equation of third order [12]:

$$u_t + 2ku_x - \alpha uu_x - \beta u_x u_{xx} - \omega u_{xxt} - \varpi uu_{xxx} = 0. \quad (1)$$

The nonlinear GP equation holds significant physical relevance due to its ability to model complex wave phenomena in various real-world systems. It serves as a generalized framework that encompasses several well-known nonlinear partial differential equations, making it applicable in diverse fields such as fluid dynamics, nonlinear optics, and plasma physics. In particular, it captures the evolution of nonlinear wave structures, including solitary waves and dispersive shock waves, which are essential in understanding water wave propagation, optical pulse dynamics in fiber optics, and wave interactions in plasmas. Its versatility and rich mathematical structure make it a valuable tool for describing and predicting behaviors in systems where nonlinearity and dispersion play crucial roles.

The GP equation was chosen due to its general form, which encompasses several important nonlinear partial differential equations frequently appearing in physical models. Topological properties and wave structures of Eq. (1) are studied in [13]. Some parametric wave solutions which are periodic, solitary, and unbounded wave solutions are reported in [14]. In [15], approximate solutions of GP equation are considered by a meshless method base on the thin plate radial basics functions. The sine Gordon expansion method utilized to contract some exact solution in the form of shock wave, topological, compound topological and nontopological soliton solutions in [16].

For arbitrary real constants, namely α, β, k, ϖ, and ω, the equation (1) transforms into well known nonlinear PDEs under specific parameter selections:

(1) Fornberg-Whitham (FW) equation [17, 18]:

$$u_t + u_x + uu_x - 3u_xu_{xx} - u_{txx} - \varpi uu_{xxx} = 0, \quad (\alpha = -1,\ \beta = 3,\ k = 0.5,\ \omega = 1). \tag{2}$$

(2) Rosenau-Hyman (RH) equation [19, 20]:

$$u_t - uu_x - 3u_xu_{xx} - \varpi uu_{xxx} = 0, \quad (\alpha = 1,\ \beta = 3,\ k = 0,\ \omega = 0). \tag{3}$$

(3) Fuchssteiner-Fokas-Camassa-Holm (FFCH) equation [21, 22]:

$$u_t + 2ku_x + uu_x - 2u_xu_{xx} - u_{txx} - \varpi uu_{xxx} = 0, \quad (\alpha = -1,\ \beta = 2,\ \omega = 1). \tag{4}$$

By employing the substitution:

$$u(x,t) = U(\xi), \quad \xi = x - \nu t, \tag{5}$$

and substituting this transformation (5) into Eq. (1), we derive the subsequent equation:

$$(2k - \nu)U' + \omega\nu U''' - \varpi UU''' - \alpha UU' - \beta U'U'' = 0. \tag{6}$$

Integrating equation (6) once, we get

$$(2k - \nu)U + \omega\nu U'' - \frac{\alpha}{2}U^2 + \left(\frac{\varpi - \beta}{2}\right)(U')^2 - \varpi UU'' = c, \tag{7}$$

where c is a constant of integration.

The remainder of this paper is organized as follows: In Section 2, we introduce the VCAEM and detail the analytical framework developed for solving the GP equation. Section 3 presents the main analytical results obtained using the proposed method, including explicit solutions and their derivations. In Section 4, we discuss the physical and mathematical implications of the results, highlighting the method's effectiveness. Finally, Section 5 concludes the paper with a summary of findings and a discussion on the broader impact and potential future applications of the method in nonlinear science and mathematical physics.

2 Methodology

In this article, we present an innovative method inspired by Hashemi's work in [23, 24], which relies on the VCAEM. Let us explore:

$$\Omega\left(\xi, \mathcal{P}, \mathcal{P}', \ldots, \mathcal{P}^{(n)}\right) = 0.$$

Here, $\Omega\left(\xi, \mathcal{P}, \mathcal{P}', \ldots, \mathcal{P}^{(n)}\right)$ denotes a polynomial w.r.t. $\mathcal{P}, \mathcal{P}', \ldots, \mathcal{P}^{(n)}$, where the coefficients vary according to the ξ.

In this research, we introduce a technique in which the solutions to the specified second order Abel's equation are also solutions to the equation represented in (7), as outlined below:

$$\mathcal{P}' = \vartheta_2(\xi)\mathcal{P}^2 + \vartheta_1(\xi)\mathcal{P} + \vartheta_0(\xi), \tag{8}$$

where $\vartheta_j(\xi) \in \mathcal{C}^n(0 \leq j \leq 2)$.

For different types of differential equations, it is possible to utilize a generalized Abel's equation with different order. In general, an n^{th} order Abel's equation has the following form:

$$\mathcal{P}' = \sum_{i=0}^{n} \vartheta_i(\xi)\mathcal{P}^i, \tag{9}$$

where $\vartheta_j(\xi) \in \mathcal{C}^n(j \in \{0, 1, ..., n\})$.

We begin by introducing a method to derive equations (8) from the initial equation (7). Letting $\mathcal{P} = \mathcal{P}(\xi)$ represent any solution of (8), differentiating w.r.t. ξ yields:

$$\mathcal{P}''(\xi) = \vartheta_2'\mathcal{P}^2 + \vartheta_1'\mathcal{P} + \vartheta_0' + \mathcal{P}'\left[2\vartheta_2\mathcal{P} + \vartheta_1\right] \tag{10}$$

Moreover, in the general case from the Eq. (9), we have

$$\mathcal{P}'' = \sum_{j=0}^{n}\left[\vartheta_j'(\xi)\mathcal{P}^j + \mathcal{P}'\left(j\vartheta_j(\xi)\mathcal{P}^{j-1}\right)\right], \tag{11}$$

and therefore

$$\mathcal{P}'' = \sum_{j=0}^{n}\left[\vartheta_j'(\xi)\mathcal{P}^j + \left(j\vartheta_j(\xi)\mathcal{P}^{j-1}\right)\sum_{i=0}^{n}\vartheta_i(\xi)\mathcal{P}^i\right]. \tag{12}$$

Therefore, in the second order case, from (8) we have

$$\mathcal{P}''(\xi) = 2\vartheta_2^2\mathcal{P}^3 + \left(\vartheta_2' + 3\vartheta_1\vartheta_2\right)\mathcal{P}^2 + \left(\vartheta_1' + \vartheta_1^2 + 2\vartheta_0\vartheta_2\right)\mathcal{P} + \left(\vartheta_0' + \vartheta_0\vartheta_1\right). \tag{13}$$

The core concept of this new approach is explained as follows. By inserting polynomial expressions for $\mathcal{P}', \mathcal{P}'', \ldots, \mathcal{P}^{(n)}$ into $\Xi\left(\xi, \mathcal{P}, \mathcal{P}', \ldots, \mathcal{P}^{(n)}\right)$, one obtains a polynomial $\Omega(\mathcal{P})$ in terms of $\mathcal{P}$. Setting the coefficients of $\Omega(\mathcal{P})$ to zero leads to an overdetermined system of algebraic and ordinary differential equations (ODEs) for $\vartheta_i(\xi)$, $0 \le i \le 2$. If this system admits solutions in the form of $\vartheta_0(\xi), \vartheta_1(\xi), \vartheta_2(\xi)$, then any solution to (8) will also satisfy (7).

3 Main results

In this present investigation, we employ the subsequent generalized Abel's equation to solve Eq. (7):

$$\frac{d}{d\xi}\mathcal{P}(\xi) = \vartheta_2(\xi)\mathcal{P}^2(\xi) + \vartheta_1(\xi)\mathcal{P}(\xi) + \vartheta_0(\xi). \tag{14}$$

Inserting (14) into Eq. (7) results in a fifth degree polynomial with respect to $\mathcal{P}$, expressed as:

$$\sum_{i=0}^{4} S_i\mathcal{P}^i = 0,$$

where

$$S_0 = 2\nu\omega\left(\frac{d}{d\xi}\vartheta_0(\xi)\right) + 2\nu\omega\vartheta_0(\xi)\,\vartheta_1(\xi) + (\varpi - \beta)\,\vartheta_0^2(\xi) - 2c,$$

$$S_1 = 4\omega\nu\,\vartheta_0(\xi)\,\vartheta_2(\xi) + 2\omega\nu\vartheta_1^2(\xi) - 2\beta\vartheta_0(\xi)\,\vartheta_1(\xi) + 2\omega\nu\left(\frac{d}{d\xi}\vartheta_1(\xi)\right) - 2\varpi\left(\frac{d}{d\xi}\vartheta_0(\xi)\right) + 4k - 2\nu,$$

$$S_2 = -2\varpi\left(\frac{d}{d\xi}\vartheta_1(\xi)\right) + 2\omega\nu\left(\frac{d}{d\xi}\vartheta_2(\xi)\right) - 2\left(-3\omega\nu\vartheta_1(\xi) + \vartheta_0(\xi)\,(\beta + \varpi)\right)\vartheta_2(\xi) - (\beta + \varpi)\,\vartheta_1^2(\xi) - \alpha,$$

$$S_3 = -2\varpi\left(\frac{d}{d\xi}\vartheta_2(\xi)\right) + 4\omega\nu\,\vartheta_2^2(\xi) - 2\,(\beta + 2\varpi)\,\vartheta_1(\xi)\,\vartheta_2(\xi),$$

$$S_4 = -\,(\beta + 3\varpi)\,\vartheta_2^2(\xi).$$

Considering $S_i = 0,\ i = 0, \ldots, 4$, leads to a system of differential algebraic equations with the subsequent sets of solutions:

Family 1:

$$\vartheta_0(\xi) = -\frac{3\left(\beta - 3\omega\alpha\right)c}{4\left(9c\,\omega^2\alpha^2 - 18k^2\omega^2\alpha - 6c\beta\omega\alpha + c\,\beta^2\right)\beta\sqrt{\frac{-3}{72c\beta\,\omega^2\alpha^2 - 144\beta\,k^2\omega^2\alpha - 48c\,\beta^2\omega\alpha + 8c\,\beta^3}}},$$
$$\vartheta_1(\xi) = 12\omega k\sqrt{\frac{-3}{72c\beta\,\omega^2\alpha^2 - 144\beta\,k^2\omega^2\alpha - 48c\,\beta^2\omega\alpha + 8c\,\beta^3}}\,\alpha,$$
$$\vartheta_2(\xi) = \sqrt{\frac{-3}{72c\beta\,\omega^2\alpha^2 - 144\beta\,k^2\omega^2\alpha - 48c\,\beta^2\omega\alpha + 8c\,\beta^3}}\,\left(-3\omega\alpha + \beta\right)\alpha,$$
$$\nu = \frac{2\beta k}{\beta - 3\omega\alpha},\ \varpi = -\frac{\beta}{3}.$$

In this scenario, (14) transforms to

$$\frac{d}{d\xi}\mathcal{P}(\xi) = \sqrt{\frac{-3}{72c\beta\,\omega^2\alpha^2 - 144\beta\,k^2\omega^2\alpha - 48c\,\beta^2\omega\alpha + 8c\,\beta^3}}\,\left(-3\omega\alpha + \beta\right)\alpha\,\mathcal{P}^2\left(\xi\right) \tag{15}$$
$$+12\omega k\sqrt{\frac{-3}{72c\beta\,\omega^2\alpha^2 - 144\beta\,k^2\omega^2\alpha - 48c\,\beta^2\omega\alpha + 8c\,\beta^3}}\,\alpha\,\mathcal{P}\left(\xi\right)$$
$$-\frac{3\left(\beta - 3\omega\alpha\right)c}{4\left(9c\,\omega^2\alpha^2 - 18k^2\omega^2\alpha - 6c\beta\omega\alpha + c\,\beta^2\right)\beta\sqrt{\frac{-3}{72c\beta\,\omega^2\alpha^2 - 144\beta\,k^2\omega^2\alpha - 48c\,\beta^2\omega\alpha + 8c\,\beta^3}}}, \tag{16}$$

with exact solution:

$$\mathcal{P}(\xi) = \frac{-\sqrt{18}\,\tan\left(\frac{\sqrt{9}\,\sqrt{\alpha\Phi}\,(C_1+\xi)\sqrt{3}}{2(9c\,\omega^2\alpha^2 - 18k^2\omega^2\alpha - 6c\beta\omega\alpha + c\,\beta^2)\beta\sqrt{\frac{-1}{9\beta\Phi}}}\right)\sqrt{\alpha\Phi} - 6k\omega\alpha}{\alpha\left(-3\omega\alpha + \beta\right)}, \tag{17}$$

where C_1 is an arbitrary constant, and $\Phi = \left(\omega\alpha - \frac{\beta}{3}\right)^2 c - 2k^2\omega^2\alpha$. Hence, based on (7), and (17), the first exact solution in the form of a Kink soliton solution can be articulated as follows:

$$u(x,t) = \frac{-\sqrt{18}\,\tan\left(\frac{\sqrt{9}\,\sqrt{\alpha\Phi}\,\left(C_1+x-\left(\frac{2\beta k}{\beta-3\omega\alpha}\right)t\right)\sqrt{3}}{2(9c\,\omega^2\alpha^2 - 18k^2\omega^2\alpha - 6c\beta\omega\alpha + c\,\beta^2)\beta\sqrt{\frac{-1}{9\beta\Phi}}}\right)\sqrt{\alpha\Phi} - 6k\omega\alpha}{\alpha\left(-3\omega\alpha + \beta\right)}. \tag{18}$$

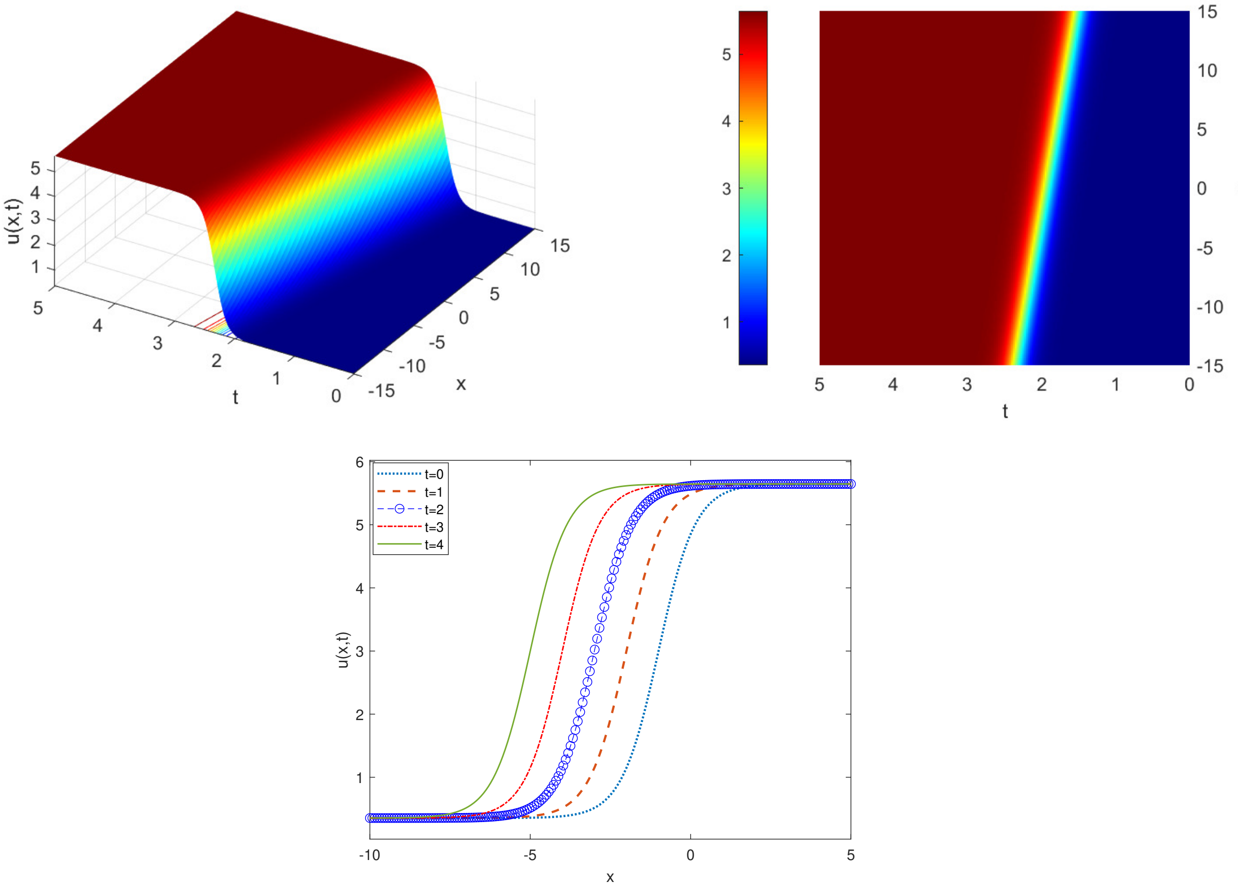

Figure 1: 2D, 3D and contour plots of (18).

Family 2:

$$\vartheta_0(\xi) = -\frac{4k\sqrt{-\frac{3\alpha}{2\beta}}}{3\alpha},\ \vartheta_1(\xi) = \sqrt{-\frac{3\alpha}{2\beta}},\ \vartheta_2(\xi) = 0,\ \nu = \frac{16k^2 - 9\alpha c}{12k},\ \varpi = -\frac{\beta}{3}.$$

Here, the ODE (14) can be written as:

$$\frac{d}{d\xi}\mathcal{P}(\xi) = \sqrt{-\frac{3\alpha}{2\beta}}\mathcal{P}(\xi) - \frac{4k\sqrt{-\frac{3\alpha}{2\beta}}}{3\alpha}, \quad (19)$$

Undoubtedly, a exact solution to (19) also serves as a solution to (7). The exact solution for (19) can be represented as follows:

$$\mathcal{P}(\xi) = \frac{4k}{3\alpha} + C_1\,\mathrm{e}^{\frac{\sqrt{-\frac{6\alpha}{\beta}}\,\xi}{2}}, \quad (20)$$

where C_1 is an arbitrary constant. Therefore, from (7), and (20), a solitary wave solution of the GP equation has the following form:

$$u(x,t) = \frac{4k}{3\alpha} + C_1\,\mathrm{e}^{\frac{\sqrt{-\frac{6\alpha}{\beta}}\left(x - \frac{16k^2 - 9\alpha c}{12k}t\right)}{2}}. \quad (21)$$

