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Intuitionistic Fuzzy Partial Metric Spaces

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Abstract

Fuzzy logic is a theory used as an alternative to classical structures in both application and algebraic fields. In particular, there are fuzzy structures in metric spaces and partial metric spaces. The most widely used of these structures are fuzzy metric spaces, fuzzy partial metric spaces and intuitionistic fuzzy metric spaces. In this paper, intuitionistic fuzzy partial metric spaces are defined, their basic properties and examples are obtained. For it, open ball, convergent sequence, and Cauchy sequence are defined and their basic properties are introduced. Furthermore, the relations between intuitionistic fuzzy partial metric spaces, classical metric spaces, fuzzy metric spaces, fuzzy partial metric spaces, and intuitionistic fuzzy metric spaces are analyzed. As a result of this investigation, it is shown that from each classical metric, classical partial metric, and intuitionistic fuzzy metric, an intuitionistic fuzzy partial metric can be obtained. Moreover, it is achieved that an intuitionistic fuzzy metric is also an intuitionistic fuzzy partial metric space. Thus, a new structure is given by transferring the partial metric structure to intuitionistic fuzzy metric spaces.

Keywords: Metric spaces; Partial metric spaces; Fuzzy metric spaces; Fuzzy partial metric spaces; Intuitionistic fuzzy metric spaces; Intuitionistic fuzzy partial metric spaces.

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Sezgisel Bulanık Kısmi Metrik Uzaylar

Öz

Bulanık mantık, hem uygulama hem de cebirsel alanlarda klasik yapılara alternatif olarak kullanılan bir teoridir. Özellikle metrik uzaylarda ve kısmi metrik uzaylarda bulanık yapılar yer almaktadır. Bu yapılardan en yaygın olarak kullanılan bulanık metrik uzaylar, bulanık kısmi metrik uzaylar ve sezgisel bulanık metrik uzaylardır. Bu çalışmada, sezgisel bulanık kısmi metrik uzaylar tanımlanmış, temel özellikleri ve örnekleri elde edilmiştir. Bunun için açık yuvar, yakınsak dizi ve Cauchy dizisi tanımlanmış ve temel özellikleri tanıtılmıştır. Ayrıca, sezgisel bulanık kısmi metrik uzaylar, klasik metrik uzaylar, bulanık metrik uzaylar, bulanık kısmi metrik uzaylar ve sezgisel bulanık metrik uzaylar arasındaki ilişkiler incelenmiştir. Bu inceleme sonucunda, her bir klasik metrik, klasik kısmi metrik ve sezgisel bulanık metrikten bir sezgisel bulanık kısmi metrik elde edilebileceği gösterilmiştir. Ayrıca, bir sezgisel bulanık metriğin aynı zamanda bir sezgisel bulanık kısmi metrik uzay olduğu elde edilmiştir. Böylece kısmi metrik yapısı sezgisel bulanık metrik uzaylara aktarılarak yeni bir yapı verilmiştir.

Anahtar Kelimeler: Metrik uzaylar; Kısmi metrik uzaylar; Bulanık metrik uzaylar; Bulanık kısmi metrik uzaylar; Sezgisel bulanık metrik uzaylar; Sezgisel bulanık kısmi metrik uzaylar.

1. Introduction

Partial metric spaces [1] were defined by Matthews in 1994. The most important feature that distinguishes this metric from the classical metric is that the distance of a point to itself is not always zero. This property leads to important properties in fixed point theories (FPTs).

Zadeh defined fuzzy logic (FL) and fuzzy sets (FS) [2] in 1965 to explain uncertainties more precisely mathematically. In FS, the degree of membership of each element of set takes a value in the range [0, 1]. Thus, unlike classical logic, the membership of each element is graded. For example, the weather can be specified with expressions such as hot, cold, warm, cool, very hot, very cold, etc., and with different degrees of membership. Thus, a more precise type of logic, including classical logic, has been defined to explain uncertainties. FL is one of the most widely used types of logic in almost every field of science, including decision-making applications and algebraic fields. Recently, Emniyet and Şahin obtained fuzzy normed rings [3] in 2018. Kum et al. studied an alternative method for determining erosion risk based on FL [4] in 2022; Wang et al. introduced a new distance measure for q-Rung Orthopair FSs [5] in 2024; Xu and Wang introduced a novel fuzzy bi-clustering algorithm for Co-Regulated Genes [6] in 2024; Plebankiewicz and Karcińska studied supporting construction using to the FSs theory [7] in 2024.

