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ON COMPLEX MODIFIED GENUINE SZÁSZ-DURRMEYER-STANCU OPERATORS

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ABSTRACT. In this paper, we introduce complex modified genuine Szász-Durrmeyer-Stancu operators to improve the results obtained in [4] and present overconvergence properties of these operators. We obtain some estimates on the rate of convergence, a Voronovskaja-type result and the exact order of approximation for these operators attached to analytic functions of exponential growth on compact disks.

1. Introduction

For a function $f \in C[0,\infty)$ satisfying an exponential growth condition, that is $|f(x)| \leq Ce^{Bx}$, $x \in [0,\infty)$, with some constants $C > 0$ and $B > 0$, Phillips [20] first defined the following operators

$$L_n(f;x) = n\sum_{j=1}^{\infty} p_{n,j}(x)\int_0^{\infty} p_{n,j-1}(t)f(t)dt + p_{n,0}(x)f(0) \quad , \quad n > B$$

where

$$p_{n,j}(x) = e^{-nx}\frac{(nx)^j}{j!}, \ j \in \mathbb{N}_0, \ x \in [0,\infty),$$

which in the literature often are known as Phillips operators. Since Phillips operators preserve constant as well as linear functions, these operators can be also named as genuine Szász-Durrmeyer operators. After that in case of real variable, these operators and their various generalizations have been widely studied by several researchers, see e.g. [1, 6, 14, 18, 19] etc.

Recently, the problem of approximation of complex operators has been one of the interesting research area. Some approximation properties of complex Bernstein polynomials in various domains in complex plane were presented by Wright [23],

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Kantorovich [16], Bernstein [2], Tonne [21] and Lorentz [17]. The first result concerning the convergence of complex Szász operators which is a generalization of the Bernstein polynomials was proved by Gergen et. al. [10]. Then, Jakimovski et. al. [15], Wood [22] and Deeba [5] studied some generalizations of Szász operators in complex domains. But all these above mentioned results were obtained without any quantitative estimate. In [7], Gal obtained quantitative estimates for the convergence and Voronovskaja's theorem in addition to the results obtained in [17] and [10]. Also, Gal compiled the results on overconvergence properties of the well known complex operators in his book [7]. Later on, a large number of authors have established approximation properties with quantitative estimates for different operators in complex domain (see e.g. [3, 8, 9, 11, 12, 13]).

Motivation for the present work is the complex modified genuine Szász-Durrmeyer operators which have been introduced and studied in [4]. In order to improve the results obtained in [4], we consider the Stancu variant of complex modified genuine Szász-Durrmeyer operators as

$$\begin{aligned} L_{a_n,b_n}^{(\alpha,\beta)}(f;z) &= \frac{a_n}{b_n}\sum_{j=1}^{\infty} p_{a_n,b_n,j}(z)\int_0^{\infty} p_{a_n,b_n,j-1}(t) f\left(\frac{a_n t+\alpha b_n}{a_n+\beta b_n}\right) dt \\ &\quad + p_{a_n,b_n,0}(z) f\left(\frac{\alpha b_n}{a_n+\beta b_n}\right), \end{aligned} \tag{1.1}$$

where

$$p_{a_n,b_n,j}(z) = e^{-\frac{a_n}{b_n}z}\frac{(a_n z)^j}{j! b_n^j}.$$

where $\{a_n\}, \{b_n\}$ are given sequences of strictly positive numbers such that $\lim_{n\to\infty}\frac{b_n}{a_n} = 0$ and $\frac{b_n}{a_n} \le \frac{1}{2}$ and also α, β are two given real parameters satisfying the condition $0 \le \alpha \le \beta$. Note that for the special case $\alpha = \beta = 0$, we get the operators defined in [4]. Also, by taking $a_n = n$, $b_n = 1$ and $\alpha = \beta = 0$, these operators become complex genuine Szász-Durrmeyer operators given in [9]. In this work, we investigate the overconvergence properties of the operators defined by (1.1) in compact disks. We obtain the rate of convergence, the Voronovskaja-type result with quantitative estimate and the exact order of approximation for these operators.

In our results, by $H(D_R)$ with $D_R = \{z \in \mathbb{C} : |z| < R,\ 1 < R < \infty\}$ we consider the class of the functions satisfying $f : [R,\infty) \cup \overline{D_R} \to \mathbb{C}$ is integrable on $[0,\infty)$, continuous in $[R,\infty) \cup \overline{D_R}$ and analytic in D_R i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$.

2. Auxiliary Results

The following auxiliary results will be very useful to prove our main results.

Lemma 1. *Let* $e_\upsilon(z) = z^\upsilon$, $\upsilon \in \mathbb{N}\cup\{0\}$, $z \in \mathbb{C}$, $n \in \mathbb{N}$ *and* $0 \le \alpha \le \beta$. *Then, we have*

$$L^{(\alpha,\beta)}_{a_n,b_n}(e_k; z) = \sum_{\upsilon=0}^{k} \binom{k}{\upsilon} \frac{a_n^\upsilon (\alpha b_n)^{k-\upsilon}}{(a_n+\beta b_n)^k} L_{a_n,b_n}(e_\upsilon; z)$$

where L_{a_n,b_n} denotes $L^{(0,0)}_{a_n,b_n}$.

Proof. From the definition (1.1), we have

$$\begin{aligned} L^{(\alpha,\beta)}_{a_n,b_n}(e_k; z) &= \frac{a_n}{b_n}\sum_{j=1}^{\infty} p_{a_n,b_n,j}(z)\int_0^\infty p_{a_n,b_n,j-1}(t)\left(\frac{a_n t + \alpha b_n}{a_n+\beta b_n}\right)^k dt \\ &\quad + p_{a_n,b_n,0}(z)(\frac{\alpha b_n}{a_n+\beta b_n})^k. \end{aligned}$$

By binomial theorem, we get

$$\begin{aligned} L^{(\alpha,\beta)}_{a_n,b_n}(e_k; z) &= \frac{a_n}{b_n}\sum_{j=1}^{\infty} p_{a_n,b_n,j}(z)\int_0^\infty p_{a_n,b_n,j-1}(t)\sum_{\upsilon=0}^{k}\binom{k}{\upsilon}\frac{a_n^\upsilon (\alpha b_n)^{k-\upsilon}}{(a_n+\beta b_n)^k} t^\upsilon dt \\ &\quad + p_{a_n,b_n,0}(z)\sum_{\upsilon=0}^{k}\binom{k}{\upsilon}\frac{a_n^\upsilon (\alpha b_n)^{k-\upsilon}}{(a_n+\beta b_n)^k} 0^\upsilon \\ &= \sum_{\upsilon=0}^{k}\binom{k}{\upsilon}\frac{a_n^\upsilon (\alpha b_n)^{k-\upsilon}}{(a_n+\beta b_n)^k} L_{a_n,b_n}(e_\upsilon; z). \end{aligned}$$