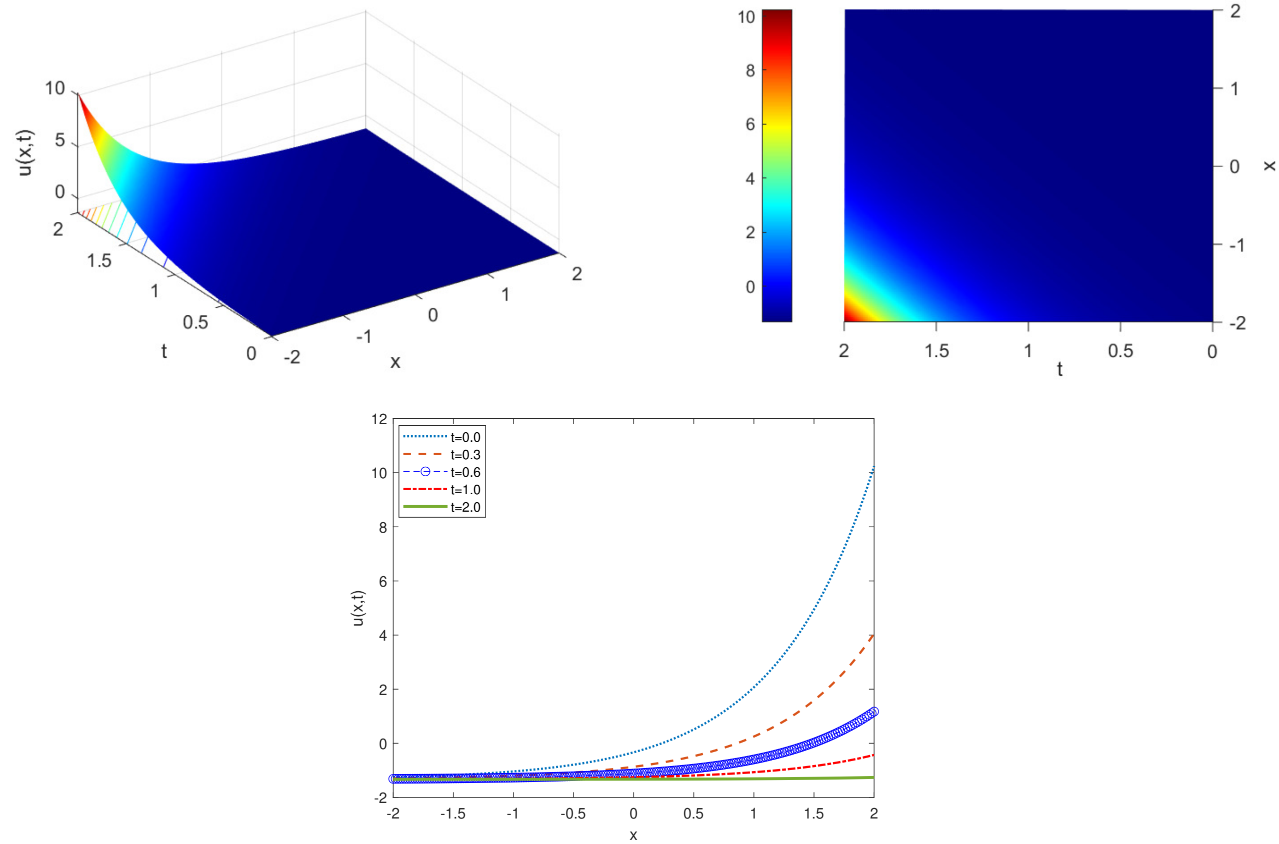

Figure 2: 2D, 3D and contour plots of (21).

Family 3:

$$\vartheta_0(\xi) = -\frac{6\sqrt{-\frac{3\alpha}{2\beta}}\left(\frac{3(6k\omega\alpha+16k\beta+3\Psi)k\,\omega^2\alpha}{-9\alpha^2\omega^2+6\omega\alpha\beta+8\beta^2} + \frac{2(6k\omega\alpha+16k\beta+3\Psi)\beta k\omega}{-9\alpha^2\omega^2+6\omega\alpha\beta+8\beta^2} + 3c\omega\alpha - 4k^2\omega - c\beta\right)}{\frac{9(6k\omega\alpha+16k\beta+3\Psi)\omega^2\alpha^2}{-9\alpha^2\omega^2+6\omega\alpha\beta+8\beta^2} - \frac{15(6k\omega\alpha+16k\beta+3\Psi)\beta\omega\alpha}{-9\alpha^2\omega^2+6\omega\alpha\beta+8\beta^2} + \frac{4(6k\omega\alpha+16k\beta+3\Psi)\beta^2}{-9\alpha^2\omega^2+6\omega\alpha\beta+8\beta^2} + 6k\omega\alpha - 8k\beta},$$

$$\vartheta_1(\xi) = \sqrt{-\frac{3\alpha}{2\beta}},\ \vartheta_2(\xi) = 0,\ \nu = \frac{(6k\omega\alpha + 16k\beta + 3\Psi)\,\beta}{-9\alpha^2\omega^2 + 6\omega\alpha\beta + 8\beta^2},\ \varpi = -\frac{\beta}{3},$$

where $\Psi = \sqrt{-18c\,\omega^2\alpha^3 + 36k^2\omega^2\alpha^2 + 12c\beta\omega\,\alpha^2 + 16c\alpha\,\beta^2}$. In this family, (14) transforms to

$$\frac{d}{d\xi}\mathcal{P}(\xi) = \sqrt{-\frac{3\alpha}{2\beta}}\mathcal{P}(\xi) - \frac{6\sqrt{-\frac{3\alpha}{2\beta}}\left(\frac{3(6k\omega\alpha+16k\beta+3\Psi)k\,\omega^2\alpha}{-9\alpha^2\omega^2+6\omega\alpha\beta+8\beta^2} + \frac{2(6k\omega\alpha+16k\beta+3\Psi)\beta k\omega}{-9\alpha^2\omega^2+6\omega\alpha\beta+8\beta^2} + 3c\omega\alpha - 4k^2\omega - c\beta\right)}{\frac{9(6k\omega\alpha+16k\beta+3\Psi)\omega^2\alpha^2}{-9\alpha^2\omega^2+6\omega\alpha\beta+8\beta^2} - \frac{15(6k\omega\alpha+16k\beta+3\Psi)\beta\omega\alpha}{-9\alpha^2\omega^2+6\omega\alpha\beta+8\beta^2} + \frac{4(6k\omega\alpha+16k\beta+3\Psi)\beta^2}{-9\alpha^2\omega^2+6\omega\alpha\beta+8\beta^2} + 6k\omega\alpha - 8k\beta}, \tag{22}$$

with exact solution:

$$\mathcal{P}(\xi) = \frac{9\left(\alpha\omega - \frac{4\beta}{3}\right)\left(\frac{\Theta\,(3\alpha\omega-\beta)\sqrt{2}}{3} + 6\omega\beta k\alpha\right)C_1 \mathrm{e}^{\frac{\sqrt{6}\sqrt{-\frac{\alpha}{\beta}}\,\xi}{2}}}{9\left(\alpha\omega - \frac{4\beta}{3}\right)\left(\frac{\Theta\,(3\alpha\omega-\beta)\sqrt{2}}{3} + 6\omega\beta k\alpha\right)} + \frac{18\left(\sqrt{2}\,\Theta\,k\omega - 3c\,\omega^2\alpha^2 + \left(6k^2\omega^2 + 5c\beta\omega\right)\alpha - \frac{4c\,\beta^2}{3}\right)\left(\alpha\omega + \frac{2\beta}{3}\right)}{9\left(\alpha\omega - \frac{4\beta}{3}\right)\left(\frac{\Theta\,(3\alpha\omega-\beta)\sqrt{2}}{3} + 6\omega\beta k\alpha\right)}, \tag{23}$$

where C_1 is an arbitrary constant, and $\Theta = \sqrt{\alpha\left(-9c\,\omega^2\alpha^2 + 18k^2\omega^2\alpha + 6\omega\beta\alpha c + 8c\,\beta^2\right)}$. Therefore, another solitary wave solution can be written as:

$$u(x,t) = \frac{9\left(\alpha\omega - \frac{4\beta}{3}\right)\left(\frac{\Theta\,(3\alpha\omega-\beta)\sqrt{2}}{3} + 6\omega\beta k\alpha\right) C_1 \mathrm{e}^{\frac{\sqrt{6}\,\sqrt{-\frac{\alpha}{\beta}}\,\left(x - \frac{(6k\omega\alpha+16k\beta+3\Psi)\beta}{-9\alpha^2\omega^2+6\omega\alpha\beta+8\beta^2}\,t\right)}{2}}}{9\left(\alpha\omega - \frac{4\beta}{3}\right)\left(\frac{\Theta\,(3\alpha\omega-\beta)\sqrt{2}}{3} + 6\omega\beta k\alpha\right)}$$
$$+\frac{18\left(\sqrt{2}\,\Theta\,k\omega - 3c\,\omega^2\alpha^2 + \left(6k^2\omega^2 + 5c\beta\omega\right)\alpha - \frac{4c\,\beta^2}{3}\right)\left(\alpha\omega + \frac{2\beta}{3}\right)}{9\left(\alpha\omega - \frac{4\beta}{3}\right)\left(\frac{\Theta\,(3\alpha\omega-\beta)\sqrt{2}}{3} + 6\omega\beta k\alpha\right)}. \tag{24}$$

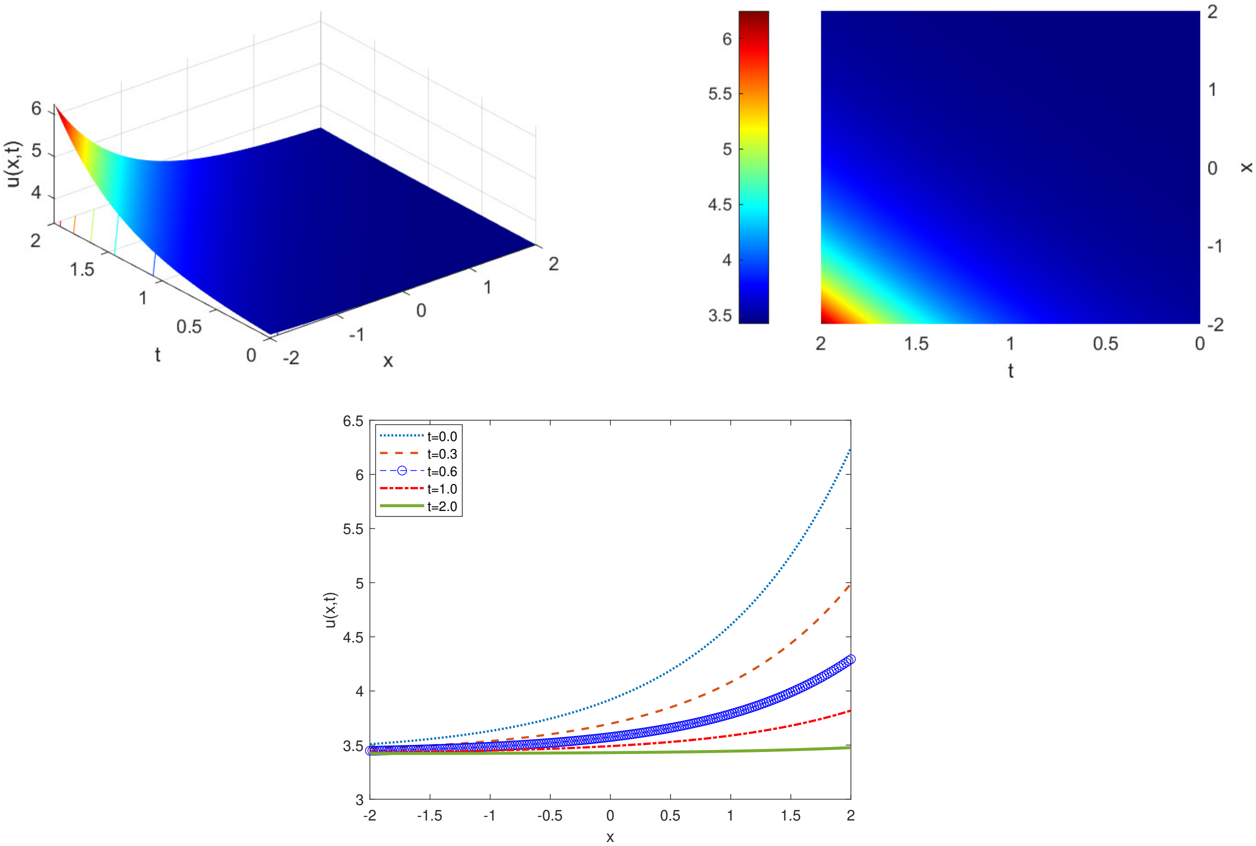

Figure 3: 2D, 3D and contour plots of (24).

Family 4:

$$\vartheta_0(\xi) = \frac{c(\alpha\omega + 2\beta)\left(2\omega k + \sqrt{2}\sqrt{-\omega(c\omega\alpha - 2k^2\omega + 2c\beta)}\right)(C_1 + C_2\sin(K\xi) + \cos(K\xi))}{\sqrt{\alpha}\sqrt{\beta}\,\omega\left(\sqrt{2}\,\alpha c\omega + 2\sqrt{2}\,\beta c - 4\sqrt{2}\,k^2\omega - 4k\sqrt{-\omega(c\omega\alpha - 2k^2\omega + 2c\beta)}\right)(-\cos(K\xi)\,C_2 + \sin(K\xi))},$$

$$\vartheta_1(\xi) = \frac{\omega\left(\left(C_2^2 - 1\right)\left(\cos^2(K\xi)\right) + (-2C_2\sin(K\xi) - C_1)\cos(K\xi) - \sin(K\xi)\,C_1C_2 - C_2^2\right)}{\sqrt{\beta}\left(\sin(K\xi)\cos(K\xi)\left(C_2^2 - 1\right) + 2\left(\cos^2(K\xi)\right)C_2 + \cos(K\xi)\,C_1C_2 - \sin(K\xi)\,C_1 - C_2\right)}$$
$$\times\frac{\sqrt{\alpha}\left(-4k\sqrt{-\omega(c\omega\alpha - 2k^2\omega + 2c\beta)} + \sqrt{2}\left(\left(\alpha c - 4k^2\right)\omega + 2c\beta\right)\right)}{\left(2\omega k + \sqrt{2}\sqrt{-\omega(c\omega\alpha - 2k^2\omega + 2c\beta)}\right)^2},$$

$$\vartheta_2(\xi) = 0,\ \nu = \frac{\left(2\omega k + \sqrt{-2c\,\omega^2\alpha + 4\omega^2k^2 - 4c\beta\omega}\right)\beta}{\omega(\alpha\omega + 2\beta)},\ \varpi = \beta.$$

where $K = \frac{\sqrt{\alpha}}{\sqrt{2\beta}}$. In this Family, the ODE (14) can be written as:

$$\frac{d}{d\xi}\mathcal{P}(\xi) = \frac{2\sqrt{2}\left(\left(-\frac{\mathcal{P}(\xi)\alpha}{2} + k\right)\cos(K\xi) + C_2\left(-\frac{\mathcal{P}(\xi)\alpha}{2} + k\right)\sin(K\xi) + kC_1\right)}{\sqrt{\beta}\sqrt{\alpha}\,(2\cos(K\xi)\,C_2 - 2\sin(K\xi))}, \tag{25}$$

by assuming $\omega = -\frac{2c\beta}{\alpha c - 2k^2}$. Definitely, any solution that satisfies (25) also meets the criteria of (7). Expressing the solution to (25) can be done as:

$$\begin{aligned}\mathcal{P}(\xi) = \quad & \Big((4\alpha C_2^3C_3 + 8kC_1C_2^2 + 8kC_2^2 - 4\alpha C_2C_3 - 8k)\cos^4(K\xi)\\ & + (-8\alpha C_2^2C_3 - 8kC_1C_2 - 16kC_2)\cos^3(K\xi)\sin(K\xi)\\ & + (4\alpha C_2C_3 - 4\alpha C_2^3C_3 - 8kC_1C_2^2 - 4kC_2^2 + 8k)\cos^2(K\xi)\\ & + (4\alpha C_2^2C_3 + 4kC_1C_2 + 4kC_2)\cos(K\xi)\sin(K\xi) + C_3\alpha C_2^3 + 2kC_1C_2^2\Big)\\ & \times\frac{1}{\alpha C_2\left(-2C_2\cos^2(K\xi) + 2\sin(K\xi)\cos(K\xi) + C_2\right)},\quad C_1 \in \mathbb{R}.\end{aligned} \tag{26}$$

Therefore

$$\begin{aligned}u(x,t) = & \Big((4\alpha C_2^3C_3 + 8kC_1C_2^2 + 8kC_2^2 - 4\alpha C_2C_3 - 8k)\cos^4(K(x - \nu t))\\ & + (-8\alpha C_2^2C_3 - 8kC_1C_2 - 16kC_2)\cos^3(K(x - \nu t))\sin(K(x - \nu t))\\ & + (4\alpha C_2C_3 - 4\alpha C_2^3C_3 - 8kC_1C_2^2 - 4kC_2^2 + 8k)\cos^2(K(x - \nu t))\\ & + (4\alpha C_2^2C_3 + 4kC_1C_2 + 4kC_2)\cos(K(x - \nu t))\sin(K(x - \nu t)) + C_3\alpha C_2^3 + 2kC_1C_2^2\Big)\\ & \times\frac{1}{\alpha C_2\left(-2C_2\cos^2(K(x - \nu t)) + 2\sin(K(x - \nu t))\cos(K(x - \nu t)) + C_2\right)}.\end{aligned} \tag{27}$$