□

Lemma 2. *Let* α, β *be satisfying* $0 \le \alpha \le \beta$ *and suppose that* $f \in H(D_R)$ *and there exist* $B, C > 0$ *such that* $|f(x)| \le Ce^{Bx}$, *for all* $x \in [R, +\infty)$. *Also let* $n_0 \in \mathbb{N}$ *be such that* $\frac{a_n}{b_n} + \beta - B > 0$ *for all* $n > n_0$. *Denoting* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, $z \in D_R$, *we have* $L^{(\alpha,\beta)}_{a_n,b_n}(f; z) = \sum_{k=0}^{\infty} c_k L^{(\alpha,\beta)}_{a_n,b_n}(e_k; z)$, *for all* $z \in D_R$ *and* $n > n_0$ *with* $n, n_0 \in \mathbb{N}$.

Proof. For any $m \in \mathbb{N}$ and $0 < r < R$, we define

$$f_m(z) = \sum_{j=0}^{m} c_j z^j \text{ if } |z| \le r \text{ and } f_m(x) = f(x) \text{ if } x \in (r, +\infty).$$

Since $|f_m(z)| \le \sum_{j=0}^{\infty} |c_j| r^j := C_r$, for all $|z| \le r$ and $m \in \mathbb{N}$, f is continuous $[r, R]$, from the hypothesis on f, there exists a constant $C_{r,R} > 0$ (independent of m) such that $|f_m(x)| \le C_{r,R} e^{Bx}$, for all $m \in \mathbb{N}$ and $x \in [0, +\infty)$. This implies that by the ratio criterion, for each fixed $m,n\in\mathbb{N}$, $z\in\mathbb{C}$ and for all $n>n_0$ with $\frac{a_n}{b_n}+\beta-B>0$,

$$\begin{aligned}
\left|L_{a_n,b_n}^{(\alpha,\beta)}(f_m;z)\right| &\leq \frac{a_n}{b_n}\sum_{j=1}^{\infty}\left|e^{-\frac{a_n}{b_n}z}\right|\frac{(a_n|z|)^j}{j!b_n^j}\int_0^{\infty}e^{-\frac{a_n}{b_n}t}\frac{(a_nt)^{j-1}}{(j-1)!b_n^{j-1}}\left|f_m\left(\frac{a_nt+\alpha b_n}{a_n+\beta b_n}\right)\right|dt \\
&\quad+\left|e^{-\frac{a_n}{b_n}z}\right|\left|f_m\left(\frac{\alpha b_n}{a_n+\beta b_n}\right)\right| \\
&\leq \frac{a_n}{b_n}\left|e^{-\frac{a_n}{b_n}z}\right|\sum_{j=1}^{\infty}\frac{(a_n|z|)^j}{j!b_n^j}\frac{a_n^{j-1}}{(j-1)!b_n^{j-1}}\int_0^{\infty}e^{-\frac{a_n}{b_n}t}t^{j-1}C_{r,R}e^{B\left(\frac{a_nt+\alpha b_n}{a_n+\beta b_n}\right)}dt \\
&\quad+\left|e^{-\frac{a_n}{b_n}z}\right|\left|c_0+c_1\frac{\alpha b_n}{a_n+\beta b_n}+...+c_m\left(\frac{\alpha b_n}{a_n+\beta b_n}\right)^m\right| \\
&\leq \frac{a_n}{b_n}\left|e^{-\frac{a_n}{b_n}z}\right|\sum_{j=1}^{\infty}\frac{(a_n|z|)^j}{j!b_n^j}\frac{a_n^{j-1}}{(j-1)!b_n^{j-1}}C_{r,R}e^{B\frac{\alpha b_n}{a_n+\beta b_n}}\int_0^{\infty}e^{-t\left(\frac{a_n}{b_n}-\frac{Ba_n}{a_n+\beta b_n}\right)}t^{j-1}dt \\
&\quad+\left|e^{-\frac{a_n}{b_n}z}\right|\{|c_0|+|c_1|+...+|c_m|\} \\
&\leq \sum_{j=1}^{\infty}\left|e^{-\frac{a_n}{b_n}z}\right|\frac{(a_n|z|)^j}{j!b_n^j}\frac{a_n^j}{b_n^j}C_{r,R}e^{B\frac{\alpha b_n}{a_n+\beta b_n}}\frac{1}{\left(\frac{a_n}{b_n}-\frac{Ba_n}{a_n+\beta b_n}\right)^j} \\
&\quad+\left|e^{-\frac{a_n}{b_n}z}\right|\{|c_0|+|c_1|+...+|c_m|\} \\
&= C_{r,R}e^{B\frac{\alpha b_n}{a_n+\beta b_n}}\left|e^{-\frac{a_n}{b_n}z}\right|\sum_{j=0}^{\infty}\frac{\left[\left(\frac{a_n}{b_n}\right)^2|z|/\left(\frac{a_n}{b_n}-\frac{Ba_n}{a_n+\beta b_n}\right)\right]^j}{j!} \\
&\quad+\left|e^{-\frac{a_n}{b_n}z}\right|\{|c_0|+|c_1|+...+|c_m|\} \\
&\leq (C_{r,R}+|c_0|+|c_1|+...+|c_m|)\left|e^{-\frac{a_n}{b_n}z}\right|e^{B\frac{\alpha b_n}{a_n+\beta b_n}}e^{\left(\frac{a_n}{b_n}\right)^2|z|/\left(\frac{a_n}{b_n}-\frac{Ba_n}{a_n+\beta b_n}\right)} \\
&< \infty.
\end{aligned}$$