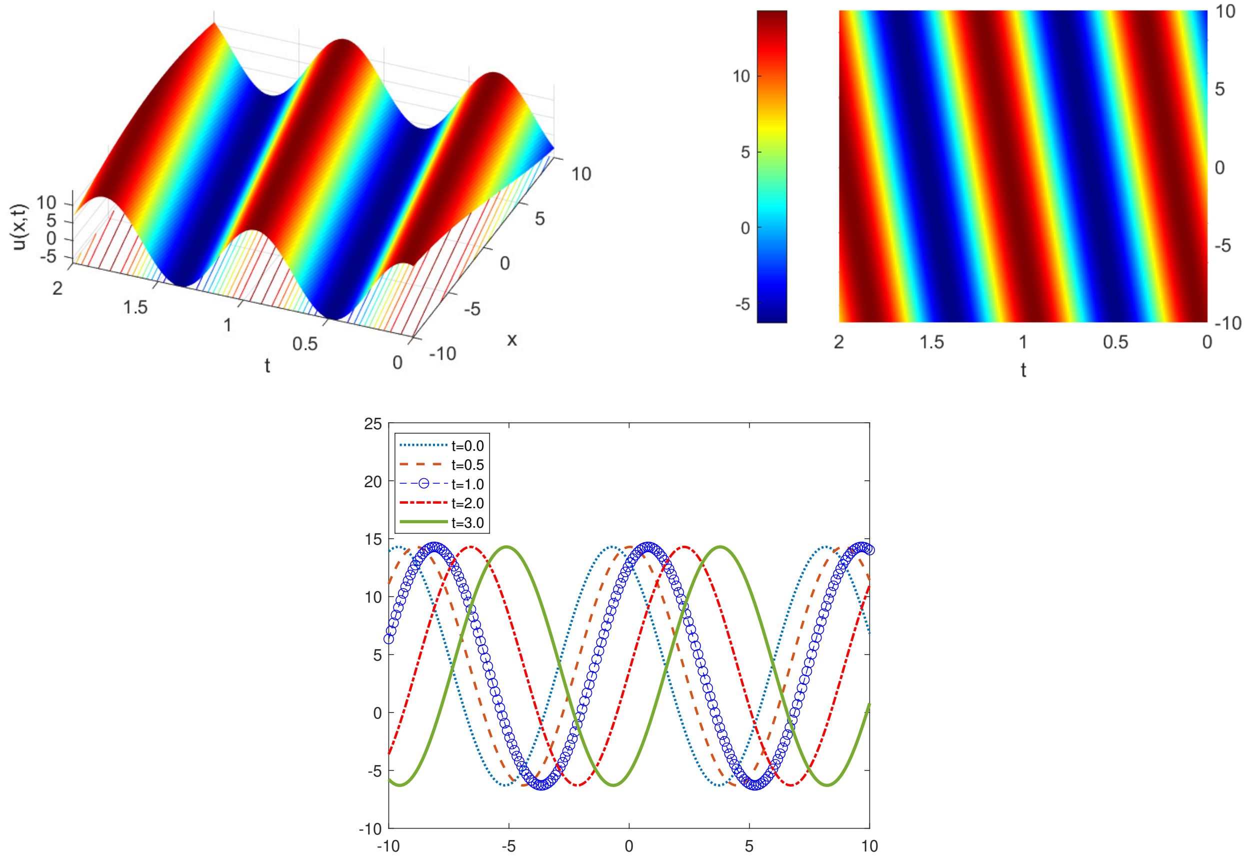

Figure 4: 2D, 3D and contour plots of (27).

Family 5:

$$\vartheta_0(\xi) = -\frac{\tan\left(\frac{\alpha\sqrt{c(\beta+\varpi)}\,(C_1+\xi)}{2\sqrt{\alpha c}\,\varpi}\right)\sqrt{c\,(\beta+\varpi)}\,\sqrt{2}}{\beta+\varpi},$$
$$\vartheta_1(\xi) = -\frac{\sqrt{\alpha c}\,\tan\left(\frac{\alpha\sqrt{c(\beta+\varpi)}\,(C_1+\xi)}{2\sqrt{\alpha c}\,\varpi}\right)\sqrt{c\,(\beta+\varpi)}}{(\beta+\varpi)\,c},$$
$$\vartheta_2(\xi) = 0,\ \nu = 2\sqrt{2\alpha c},\ \omega = -\frac{\varpi}{2\alpha},\ ,\ k = \frac{\sqrt{2}\,\sqrt{\alpha c}}{2}.$$

Here, the ODE (14) takes the following form:

$$\frac{d}{d\xi}\mathcal{P}(\xi) = -\frac{\tan\left(\frac{\alpha\sqrt{c(\beta+\varpi)}\,(C_1+\xi)}{2\sqrt{\alpha c}\,\varpi}\right)\sqrt{c\,(\beta+\varpi)}\,\left(\mathcal{P}(\xi)\,\sqrt{\alpha c}+\sqrt{2}\,c\right)}{c\,(\beta+\varpi)}, \quad (28)$$

with exact solution

$$\mathcal{P}(\xi) = \left(-\frac{\sqrt{2}\left(\frac{2}{\cos\left(\frac{\alpha\sqrt{c(\beta+\varpi)}\,(C_1+\xi)}{\sqrt{\alpha c}\,\varpi}\right)+1}\right)^{\frac{\varpi}{\beta+\varpi}}\sqrt{\alpha c}}{\alpha} + C_3 \right) \times \left(1+\tan^2\left(\frac{\alpha\sqrt{c\,(\beta+\varpi)}\,(C_1+\xi)}{2\sqrt{\alpha c}\,\varpi}\right)\right)^{-\frac{\varpi}{\beta+\varpi}}, \quad C_1 \in \mathbb{R}. \tag{29}$$

Thus, a periodic wave solution can be expressed in the following manner:

$$u(x,t) = \left(-\frac{\sqrt{2}\left(\frac{2}{\cos\left(\frac{\alpha\sqrt{c(\beta+\varpi)}\,(C_1-2\sqrt{2}\,\sqrt{\alpha c}\,t+x)}{\sqrt{\alpha c}\,\varpi}\right)+1}\right)^{\frac{\varpi}{\beta+\varpi}}\sqrt{\alpha c}}{\alpha} + C_3 \right) \times \left(1+\tan^2\left(\frac{\alpha\sqrt{c\,(\beta+\varpi)}\,\left(C_1-2\sqrt{2}\,\sqrt{\alpha c}\,t+x\right)}{2\sqrt{\alpha c}\,\varpi}\right)\right)^{-\frac{\varpi}{\beta+\varpi}}, \quad C_3 \in \mathbb{R}. \tag{30}$$

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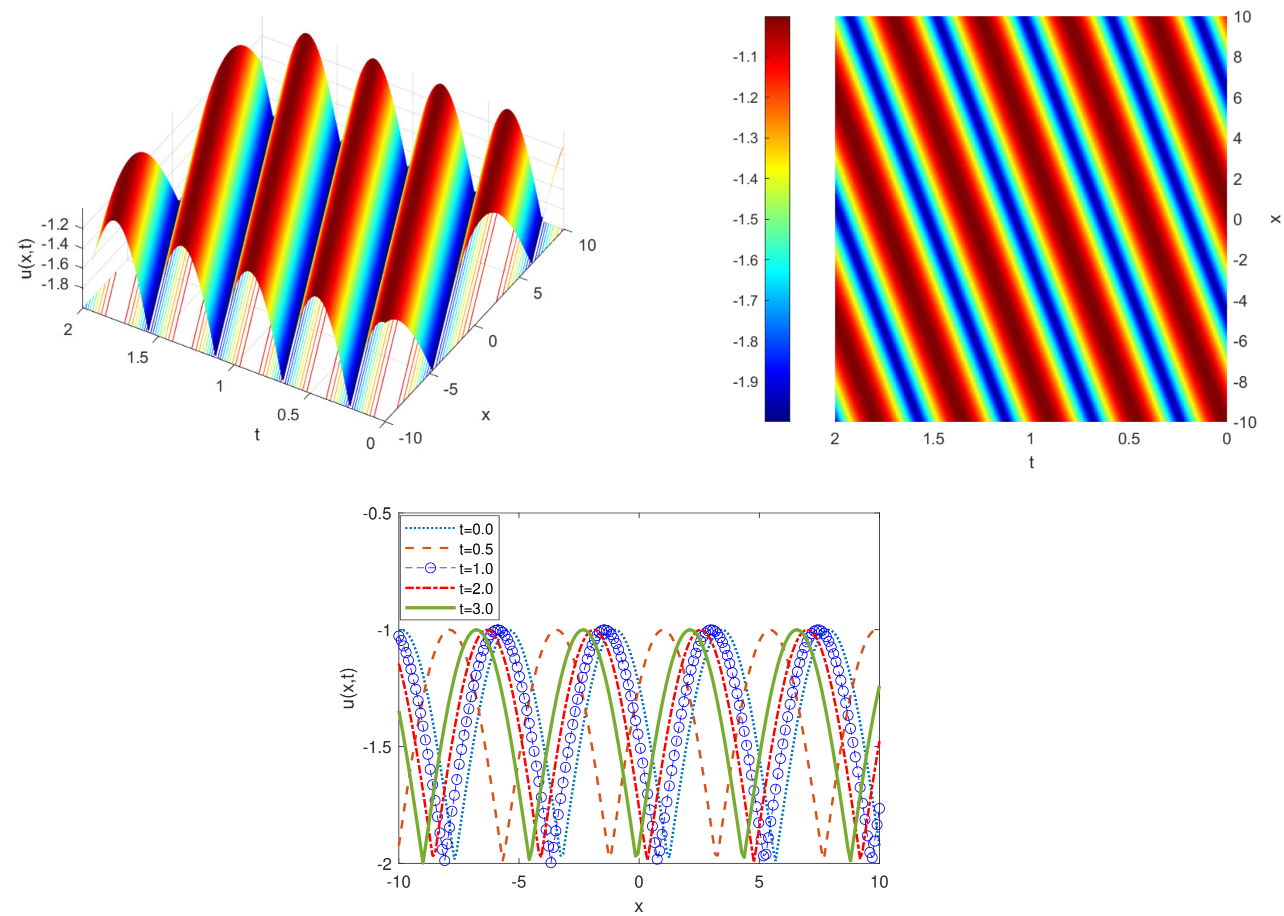

Figure 5: 2D, 3D and contour plots of (30).

Family 6:

$$\vartheta_0(\xi) = \frac{\sqrt{-\frac{\alpha}{\beta+\varpi}}\left(\alpha\omega+\beta+\varpi\right)}{\left(\alpha\omega-\beta+\varpi\right)\left(k\varpi\,\omega^2\alpha^2+2\alpha\left(k\,\varpi^2+\frac{\Phi}{4}\right)\omega-\varpi\left(k\,\beta^2-k\,\varpi^2-\frac{1}{2}\Phi\right)\right)}$$
$$\times\Bigg(\left(\left(-2k^2\varpi+\beta\left(\alpha c-2k^2\right)\right)\alpha\,\omega^2+\left(-\alpha\,\beta^2c+2c\beta\varpi\alpha+2k^2\beta^2-2k^2\varpi^2-k\Phi\right)\omega-c\beta\varpi\left(\beta-\varpi\right)\right)\Bigg),$$

$$\vartheta_1(\xi) = \sqrt{-\frac{\alpha}{\beta+\varpi}},\ \vartheta_2(\xi)=0,$$

$$\nu = \frac{-2\alpha k\varpi\omega+2k\,\beta^2-2k\,\varpi^2+\sqrt{-2c\,\beta^2\omega^2\alpha^3+4\beta^2k^2\omega^2\alpha^2-4c\,\beta^2\varpi\omega\,\alpha^2+2c\,\beta^4\alpha-2c\,\beta^2\varpi^2\alpha}}{-\alpha^2\omega^2-2\alpha\varpi\omega+\beta^2-\varpi^2},$$

where

$$\Phi=-2\alpha k\varpi\omega+2k\,\beta^2-2k\,\varpi^2+\sqrt{-2c\,\beta^2\omega^2\alpha^3+4\beta^2k^2\omega^2\alpha^2-4c\,\beta^2\varpi\omega\,\alpha^2+2c\,\beta^4\alpha-2c\,\beta^2\varpi^2\alpha}.$$

In this instance, the Eq. (14) transforms into

$$\frac{d}{d\xi}\mathcal{P}(\xi) = \sqrt{-\frac{\alpha}{\beta+\varpi}}\mathcal{P}(\xi)+\frac{\sqrt{-\frac{\alpha}{\beta+\varpi}}\left(\alpha\omega+\beta+\varpi\right)}{\left(\alpha\omega-\beta+\varpi\right)\left(k\varpi\,\omega^2\alpha^2+2\alpha\left(k\,\varpi^2+\frac{\Phi}{4}\right)\omega-\varpi\left(k\,\beta^2-k\,\varpi^2-\frac{1}{2}\Phi\right)\right)}$$
$$\times\Bigg(\left(\left(-2k^2\varpi+\beta\left(\alpha c-2k^2\right)\right)\alpha\,\omega^2+\left(-\alpha\,\beta^2c+2c\beta\varpi\alpha+2k^2\beta^2-2k^2\varpi^2-k\Phi\right)\omega-c\beta\varpi\left(\beta-\varpi\right)\right)\Bigg), \quad (31)$$

with exact solution

$$\begin{aligned}\mathcal{P}(\xi) = -\mathrm{e}^{\sqrt{-\frac{\alpha}{\beta+\varpi}}\,\xi}\Big[\mathrm{e}^{-\sqrt{-\frac{\alpha}{\beta+\varpi}}\,\xi}\big(-2c\beta\,\omega^3\alpha^3 + 4\beta\,k^2\omega^3\alpha^2 - 6c\beta\varpi\,\omega^2\alpha^2 + 4\beta^2k^2\omega^2\alpha + 4\beta\,k^2\varpi\,\omega^2\alpha \\ +2c\,\beta^3\omega\alpha - 6c\beta\,\varpi^2\omega\alpha + 2c\,\beta^3\varpi - 2c\beta\,\varpi^3\big) + 2\beta^2k\,\omega^2\alpha^2C_1 - 2\beta^3k\omega\alpha C_1 + 2\beta^2k\varpi\omega\alpha C_1 \\ +\sqrt{2}\,\sqrt{\beta^2\alpha\left(-c\,\omega^2\alpha^2 + 2k^2\omega^2\alpha - 2\alpha\varpi\omega c + \beta^2c - \varpi^2c\right)}\big(2\,\mathrm{e}^{-\sqrt{-\frac{\alpha}{\beta+\varpi}}\,\xi}\,k\,\omega^2\alpha + \omega^2\alpha^2C_1 - \beta\varpi C_1 \\ +2\varpi\omega\alpha C_1 - \beta\omega\alpha C_1 + 2\,\mathrm{e}^{-\sqrt{-\frac{\alpha}{\beta+\varpi}}\,\xi}\beta k\omega + 2\,\mathrm{e}^{-\sqrt{-\frac{\alpha}{\beta+\varpi}}\,\xi}k\varpi\omega + \varpi^2C_1\big)\Big] \\ \times\left(-\alpha\omega + \beta - \varpi\right)^{-1}\left(2k\omega\,\beta^2\alpha + \sqrt{2}\,\sqrt{\beta^2\alpha\left(-c\,\omega^2\alpha^2 + 2k^2\omega^2\alpha - 2\alpha\varpi\omega c + \beta^2c - \varpi^2c\right)}\left(\omega\alpha + \varpi\right)\right)^{-1} \quad (32)\end{aligned}$$

where $C_1 \in \mathbb{R}$. Therefore, an additional exact solution in the shape of a solitary wave solution can be expressed as follows:

$$\begin{aligned}u(x,t) \quad &= -\mathrm{e}^{\sqrt{-\frac{\alpha}{\beta+\varpi}}\,(x-\nu t)}\Big[\mathrm{e}^{-\sqrt{-\frac{\alpha}{\beta+\varpi}}\,(x-\nu t)}\big(-2c\beta\,\omega^3\alpha^3 + 4\beta\,k^2\omega^3\alpha^2 - 6c\beta\varpi\,\omega^2\alpha^2 + 4\beta^2k^2\omega^2\alpha + 4\beta\,k^2\varpi\,\omega^2\alpha \\ &+2c\,\beta^3\omega\alpha - 6c\beta\,\varpi^2\omega\alpha + 2c\,\beta^3\varpi - 2c\beta\,\varpi^3\big) + 2\beta^2k\,\omega^2\alpha^2C_1 - 2\beta^3k\omega\alpha C_1 + 2\beta^2k\varpi\omega\alpha C_1 \\ &+\sqrt{2}\,\sqrt{\beta^2\alpha\left(-c\,\omega^2\alpha^2 + 2k^2\omega^2\alpha - 2\alpha\varpi\omega c + \beta^2c - \varpi^2c\right)}\big(2\,\mathrm{e}^{-\sqrt{-\frac{\alpha}{\beta+\varpi}}\,(x-\nu t)}\,k\,\omega^2\alpha + \omega^2\alpha^2C_1 - \beta\varpi C_1 \\ &+2\varpi\omega\alpha C_1 - \beta\omega\alpha C_1 + 2\,\mathrm{e}^{-\sqrt{-\frac{\alpha}{\beta+\varpi}}\,(x-\nu t)}\beta k\omega + 2\,\mathrm{e}^{-\sqrt{-\frac{\alpha}{\beta+\varpi}}\,(x-\nu t)}k\varpi\omega + \varpi^2C_1\big)\Big] \\ &\times\left(-\alpha\omega + \beta - \varpi\right)^{-1}\left(2k\omega\,\beta^2\alpha + \sqrt{2}\,\sqrt{\beta^2\alpha\left(-c\,\omega^2\alpha^2 + 2k^2\omega^2\alpha - 2\alpha\varpi\omega c + \beta^2c - \varpi^2c\right)}\left(\omega\alpha + \varpi\right)\right)^{-1} \quad (33)\end{aligned}$$

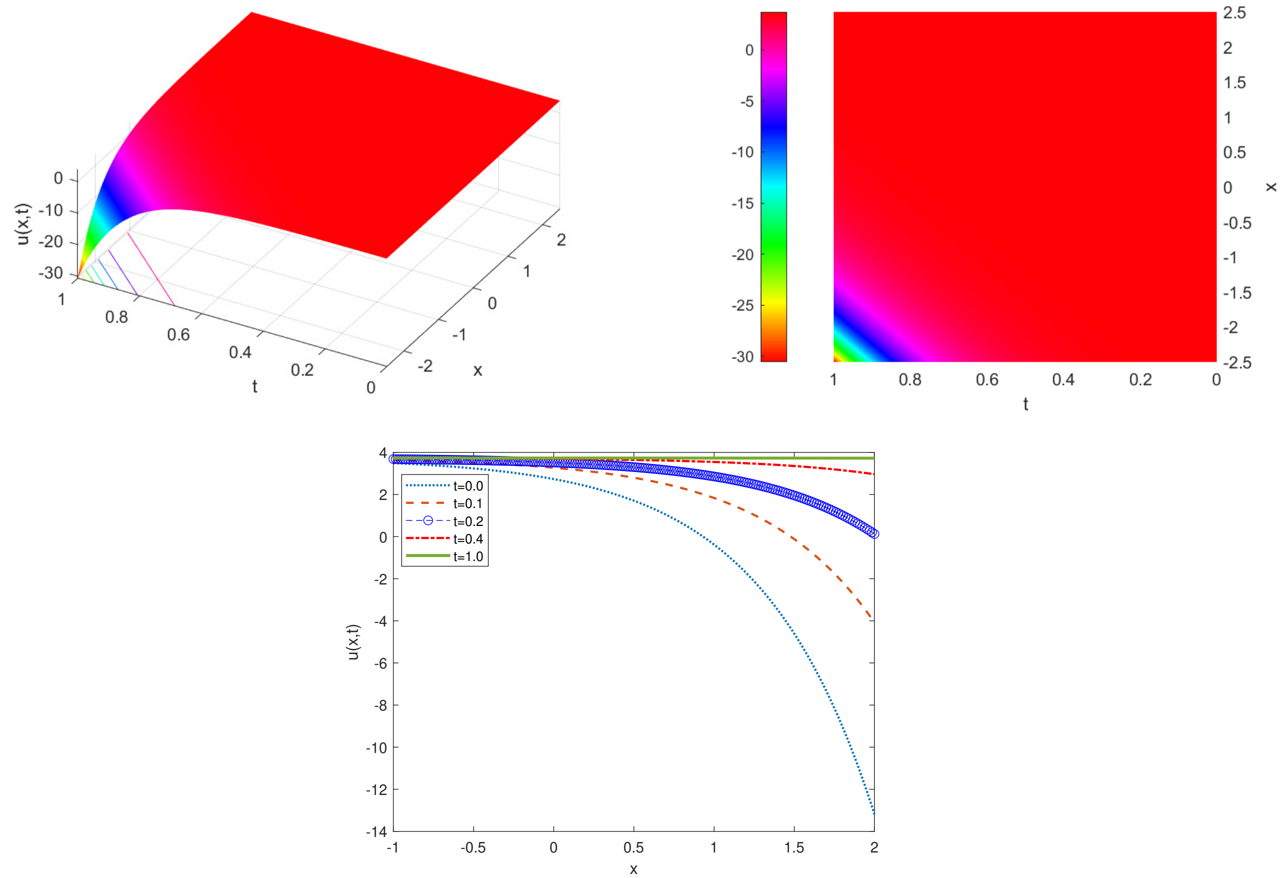

Figure 6: 2D, 3D and contour plots of (33).