Therefore, $L_{a_n,b_n}^{(\alpha,\beta)}(f_m;z)$ is well-defined. Denoting

$$f_{m,k}(z)=c_ke_k(z)\ if\ \ |z|\leq r \text{ and } f_{m,k}(x)=\frac{f(x)}{m+1} \text{ if } x\in(r,\infty),$$

it is clear that each $f_{m,k}$ is of exponential growth on $[0,\infty)$ and $f_m(z)=\sum_{k=0}^{m} f_{m,k}(z)$. Since $L_{a_n,b_n}^{(\alpha,\beta)}$ is linear, we have

$$L_{a_n,b_n}^{(\alpha,\beta)}(f_m;z)=\sum_{k=0}^{m} c_k L_{a_n,b_n}^{(\alpha,\beta)}(e_k;z),\ \text{for all } |z|\le r,$$

it suffices to prove that $\lim_{m\to\infty} L_{a_n,b_n}^{(\alpha,\beta)}(f_m;z)=L_{a_n,b_n}^{(\alpha,\beta)}(f;z)$ for any fixed $n\in\mathbb{N}$ and $|z|\le r$. But this is immediate from $\lim_{m\to\infty}\|f_m-f\|_r=0$, from $\|f_m-f\|_{B[0,+\infty)}\le \|f_m-f\|_r$ and from the following inequality

$$\begin{aligned}
&\left|L_{a_n,b_n}^{(\alpha,\beta)}(f_m;z)-L_{a_n,b_n}^{(\alpha,\beta)}(f;z)\right|\\
\le\ & \frac{a_n}{b_n}\left|e^{-\frac{a_n}{b_n}z}\right|\sum_{j=1}^{\infty}\frac{(a_n|z|)^j}{j!b_n^j}\int_0^{\infty} e^{-\frac{a_n}{b_n}t}\frac{(a_nt)^{j-1}}{(j-1)!b_n^{j-1}}\left|f_m\left(\frac{a_nt+\alpha b_n}{a_n+\beta b_n}\right)-f\left(\frac{a_nt+\alpha b_n}{a_n+\beta b_n}\right)\right|dt\\
&+\left|e^{-\frac{a_n}{b_n}z}\right|\left|f_m\left(\frac{\alpha b_n}{a_n+\beta b_n}\right)-f\left(\frac{\alpha b_n}{a_n+\beta b_n}\right)\right|\\
\le\ & \frac{a_n}{b_n}\left|e^{-\frac{a_n}{b_n}z}\right|\sum_{j=1}^{\infty}\frac{(a_n|z|)^j}{j!b_n^j}\frac{a_n^{j-1}}{(j-1)!b_n^{j-1}}\|f_m-f\|_{B[0,+\infty)}\int_0^{\infty} e^{-\frac{a_n}{b_n}t}t^{j-1}dt\\
&+\left|e^{-\frac{a_n}{b_n}z}\right|\|f_m-f\|_{B[0,+\infty)}\\
\le\ & \left|e^{-\frac{a_n}{b_n}z}\right|\sum_{j=1}^{\infty}\frac{(a_n|z|)^j}{j!b_n^j}\frac{a_n^j}{b_n^j}\|f_m-f\|_{B[0,+\infty)}\frac{b_n^j}{a_n^j}+\left|e^{-\frac{a_n}{b_n}z}\right|\|f_m-f\|_{B[0,+\infty)}\\
=\ & \left|e^{-\frac{a_n}{b_n}z}\right|e^{\frac{a_n}{b_n}|z|}\|f_m-f\|_{B[0,+\infty)}\\
\le\ & \left|e^{-\frac{a_n}{b_n}z}\right|e^{\frac{a_n}{b_n}|z|}\|f_m-f\|_r
\end{aligned}$$

valid for all $|z|\le r$. Here $\|\cdot\|_{B[0,+\infty)}$ denotes the uniform norm on $C[0,+\infty)-$ the space of all real-valued bounded functions on $[0,+\infty)$. □

Lemma 3. *If we denote $L_{a_n,b_n}(e_k;z):=L_{a_n,b_n}^{(0,0)}(e_k;z)$, where $e_k(z)=z^k$, then for all $|z|\le r$ with $r\ge 1$, $n\in\mathbb{N}$ and $k\in\mathbb{N}\cup\{0\}$, we have*

$$|\ L_{a_n,b_n}(e_k;z)|\le k!r^k.$$

Proof. We will use the following recurrence formula obtained in the proof of Theorem 1(i) in [4], that is

$$L_{a_n,b_n}(e_{k+1};z)=\frac{b_n}{a_n}z\left(L_{a_n,b_n}(e_k;z)\right)'+\left(z+\frac{b_n}{a_n}k\right)L_{a_n,b_n}(e_k;z)$$

for all $z \in \mathbb{C}$, $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$. Since $L_{a_n,b_n}(e_0; z) = 1$, for $k = 0$ we have

$$|L_{a_n,b_n}(e_1; z)| \leq r$$

for all $|z| \leq r$. Then, for $k = 1$ we get

$$|L_{a_n,b_n}(e_2; z)| \leq \frac{b_n}{a_n} r \left|(L_{a_n,b_n}(e_1; z))'\right| + \left(r + \frac{b_n}{a_n}\right) |L_{a_n,b_n}(e_1; z)| .$$

Taking into account that by Lemma 1 in [4] $L_{a_n,b_n}(e_k; z)$ is a polynomial of degree k, the well-known Bernstein's inequality gives

$$\left|(L_{a_n,b_n}(e_k; z))'\right| \leq \frac{k}{r} \|L_{a_n,b_n}(e_k; z)\|_r .$$

By the last inequality, we find

$$\begin{aligned} |L_{a_n,b_n}(e_2; z)| &\leq \frac{b_n}{a_n} \|L_{a_n,b_n}(e_1; z)\|_r + \left(r + \frac{b_n}{a_n}\right) |L_{a_n,b_n}(e_1; z)| \\ &\leq r \left(r + 2\frac{b_n}{a_n}\right). \end{aligned}$$

By writing for $k = 2, 3, ...$, step by step we easily obtain

$$\begin{aligned} |L_{a_n,b_n}(e_k; z)| &\leq r \left(r + 2\frac{b_n}{a_n}\right) ... \left(r + 2(k-1)\frac{b_n}{a_n}\right) \\ &= \prod_{j=1}^{k} \left(r + 2(j-1)\frac{b_n}{a_n}\right) \\ &\leq r^k \prod_{j=1}^{k} \left(1 + 2(j-1)\frac{b_n}{a_n}\right). \end{aligned}$$

Using $\frac{b_n}{a_n} \leq \frac{1}{2}$, the last inequality follows that

$$|L_{a_n,b_n}(e_k; z)| \leq r^k \prod_{j=1}^{k} j = r^k k!$$

for all $|z| \leq r$, $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$. □

3. Main Results

We obtain upper quantitative estimates for the operator (1.1) in the following theorem.