To discover further exact solutions for the FW equation (2), by substituting (14) into Eq. (7) (with parameters $\alpha = -1$, $\beta = 3$, $k = 0.5$, and $\omega = 1$), we obtain a fifth degree polynomial in terms of $\mathcal{P}$, formulated as:

$$\sum_{i=0}^{4} S_i \mathcal{P}^i = 0,$$

where

$$S_0 = \nu \left(\frac{d}{d\xi} \vartheta_0(\xi) \right) + \vartheta_1(\xi)\, \vartheta_0(\xi)\, \nu + \frac{(\varpi - 3)\, \vartheta_0^2(\xi)}{2} - c,$$

$$S_1 = 2\nu \vartheta_0(\xi)\, \vartheta_2(\xi) + \nu \vartheta_1^2(\xi) + \nu \left(\frac{d}{d\xi} \vartheta_1(\xi) \right) - 3\vartheta_0(\xi)\, \vartheta_1(\xi) - \varpi \left(\frac{d}{d\xi} \vartheta_0(\xi) \right) - \nu + 1,$$

$$S_2 = -\varpi \left(\frac{d}{d\xi} \vartheta_1(\xi) \right) + \nu \left(\frac{d}{d\xi} \vartheta_2(\xi) \right) + \frac{(-\varpi - 3)\, \vartheta_1^2(\xi)}{2} + 3\nu \vartheta_2(\xi)\, \vartheta_1(\xi) + \frac{1}{2} - (\varpi + 3)\, \vartheta_0(\xi)\, \vartheta_2(\xi),$$

$$S_3 = -\varpi \left(\frac{d}{d\xi} \vartheta_2(\xi) \right) + 2 \left(\nu \vartheta_2(\xi) - \left(\varpi + \frac{3}{2} \right) \vartheta_1(\xi) \right) \vartheta_2(\xi),$$

$$S_4 = -\frac{3\,(\varpi + 1)\, \vartheta_2^2(\xi)}{2}.$$

Considering $S_i = 0,\ i = 0, \ldots, 4$, leads to a system of differential algebraic equations with the subsequent sets of solutions:

Case 1:

$$\vartheta_0(\xi) = \frac{C_1\sqrt{2}\,\mathrm{e}^{\frac{\sqrt{2}\,\xi}{2}}}{8C_2\mathrm{e}^{\sqrt{2}\,\xi}-8} - \frac{\sqrt{2}\left(C_1 + C_2\mathrm{e}^{\frac{\sqrt{2}\,\xi}{2}} + \mathrm{e}^{-\frac{\sqrt{2}\,\xi}{2}}\right)}{2\left(2C_2\mathrm{e}^{\frac{\sqrt{2}\,\xi}{2}} - 2\,\mathrm{e}^{-\frac{\sqrt{2}\,\xi}{2}}\right)},\quad \vartheta_1(\xi) = -\frac{\sqrt{2}\left(C_1 + C_2\mathrm{e}^{\frac{\sqrt{2}\,\xi}{2}} + \mathrm{e}^{-\frac{\sqrt{2}\,\xi}{2}}\right)}{2C_2\mathrm{e}^{\frac{\sqrt{2}\,\xi}{2}} - 2\,\mathrm{e}^{-\frac{\sqrt{2}\,\xi}{2}}},$$

$$\vartheta_2(\xi) = -\frac{C_1\sqrt{2}\,\mathrm{e}^{\frac{\sqrt{2}\,\xi}{2}}}{2C_2\mathrm{e}^{\sqrt{2}\,\xi}-2},\quad \nu = \frac{1}{2},\quad \varpi = -1,\quad c = -\frac{1}{8}.$$

Here, from the ODE (14) we have

$$\frac{d}{d\xi}\mathcal{P}(\xi) = -\frac{\sqrt{2}\left(C_1\left(\mathcal{P}(\xi)+\frac{1}{2}\right)\mathrm{e}^{\frac{\sqrt{2}\,\xi}{2}} + C_2\mathrm{e}^{\sqrt{2}\,\xi}+1\right)\left(\mathcal{P}(\xi)+\frac{1}{2}\right)}{2C_2\mathrm{e}^{\sqrt{2}\,\xi}-2}, \tag{34}$$

with exact solution

$$\mathcal{P}(\xi) = \frac{-C_1\left(C_3-1\right)\mathrm{e}^{-\frac{\sqrt{2}\,\xi}{2}} - C_1\left(C_3+1\right)\mathrm{e}^{\frac{\sqrt{2}\,\xi}{2}} + \left(-4C_2-4\right)C_3 + 4C_2 - 4}{4C_1\left(C_3\cosh\left(\frac{\sqrt{2}\,\xi}{2}\right) + \sinh\left(\frac{\sqrt{2}\,\xi}{2}\right)\right)}. \tag{35}$$

Therefore, utilizing (7) and (35), the ensuing dark soliton can be expressed as:

$$u(x,t) = \frac{-C_1\left(C_3-1\right)\mathrm{e}^{-\frac{\sqrt{2}\left(-\frac{t}{2}+x\right)}{2}} - C_1\left(C_3+1\right)\mathrm{e}^{\frac{\sqrt{2}\left(-\frac{t}{2}+x\right)}{2}} + \left(-4C_2-4\right)C_3 + 4C_2 - 4}{4C_1\left(\cosh\left(\frac{\sqrt{2}\left(-\frac{t}{2}+x\right)}{2}\right)C_3 + \sinh\left(\frac{\sqrt{2}\left(-\frac{t}{2}+x\right)}{2}\right)\right)}. \tag{36}$$

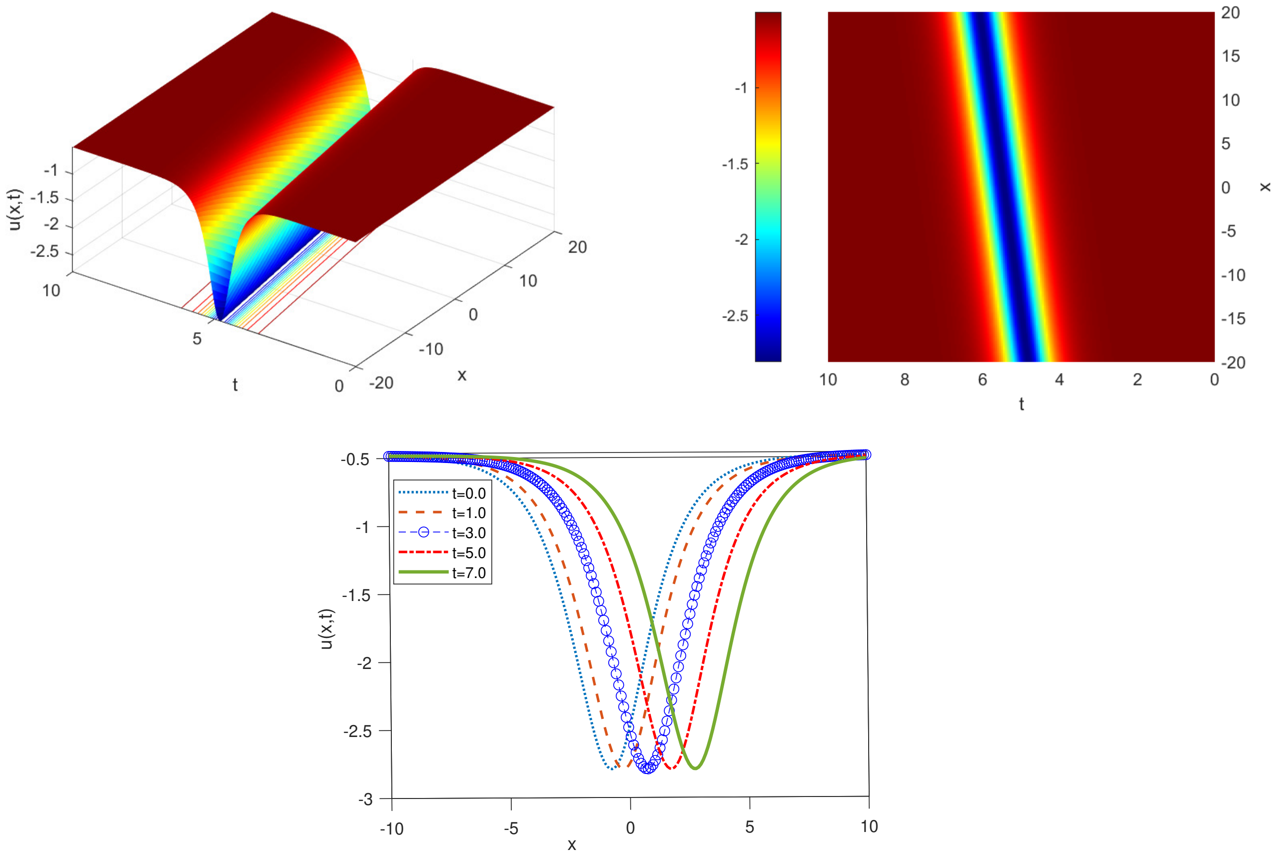

Figure 7: 2D, 3D and contour plots of (36).