Theorem 1. *Let* $0 \leq \alpha \leq \beta$, $f \in H(D_R)$ *and suppose that there exist* $M, C, B > 0$ *and* $A \in \left(\frac{1}{R}, 1\right)$, *with the property* $|c_k| \leq M\frac{A^k}{k!}$, *for all* $k = 0, 1, ...,$ *(which implies* $|f(z)| \leq Me^{A|z|}$ *for all* $z \in D_R$ *) and* $|f(x)| \leq Ce^{Bx}$, *for all* $x \in [R, \infty)$. *Also let* $n_0 \in \mathbb{N}$ *be such that* $\frac{a_n}{b_n} + \beta - B > 0$ *for all* $n > n_0$.

i) Let $1 \leq r < \frac{1}{A}$ be arbitrarily fixed. For all $|z| \leq r$ and $n > n_0$ with $n, n_0 \in \mathbb{N}$, we have

$$\left| L_{a_n,b_n}^{(\alpha,\beta)}(f;z) - f(z) \right| \leq C_{r,A} \frac{b_n \left[a_n \left(1+\beta \right) + \beta b_n \right]}{a_n \left(a_n + \beta b_n \right)} ,$$

where

$$C_{r,A} = 2M \sum_{k=1}^{\infty} \left(k+1 \right) \left(rA \right)^k < \infty.$$

ii) For the simultaneous approximation, we have: if $1 \leq r < r_1 < \frac{1}{A}$ are arbitrarily fixed, then for all $n > n_0$, $|z| \leq r$ and $n, p \in \mathbb{N}$,

$$\left| \left(L_{a_n,b_n}^{(\alpha,\beta)}(f;z) \right)^{(p)} - f^{(p)}(z) \right| \leq C_{r_1,A} \frac{b_n \left[a_n \left(1+\beta \right) + \beta b_n \right]}{a_n \left(a_n + \beta b_n \right)} \frac{p! r_1}{(r_1 - r)^{p+1}},$$

where $C_{r_1,A}$ is given in (i).

Proof. (i) By Lemma 1, we can write

$$\begin{aligned} & L_{a_n,b_n}^{(\alpha,\beta)}(e_k;z) - e_k\left(z\right) \\ = & \sum_{j=0}^{k-1} \binom{k}{j} \frac{a_n^j \left(\alpha b_n\right)^{k-j}}{\left(a_n+\beta b_n\right)^k} \left[L_{a_n,b_n}(e_j;z) - e_j\left(z\right) \right] + \sum_{j=0}^{k-1} \binom{k}{j} \frac{a_n^j \left(\alpha b_n\right)^{k-j}}{\left(a_n+\beta b_n\right)^k} e_j\left(z\right) \\ & + \frac{a_n^k}{\left(a_n+\beta b_n\right)^k} \left[L_{a_n,b_n}(e_k;z) - e_k\left(z\right) \right] - \left[1 - \frac{a_n^k}{\left(a_n+\beta b_n\right)^k} \right] e_k\left(z\right), \end{aligned}$$

which follows that

$$\begin{aligned} & \left\| L_{a_n,b_n}^{(\alpha,\beta)}(e_k) - e_k \right\|_r \\ \leq & \sum_{j=0}^{k-1} \binom{k}{j} \frac{a_n^j \left(\alpha b_n\right)^{k-j}}{\left(a_n+\beta b_n\right)^k} \left\| L_{a_n,b_n}(e_j) - e_j \right\|_r + \sum_{j=0}^{k-1} \binom{k}{j} \frac{a_n^j \left(\alpha b_n\right)^{k-j}}{\left(a_n+\beta b_n\right)^k} r^j \\ & + \frac{a_n^k}{\left(a_n+\beta b_n\right)^k} \left\| L_{a_n,b_n}(e_k) - e_k \right\|_r + \left[1 - \frac{a_n^k}{\left(a_n+\beta b_n\right)^k} \right] r^k. \end{aligned}$$

Using the following inequality

$$\left| L_{a_n,b_n}(e_k) - e_k \right| \leq \frac{b_n}{a_n} \left(k+1 \right)! r^{k-1} \tag{3.1}$$

obtained in the proof of Theorem 1(i) in [4], by some calculations the last inequality can be written

$$\begin{aligned}
&\left\|L_{a_n,b_n}^{(\alpha,\beta)}(e_k)-e_k\right\|_r \\
\leq\ & \frac{b_n}{a_n}(k+1)!r^{k-1}\sum_{j=0}^{k-1}\binom{k}{j}\frac{a_n^j(\alpha b_n)^{k-j}}{(a_n+\beta b_n)^k}+r^k\sum_{j=0}^{k-1}\binom{k}{j}\frac{a_n^j(\alpha b_n)^{k-j}}{(a_n+\beta b_n)^k} \\
&+\frac{a_n^k}{(a_n+\beta b_n)^k}\frac{b_n}{a_n}(k+1)!r^{k-1}+\left[1-\frac{a_n^k}{(a_n+\beta b_n)^k}\right]r^k \\
\leq\ & \left(\frac{a_n+\alpha b_n}{a_n+\beta b_n}\right)^k\frac{b_n}{a_n}(k+1)!r^{k-1}+\left[\left(\frac{a_n+\alpha b_n}{a_n+\beta b_n}\right)^k-\frac{a_n^k}{(a_n+\beta b_n)^k}\right]r^k \\
&+\frac{a_n^k}{(a_n+\beta b_n)^k}\frac{b_n}{a_n}(k+1)!r^{k-1}+\left[1-\frac{a_n^k}{(a_n+\beta b_n)^k}\right]r^k \\
\leq\ & 2\frac{b_n}{a_n}(k+1)!r^{k-1}+2\left(1-\frac{a_n^k}{(a_n+\beta b_n)^k}\right)r^k \\
\leq\ & 2\frac{b_n}{a_n}(k+1)!r^{k-1}+2\frac{k\beta b_n}{a_n+\beta b_n}r^k \\
\leq\ & 2(k+1)!r^kb_n\left(\frac{1}{a_n}+\frac{\beta}{a_n+\beta b_n}\right)=2(k+1)!r^kb_n\frac{a_n(1+\beta)+\beta b_n}{a_n(a_n+\beta b_n)}. \quad (3.2)
\end{aligned}$$

As a consequence, from Lemma 2, (3.2) and the hypothesis on c_k, we get

$$\begin{aligned}
\left|L_{a_n,b_n}^{(\alpha,\beta)}(f;z)-f(z)\right| &\leq \sum_{k=1}^{\infty}|c_k|\left|L_{a_n,b_n}^{(\alpha,\beta)}(e_k;z)-e_k(z)\right| \\
&\leq \sum_{k=1}^{\infty}M\frac{A^k}{k!}2(k+1)!r^kb_n\frac{a_n(1+\beta)+\beta b_n}{a_n(a_n+\beta b_n)} \\
&= \frac{b_n[a_n(1+\beta)+\beta b_n]}{a_n(a_n+\beta b_n)}2M\sum_{k=1}^{\infty}(k+1)(rA)^k \\
&= \frac{b_n[a_n(1+\beta)+\beta b_n]}{a_n(a_n+\beta b_n)}C_{r,A},
\end{aligned}$$

where

$$C_{r,A}=2M\sum_{k=1}^{\infty}(k+1)(rA)^k<\infty$$

for all $1\leq r<\frac{1}{A}$. Note that $f(z)=\sum_{k=1}^{\infty}z^{k+1}$ and its derivative $f'(z)=\sum_{k=1}^{\infty}(k+1)z^k$ are absolutely and uniformly convergent in $|z|\leq r$, for any $1\leq r<\frac{1}{A}$.