Case 2:

$$\vartheta_0(\xi) = -2c\sqrt{-\frac{1}{8c+1}}, \quad \vartheta_1(\xi) = \vartheta_2(\xi) = \sqrt{-\frac{1}{8c+1}}, \quad \nu = \frac{1}{2}, \quad \varpi = -1.$$

Eq. (14) in the current case can be written as

$$\frac{d}{d\xi}\mathcal{P}(\xi) = \sqrt{-\frac{1}{8c+1}}\left(\mathcal{P}^2(\xi) + \mathcal{P}(\xi) - 2c\right). \tag{37}$$

The solution of equation (37) can be formulated as:

$$\mathcal{P}(\xi) = -\frac{\tanh\left(\frac{\sqrt{-\frac{1}{8c+1}}\sqrt{8c+1}\,(C_1+\xi)}{2}\right)\sqrt{8c+1}}{2} - \frac{1}{2}, \quad C_1 \in \mathbb{R}. \tag{38}$$

Hence, based on (7), and (38), we obtain the following periodic singular solution:

$$u(x,t) = -\frac{\tanh\left(\frac{\sqrt{-\frac{1}{8c+1}}\sqrt{8c+1}\,\left(C_1-\frac{t}{2}+x\right)}{2}\right)\sqrt{8c+1}}{2} - \frac{1}{2}. \tag{39}$$

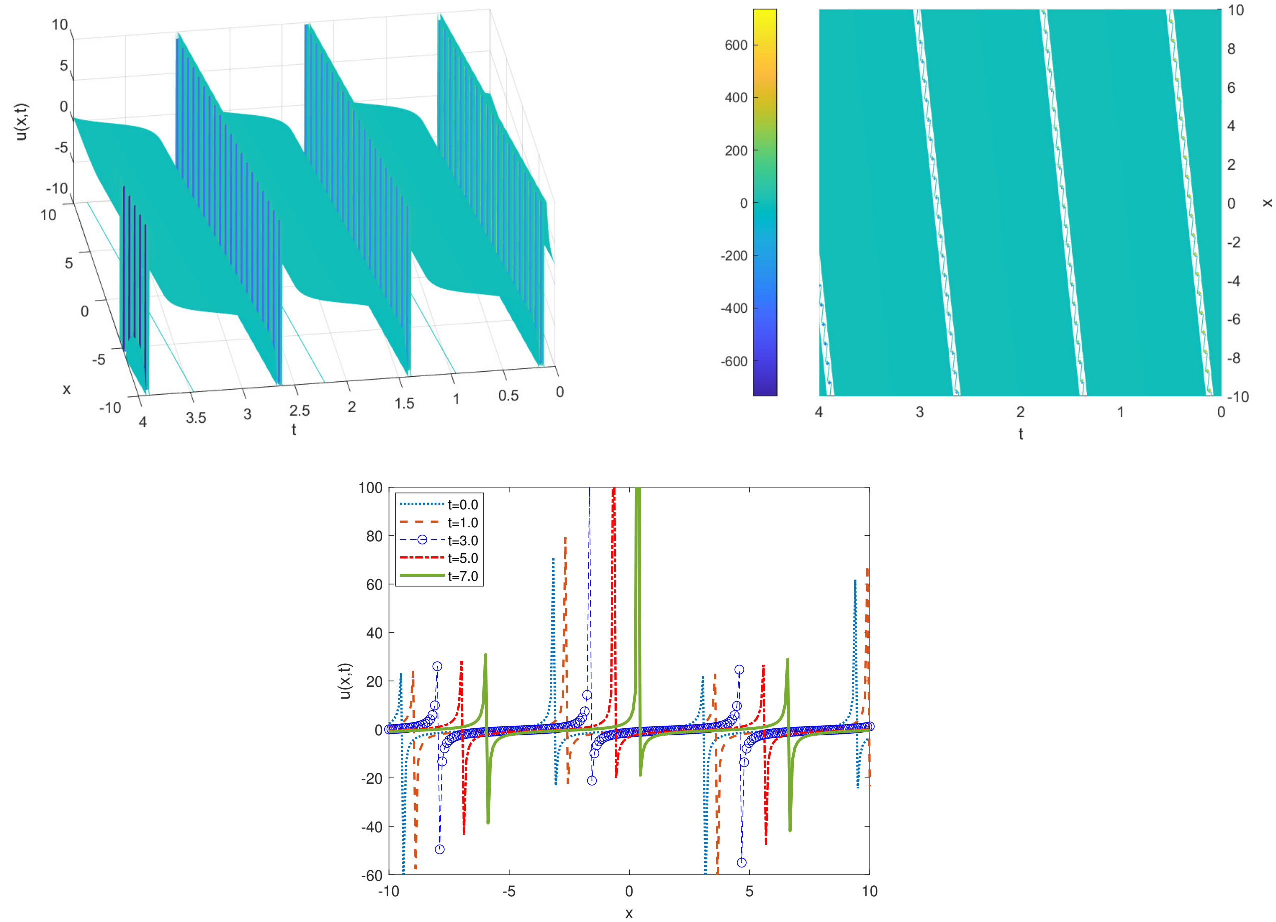

Figure 8: 2D, 3D and contour plots of (39).

4 Results and Discussions

The Results section presents the findings obtained from the application of the VCAEM to the study of exact solutions for the GP equation. Through rigorous mathematical analysis and computational techniques, this study explores the behavior and characteristics of the GP equation under various conditions and parameters. In this paper, all computations have been carried out using Maple 2022. The section begins by detailing the exact solutions derived using the proposed technique, showcasing their forms and properties. Subsequently, these solutions are analyzed and discussed in depth, shedding light on the implications and significance of the results within the context of the GP equation. Insights into the behavior of the equation, its solutions, and their relevance to theoretical and applied research are thoroughly examined, providing a comprehensive understanding of the findings presented in this manuscript.

Fig. 1 illustrates the Kink soliton solution of equation (18) under the conditions $c = k = \omega = \alpha = \beta = C_1 = 1$, presented in three dimensional, contour, and two dimensional formats.

Figure 2 depicts the solitary wave solution of equation (21) with the parameters set as $c = k = \omega = \beta = C_1 = 1$, and $\alpha = -1$, showcased through three dimensional, contour, and two dimensional visualizations.

Figure 3 illustrates an alternative solitary wave solution of equation (24), where the parameters are specified as $c = 0.1$, $k = \omega = 2$, $\beta = -2$, $C_1 = 0.5$, and $\alpha = 1$, presented across three dimensional, contour, and two dimensional representations.

In Figure 4, we depict a periodic wave solution corresponding to equation (27). Here, the parameters take on specific values: $c = k = \omega = 2$, while

$\alpha = \beta = C_1 = C_2 = C_3 = 1$. The illustration showcases the solution through three dimensional, contour, and two dimensional representations.

In Figure 5, we illustrate a periodic wave solution associated with equation (30). The parameters are assigned specific values, with $\alpha = \beta = \varpi = C_1 = C_3 = 1$, and $c = 2$. The visual representation showcases the solution across three dimensional, contour, and two dimensional depictions.

In Figure 6, we depict a solitary wave solution corresponding to equation (33). Specific parameter values are assigned, with $\alpha = \beta = \varpi = C_1 = 1$, and $c = k = 2$. The visual presentation exhibits the solution through three dimensional, contour, and two dimensional renderings.

In Figure 7, we illustrate a dark soliton associated with equation (36). Specific parameter values are assigned, with $C_1 = C_2 = 1$, and $C_3 = 2$. The visual representation displays the solution through three dimensional, contour, and two dimensional depictions.

In Figure 8, we depict a periodic singular solution corresponding to equation (39). Assigning specific parameter values, with $C_1 = 1$ and $c = -0.5$. The visual representation showcases the solution across three dimensional, contour, and two dimensional depictions.

5 Conclusion

This study introduces a novel approach, namely the VCAEM, to investigate the GP equation. Through this method, various types of solutions including Kink solitons, dark solitons, periodic wave solutions, periodic singular solutions, and solitary wave solutions have been derived. These findings significantly contribute to the understanding of the dynamics governed by the GP equation. The versatility of the proposed method enables the exploration of a wide range of phenomena and opens avenues for further research in nonlinear dynamics and related fields. The derived solutions offer valuable insights into the behavior and characteristics of the system under consideration, providing a foundation for future theoretical and experimental studies. Overall, the VCAEM proves to be a powerful tool for analyzing complex nonlinear systems, offering new perspectives and opportunities for advancing our understanding of nonlinear phenomena.

Declarations

Ethical Approval:

Not applicable.

Competing interests:

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding:

No funding received for this paper.

Availability of data and materials:

No applicable.

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