(ii) For simultaneous approximation, denoting by γ the circle of radius $r_1 > r$ and center 0, since for any $|z| \leq r$ and $\upsilon \in \gamma$, we have $|\upsilon - z| \geq r_1 - r$, by Cauchy's formulas it follows that for all $|z| \leq r$ and $n > n_0$ with $\frac{a_n}{b_n} + \beta - B > 0$, we have

$$\begin{aligned}\left|\left(L_{a_n,b_n}^{(\alpha,\beta)}(f;z)\right)^{(p)} - f^{(p)}(z)\right| &\leq \frac{p!}{2\pi}\int_{\gamma} \frac{\left|L_{a_n,b_n}^{(\alpha,\beta)}(f;\upsilon) - f(\upsilon)\right|}{|\upsilon - z|^{p+1}}|d\upsilon| \\ &\leq \frac{p!}{2\pi} C_{r_1,A}\frac{b_n\left[a_n(1+\beta)+\beta b_n\right]}{a_n(a_n+\beta b_n)}\frac{2\pi r_1}{(r_1-r)^{p+1}} \\ &= C_{r_1,A}\frac{b_n\left[a_n(1+\beta)+\beta b_n\right]}{a_n(a_n+\beta b_n)}\frac{p! r_1}{(r_1-r)^{p+1}}.\end{aligned}$$

□

For the Voronovskaja-type formula with a quantitative estimate, we present the following.

Theorem 2. *Suppose that the hypotheses on the function f and on the constants n_0, R, M, C, B, A in the statement of Theorem 1 hold. Also, let $0 \leq \alpha \leq \beta$ and $1 \leq r < \frac{1}{A}$. Then for all $n > n_0$ and $|z| \leq r$, we have*

$$\begin{aligned}&\left|L_{a_n,b_n}^{(\alpha,\beta)}(f;z) - f(z) - \frac{(\alpha-\beta z)b_n}{a_n+\beta b_n}f'(z) - \frac{b_n z}{a_n}f''(z)\right| \\ \leq\ &\left(\frac{b_n}{a_n}\right)^2 K_{r,A} + \frac{b_n^2}{(a_n+\beta b_n)^2}C_{r,1}^{(\alpha,\beta)} + \frac{b_n^2}{a_n(a_n+\beta b_n)}C_{r,2}^{(\alpha,\beta)},\end{aligned}$$

where

$$\begin{aligned}K_{r,A} &= \frac{4M}{r^2}\sum_{k=2}^{\infty}(k+2)(k+1)(rA)^k < \infty, \\ C_{r,1}^{(\alpha,\beta)} &= \left(\alpha^2+\alpha\beta+2\beta^2\right)M\sum_{k=0}^{\infty}k(k-1)(rA)^k < \infty, \\ C_{r,2}^{(\alpha,\beta)} &= (\alpha+\beta)MA\sum_{k=0}^{\infty}(k+1)k(rA)^{k-1} < \infty.\end{aligned}$$

Proof. For all $z \in D_R$, we can write

$$L_{a_n,b_n}^{(\alpha,\beta)}(f;z) - f(z) - \frac{(\alpha-\beta z)b_n}{a_n+\beta b_n}f'(z) - \frac{b_n}{a_n}zf''(z)$$

$$= L_{a_n,b_n}(f;z) - f(z) - \frac{b_n}{a_n}zf''(z) + L_{a_n,b_n}^{(\alpha,\beta)}(f;z) - L_{a_n,b_n}(f;z) - \frac{(\alpha-\beta z)b_n}{a_n+\beta b_n}f'(z).$$

Taking $f(z)=\sum\limits_{k=0}^{\infty}c_kz^k$, we obtain

$$\begin{aligned}
&L_{a_n,b_n}^{(\alpha,\beta)}(f;z)-f(z)-\frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n}f'(z)-\frac{b_n}{a_n}zf''(z)\\
=\;&\sum_{k=0}^{\infty}c_k\left(L_{a_n,b_n}(e_k;z)-e_k(z)-\frac{b_n}{a_n}zk(k-1)z^{k-2}\right)\\
&+\sum_{k=0}^{\infty}c_k\left(L_{a_n,b_n}^{(\alpha,\beta)}(e_k;z)-L_{a_n,b_n}(e_k;z)-\frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n}kz^{k-1}\right).
\end{aligned}$$

By Theorem 2 in [4], for all $n>n_0$ and $|z|\le r$, we have

$$\left|L_{a_n,b_n}(f;z)-f(z)-\frac{b_n}{a_n}zf''(z)\right|\le\left(\frac{b_n}{a_n}\right)^2K_{r,A},$$

where $K_{r,A}=\frac{4M}{r^2}\sum\limits_{k=2}^{\infty}(k+2)(k+1)(rA)^k<\infty$.

To estimate the second sum, using Lemma 1 and making rearrangements, we easily get

$$\begin{aligned}
&L_{a_n,b_n}^{(\alpha,\beta)}(e_k;z)-L_{a_n,b_n}(e_k;z)-\frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n}kz^{k-1}\\
=\;&\sum_{j=0}^{k-1}\binom{k}{j}\frac{a_n^j(\alpha b_n)^{k-j}}{(a_n+\beta b_n)^k}L_{a_n,b_n}(e_j;z)\\
&-\left(1-\frac{a_n^k}{(a_n+\beta b_n)^k}\right)L_{a_n,b_n}(e_k;z)-\frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n}kz^{k-1}\\
=\;&\sum_{j=0}^{k-2}\binom{k}{j}\frac{a_n^j(\alpha b_n)^{k-j}}{(a_n+\beta b_n)^k}L_{a_n,b_n}(e_j;z)+\frac{ka_n^{k-1}\alpha b_n}{(a_n+\beta b_n)^k}L_{a_n,b_n}(e_{k-1};z)\\
&-\sum_{j=0}^{k-2}\binom{k}{j}\frac{a_n^j(\beta b_n)^{k-j}}{(a_n+\beta b_n)^k}L_{a_n,b_n}(e_k;z)-\frac{ka_n^{k-1}\beta b_n}{(a_n+\beta b_n)^k}L_{a_n,b_n}(e_k;z)-\frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n}kz^{k-1}\\
=\;&\sum_{j=0}^{k-2}\binom{k}{j}\frac{a_n^j(\alpha b_n)^{k-j}}{(a_n+\beta b_n)^k}L_{a_n,b_n}(e_j;z)+\frac{ka_n^{k-1}\alpha b_n}{(a_n+\beta b_n)^k}\left[L_{a_n,b_n}(e_{k-1};z)-e_{k-1}(z)\right]\\
&-\sum_{j=0}^{k-2}\binom{k}{j}\frac{a_n^j(\beta b_n)^{k-j}}{(a_n+\beta b_n)^k}L_{a_n,b_n}(e_k;z)-\frac{ka_n^{k-1}\beta b_n}{(a_n+\beta b_n)^k}\left[L_{a_n,b_n}(e_k;z)-e_k(z)\right]\\
&-\frac{k\alpha b_n}{a_n+\beta b_n}z^{k-1}\left(1-\frac{a_n^{k-1}}{(a_n+\beta b_n)^{k-1}}\right)+\frac{k\beta b_n}{a_n+\beta b_n}z^k\left(1-\frac{a_n^{k-1}}{(a_n+\beta b_n)^{k-1}}\right).
\end{aligned}\tag{3.3}$$

By applying (3.1), Lemma 3 and the following inequalities

$$1-\frac{a_n^k}{(a_n+\beta b_n)^k}\le\sum_{j=1}^{k}\left(1-\frac{a_n}{a_n+\beta b_n}\right)=\frac{k\beta b_n}{a_n+\beta b_n}$$

and

$$\begin{aligned}
&\left|\sum_{j=0}^{k-2}\binom{k}{j}\frac{a_n^j(\alpha b_n)^{k-j}}{(a_n+\beta b_n)^k}L_{a_n,b_n}(e_j;z)\right|\\
\le\ & \sum_{j=0}^{k-2}\binom{k}{j}\frac{a_n^j(\alpha b_n)^{k-j}}{(a_n+\beta b_n)^k}\left|L_{a_n,b_n}(e_j;z)\right|\\
\le\ & \sum_{j=0}^{k-2}\frac{k(k-1)}{(k-j)(k-j-1)}\binom{k-2}{j}\frac{a_n^j(\alpha b_n)^{k-j}}{(a_n+\beta b_n)^k}j!r^j\\
\le\ & \frac{k(k-1)}{2}\frac{(\alpha b_n)^2}{(a_n+\beta b_n)^2}(k-2)!r^{k-2}\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{a_n^j(\alpha b_n)^{k-j-2}}{(a_n+\beta b_n)^{k-2}}\\
\le\ & \frac{k(k-1)}{2}\frac{(\alpha b_n)^2}{(a_n+\beta b_n)^2}(k-2)!r^{k-2},
\end{aligned}$$

in (3.3) we immediately obtain

$$\left|L_{a_n,b_n}^{(\alpha,\beta)}(e_k;z)-L_{a_n,b_n}(e_k;z)-\frac{(\alpha-\beta z)b_n}{a_n+\beta b_n}kz^{k-1}\right|$$

$$\begin{aligned}
\le\ & \frac{k(k-1)}{2}\frac{(\alpha b_n)^2}{(a_n+\beta b_n)^2}(k-2)!r^{k-2}+\frac{k\alpha a_n^{k-1}b_n}{(a_n+\beta b_n)^k}\frac{b_n}{a_n}k!r^{k-2}\\
&+\frac{k(k-1)}{2}\frac{(\beta b_n)^2}{(a_n+\beta b_n)^2}k!r^k+\frac{ka_n^{k-1}\beta b_n}{(a_n+\beta b_n)^k}\frac{b_n}{a_n}(k+1)!r^{k-1}\\
&+\frac{k\alpha b_n}{a_n+\beta b_n}r^{k-1}\frac{(k-1)\beta b_n}{a_n+\beta b_n}+\frac{k\beta b_n}{a_n+\beta b_n}r^k\frac{(k-1)\beta b_n}{a_n+\beta b_n}\\
\le\ & \frac{k(k-1)(\alpha b_n)^2}{2(a_n+\beta b_n)^2}(k-2)!r^{k-2}+\frac{k\alpha b_n^2}{a_n(a_n+\beta b_n)}k!r^{k-2}+\frac{k(k-1)(\beta b_n)^2}{2(a_n+\beta b_n)^2}k!r^k\\
&+\frac{k\beta b_n^2}{a_n(a_n+\beta b_n)}(k+1)!r^{k-1}+\frac{k(k-1)\alpha\beta b_n^2}{(a_n+\beta b_n)^2}r^{k-1}+\frac{k(k-1)(\beta b_n)^2}{(a_n+\beta b_n)^2}r^k\\
\le\ & \frac{b_n^2}{(a_n+\beta b_n)^2}r^k k(k-1)k!\left[\alpha^2+\alpha\beta+2\beta^2\right]+\frac{b_n^2}{a_n(a_n+\beta b_n)}r^{k-1}k(k+1)!\left[\alpha+\beta\right].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\left|\sum_{k=0}^{\infty} c_k \left(L_{a_n,b_n}^{(\alpha,\beta)}(e_k;z) - L_{a_n,b_n}(e_k;z) - \frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n} k z^{k-1}\right)\right| \\
\leq\ & \sum_{k=0}^{\infty} |c_k| \left|L_{a_n,b_n}^{(\alpha,\beta)}(e_k;z) - L_{a_n,b_n}(e_k;z) - \frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n} k z^{k-1}\right| \\
\leq\ & \frac{b_n^2}{(a_n+\beta b_n)^2}\left(\alpha^2+\alpha\beta+2\beta^2\right) M \sum_{k=0}^{\infty} k\,(k-1)\,(rA)^k \\
& + \frac{b_n^2}{a_n\,(a_n+\beta b_n)}\,(\alpha+\beta)\,MA \sum_{k=0}^{\infty} (k+1)\,k\,(rA)^{k-1},
\end{aligned}$$

where the series are convergent for $1 \leq r < \frac{1}{A}$. This proves the theorem. □

Now we obtain the exact orders in approximation by the operators (1.1) and their derivatives on compact disks, respectively.

Theorem 3. *Suppose that the hypothesis of Theorem 1 holds. If f is not a polynomial of degree ≤ 0 for $0 < \alpha \leq \beta$, if f is not a polynomial of degree ≤ 1 for $\alpha = \beta = 0$ and if f is not of the form $f(z) = Ce^{\beta z}$ with $A \neq \beta$ for $0 = \alpha < \beta$, then for all $1 \leq r < \frac{1}{A}$ and $n > n_0$, we have*

$$\left\|L_{a_n,b_n}^{(\alpha,\beta)}(f) - f\right\|_r \sim \frac{b_n}{a_n},$$

where the constants in the equivalence depend only on f, α, β and r.

Proof. For all $|z| \leq r$ and $n \in \mathbb{N}$, we get

$$\begin{aligned}
L_{a_n,b_n}^{(\alpha,\beta)}(f;z) - f(z) \ =\ & \frac{b_n}{a_n}\Bigg\{(\alpha-\beta z)\,f'(z) + z f''(z) + \frac{b_n}{a_n}\left(\frac{a_n}{b_n}\right)^2 \Big[L_{a_n,b_n}^{(\alpha,\beta)}(f;z) - f(z) \\
& - \frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n} f'(z) - \frac{b_n}{a_n} z f''(z) - \frac{b_n}{a_n}\frac{\beta b_n}{(a_n+\beta b_n)}(\alpha-\beta z)\,f'(z)\Big]\Bigg\}.
\end{aligned}$$

Applying the following inequality

$$\|F+G\| \geq |\|F\| - \|G\|| \geq \|F\| - \|G\|,$$

we immediately obtain

$$\begin{aligned}
\left\|L_{a_n,b_n}^{(\alpha,\beta)}(f) - f\right\|_r \ \geq\ & \frac{b_n}{a_n}\Bigg[\|(\alpha-\beta e_1)\,f' + e_1 f''\|_r - \frac{b_n}{a_n}\left(\frac{a_n}{b_n}\right)^2 \Big\|L_{a_n,b_n}^{(\alpha,\beta)}(f) - f \\
& - \frac{(\alpha-\beta e_1)\,b_n}{a_n+\beta b_n} f' - \frac{b_n}{a_n} e_1 f'' - \frac{\beta b_n^2}{a_n\,(a_n+\beta b_n)}(\alpha-\beta e_1)\,f'\Big\|_r\Bigg].
\end{aligned}$$

Considering the hypotheses on f, it is immediate that $\|(\alpha-\beta e_1)f'+e_1 f''\|_r>0$. Indeed, let us suppose the contrary. It follows that

$$(\alpha-\beta z)f'(z)+zf''(z)=0$$

for all $z\in\overline{D_r}$. Here, we have three possible cases. If $0<\alpha\le\beta$, denoting $y(z)=f'(z)$, searching $y(z)$ in the form $y(z)=\sum_{k=0}^{\infty}\delta_k z^k$ and replacing in the above differential equation, we easily obtain $\delta_k=0$ for all $k=0,1,...$, which implies that $f(z)$ is a polynomial of degree≤ 0, a contradiction. If $\alpha=\beta=0$, then we immediately get $f''(z)=0$ for all $|z|\le r$, i.e. f is a polynomial of degree≤ 1, a contradiction. If $0=\alpha<\beta$, the differential equation easily gives the solution $f(z)=Ce^{\beta z}, C\in\mathbb{C}$ arbitrary complex constant, which is a contradiction.

By Theorem 2, it follows that

$$\begin{aligned}
&\left(\frac{a_n}{b_n}\right)^2\left\|L_{a_n,b_n}^{(\alpha,\beta)}(f)-f-\frac{(\alpha-\beta e_1)b_n}{a_n+\beta b_n}f'-\frac{b_n}{a_n}e_1f''-\frac{\beta b_n^2}{a_n(a_n+\beta b_n)}(\alpha-\beta e_1)f'\right\|_r\\
\le\ &\left(\frac{a_n}{b_n}\right)^2\left\|L_{a_n,b_n}^{(\alpha,\beta)}(f)-f-\frac{(\alpha-\beta e_1)b_n}{a_n+\beta b_n}f'-\frac{b_n}{a_n}e_1f''\right\|_r+\frac{a_n}{a_n+\beta b_n}\|\beta(\alpha-\beta e_1)f'\|_r\\
\le\ &K_{r,A}+\frac{a_n^2}{(a_n+\beta b_n)^2}C_{r,1}^{(\alpha,\beta)}+\frac{a_n}{a_n+\beta b_n}C_{r,2}^{(\alpha,\beta)}+\frac{a_n}{a_n+\beta b_n}\beta(\alpha+\beta r)\|f'\|_r\\
\le\ &K_{r,A}+C_{r,1}^{(\alpha,\beta)}+C_{r,2}^{(\alpha,\beta)}+\beta(\alpha+\beta r)\|f'\|_r.
\end{aligned}$$

Consequently, there exists an index $n_1>n_0$ (depending on f,α,β and r only) such that for all $n\ge n_1$, we get

$$\begin{aligned}
&\|(\alpha-\beta e_1)f'+e_1f''\|_r\\
&-\frac{b_n}{a_n}\left(\frac{a_n}{b_n}\right)^2\left\|L_{a_n,b_n}^{(\alpha,\beta)}(f)-f-\frac{(\alpha-\beta e_1)b_n}{a_n+\beta b_n}f'-\frac{b_n}{a_n}e_1f''-\frac{\beta b_n^2}{a_n(a_n+\beta b_n)}(\alpha-\beta e_1)f'\right\|_r\\
&\ge\frac{1}{2}\|(\alpha-\beta e_1)f'+e_1f''\|_r,
\end{aligned}$$

which implies that

$$\left\|L_{a_n,b_n}^{(\alpha,\beta)}(f)-f\right\|_r\ge\frac{b_n}{2a_n}\|(\alpha-\beta e_1)f'+e_1f''\|_r$$

for all $n\ge n_1$. For $n\in\{n_0+1,...,n_1-1\}$, we have

$$\left\|L_{a_n,b_n}^{(\alpha,\beta)}(f)-f\right\|_r\ge\frac{b_n}{a_n}M_{r,n}(f)$$

with $M_{r,n}(f)=\frac{a_n}{b_n}\left\|L_{a_n,b_n}^{(\alpha,\beta)}(f)-f\right\|_r>0$. Therefore, for all $n>n_0$, finally we get

$$\left\|L_{a_n,b_n}^{(\alpha,\beta)}(f)-f\right\|_r\ge\frac{b_n}{a_n}C_r(f)$$

where

$$C_r(f) = \min\left\{ M_{r,n_0+1}(f), ..., M_{r,n_1-1}(f), \frac{1}{2}\left\|(\alpha-\beta e_1) f' + e_1 f''\right\|_r \right\}.$$

Combining the last inequality with Theorem 1(i), it gives the desired conclusion. □

Theorem 4. *Suppose that the hypothesis of Theorem 1 holds and let* $1 \le r < r_1 < \frac{1}{A}$ *and* $p \in \mathbb{N}$ *be fixed. If* f *is not a polynomial of degree* $\le p-1$ *for* $0 < \alpha \le \beta$, *if* f *is not a polynomial of degree* $\le p$ *for* $\alpha = \beta = 0$ *and if* f *is not of the form* $f(z) = Ce^{\beta z}$ *with* $A \ne \beta$ *for* $0 = \alpha < \beta$, *then for all* $n > n_0$ *and* $|z| \le r$, *we have*

$$\left\| \left(L_{a_n,b_n}^{(\alpha,\beta)}(f)\right)^{(p)} - f^{(p)} \right\|_r \sim \frac{b_n}{a_n},\ n \in \mathbb{N}$$

where the constants in the equivalence depend only on f, α, β, p, r_1 and r.

Proof. Taking the upper estimate in Theorem 1 (ii) into consideration, it remains to prove the lower estimate for $\left\| \left(L_{a_n,b_n}^{(\alpha,\beta)}(f)\right)^{(p)} - f^{(p)} \right\|_r$. Denoting by Γ the circle of radius r_1 and center 0 (where $r_1 > r \ge 1$), we have $|\upsilon - z| \ge r_1 - r$ valid for all $|z| \le r$ and $\upsilon \in \Gamma$.

For all $\upsilon \in \Gamma$ and $n > n_0$, we have

$$\begin{aligned} & L_{a_n,b_n}^{(\alpha,\beta)}(f;\upsilon) - f(\upsilon) \\ = & \frac{b_n}{a_n}\left\{ (\alpha - \beta\upsilon) f'(\upsilon) + \upsilon f''(\upsilon) + \frac{b_n}{a_n}\left[\left(\frac{a_n}{b_n}\right)^2 \left(L_{a_n,b_n}^{(\alpha,\beta)}(f;\upsilon) - f(\upsilon) \right.\right.\right. \\ & \left.\left.\left. - \frac{(\alpha-\beta\upsilon) b_n}{a_n + \beta b_n} f'(\upsilon) - \frac{b_n}{a_n}\upsilon f''(\upsilon) \right) - \frac{\beta a_n}{(a_n+\beta b_n)}(\alpha - \beta\upsilon) f'(\upsilon) \right] \right\}. \end{aligned}$$

Applying Cauchy's formula for derivatives, we can write

$$\left(L_{a_n,b_n}^{(\alpha,\beta)}(f;z)\right)^{(p)} - f^{(p)}(z) = \frac{b_n}{a_n}\left\{ \left[(\alpha-\beta z) f'(z) + z f''(z)\right]^{(p)} \right.$$

$$+ \frac{b_n}{a_n}\left[\frac{p!}{2\pi i}\int_\Gamma \frac{\left(\frac{a_n}{b_n}\right)^2 \left(L_{a_n,b_n}^{(\alpha,\beta)}(f;\upsilon) - f(\upsilon) - \frac{(\alpha-\beta\upsilon)b_n}{a_n+\beta b_n} f'(\upsilon) - \frac{b_n}{a_n}\upsilon f''(\upsilon)\right)}{(\upsilon - z)^{p+1}} d\upsilon \right.$$

$$\left.\left. - \frac{p!}{2\pi i}\int_\Gamma \frac{\frac{\beta a_n}{(a_n+\beta b_n)}(\alpha-\beta\upsilon) f'(\upsilon)}{(\upsilon-z)^{p+1}} d\upsilon \right]\right\}.$$

For all $|z| \le r$ and $n > n_0$, we obtain

$$\left\| \left(L_{a_n,b_n}^{(\alpha,\beta)}(f)\right)^{(p)} - f^{(p)} \right\|_r \ge \frac{b_n}{a_n}\left\{ \left\| \left[(\alpha - \beta e_1) f' + e_1 f''\right]^{(p)} \right\|_r \right.$$

$$-\frac{b_n}{a_n}\left\|\frac{p!}{2\pi i}\int_{\Gamma}\frac{\left(\frac{a_n}{b_n}\right)^2\left(L_{a_n,b_n}^{(\alpha,\beta)}(f;\upsilon)-f(\upsilon)-\frac{(\alpha-\beta\upsilon)b_n}{a_n+\beta b_n}f'(\upsilon)-\frac{b_n}{a_n}\upsilon f''(\upsilon)\right)}{(\upsilon-z)^{p+1}}d\upsilon\right.$$
$$\left.\left.-\frac{p!}{2\pi i}\int_{\Gamma}\frac{\frac{\beta a_n}{(a_n+\beta b_n)}(\alpha-\beta\upsilon)f'(\upsilon)}{(\upsilon-z)^{p+1}}d\upsilon\right\|_r\right\}.$$

From Theorem 2, for all $n>n_0$, it follows

$$\left\|\frac{p!}{2\pi i}\int_{\Gamma}\frac{\left(\frac{a_n}{b_n}\right)^2\left(L_{a_n,b_n}^{(\alpha,\beta)}(f;\upsilon)-f(\upsilon)-\frac{(\alpha-\beta\upsilon)b_n}{a_n+\beta b_n}f'(\upsilon)-\frac{b_n}{a_n}\upsilon f''(\upsilon)\right)}{(\upsilon-z)^{p+1}}d\upsilon\right.$$
$$\left.-\frac{p!}{2\pi i}\int_{\Gamma}\frac{\frac{\beta a_n}{(a_n+\beta b_n)}(\alpha-\beta\upsilon)f'(\upsilon)}{(\upsilon-z)^{p+1}}d\upsilon\right\|_r$$
$$\leq \frac{p!}{2\pi}\frac{2\pi r_1}{(r_1-r)^{p+1}}\left[K_{r_1,A}+C_{r_1,1}^{(\alpha,\beta)}+C_{r_1,2}^{(\alpha,\beta)}\right]+\frac{p!}{2\pi}\frac{2\pi r_1}{(r_1-r)^{p+1}}\beta(\alpha+\beta r_1)\|f'\|_{r_1}.$$

By the hypothesis on f , we have $\left\|\left[(\alpha-\beta e_1)f'+e_1f''\right]^{(p)}\right\|_r>0$. In continuation, by exactly the lines in the proof of Theorem 3, we easily prove our assertion. □

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