In 1975, Kramosil and Michálek defined fuzzy metric space (FMS) [8]. Also, Grabiec obtained fixed points in FMSs [9] in 1989. This new type of metric, defined as an alternative to classical metric spaces, brings the advantages of FL to the theory of metric spaces. Thus, the distance between two points is graded with the help of a fuzzy membership function. Many classical metric types have been similarly redefined according to FL. Recently, Shukla et al. studied vector-valued FMS [10] in 2024; Gregori et al. achieved completeness FPT based on FMSs [11] in 2024; Huang introduced properties for some metric spaces based on FSs [12] in 2024.

Moreover, fuzzy partial metric space (FPMS) has been defined differently in three different works [13-15] in different years. Olgun et al. introduced the basic definition and properties were given and provided an FPT for FPMS [13]. Recently, Aygün et al. studied FPTs based on FPMS [16] in 2022; Gregori et al. studied relationship between FPMS and fuzzy quasi-metrics [17] in 2020.

In 1986, Atanassov [18] defined intuitionistic fuzzy sets (IFS) by including the degree of uncertainty in addition to the degree of membership and non-membership in FL. In this type of logic, the degree of uncertainty is defined depending on the degree of membership and the degree of non-membership so that the sum of the degree of membership, degree of non-membership, and degree of uncertainty is 1. Thus, a new type of logic that can give more precise results than FL has emerged. Intuitionistic fuzzy logic is frequently used in areas where FL is insufficient. Recently, Ngan studied creating operators and functions based on IFSs [19] in 2024; Gerogiannis et al. obtained an approach for IFSs [20] in 2024; Rajafillah et al. defined intuitionistic fuzzy pooling [21] in 2024.

In 2004, Park defined intuitionistic fuzzy metric spaces (IFMS) [22]. In this study, open balls, convergent sequence, Cauchy sequence, and complete space are defined for IFMS and given an FPT. Also, thanks to this metric space, the advantage of using IFSs is brought to the theory of metric spaces. While defining this metric space, the relationship between FL and intuitionistic FL was transferred to metric spaces. Thus, many researchers are working on IFMS [23-26]. Recently, Wong et al. studied complex-valued IFMS [27] in 2024; Singh et al. defined fuzzy differential equations based on IFMS [28] in 2024.

In this paper, we define intuitionistic fuzzy partial metric spaces (IFPMS) for the first time and give their basic properties. These definitions and properties are obtained by considering the basic definitions and properties given for FPMS in the study [13] of Olgun et al. and in the study [22] of Park. The IFPMS defined in Section 3 have new properties different from the other structures, but they also provide some basic properties of FMS, FPMS, and IFMS. For this reason, to make the definitions, properties, and results obtained in Section 3 more comprehensible and to easily show which definitions, properties, and results are given by making use of which definitions, properties, and results, the basic information in Section 2 is included. In Section 3, IFPMS are defined and some examples are given. It is also shown that every IFMS is also an IFPMS. It is also shown that one can obtain an IFPMS from every classical partial metric and every IFMS. For IFPMS, basic properties are given. Also, open balls, convergent sequence, Cauchy sequence, and complete space are defined for IFPMS. The basic properties of these structures are given. Thus, existence of IFPMS is proved for each classical, classical partial, and IFMS. In the last section, the conclusions of this study are given and suggestions are made to researchers about the structures that they can obtain by using this study.

2. Preliminaries

In this section, the basic definitions, examples, lemmas, and properties are given based on FS, IFS, FMS and IFMS. These basic contractures are used in Section 3.

Definition 1. [2] Let E be a non-empty set. Fuzzy set K is denoted by

$$\mathrm{K} = \{\langle \gamma, \mu_K(\gamma)\rangle : \gamma \in E\}.$$

where

$$\mu_K : E \to [0,1]$$

is the membership function of K. For example, $\mu_K(\gamma)$ is the membership value of $\gamma \in E$.

Definition 2. [18] Let E be a non-empty set. Intuitionistic fuzzy set L is denoted by

$$\mathrm{L} = \{\langle \gamma, \mu_L(\gamma), \nu_L(\gamma)\rangle : \gamma \in E\}.$$

where

$$\mu_L : E \to [0,1]$$

is the membership function of L and

$$\nu_L : E \to [0,1]$$

is the non-membership function of L such that

$$0 \leq \mu_L(\gamma) + \nu_L(\gamma) \leq 1.$$

Also, $\mu_L(\gamma)$ and $\nu_L(\gamma)$ are membership values of γ and non-membership values of γ; respectively.

Definition 3. [29] Let $\oplus$ be a binary operation such that

$\oplus: [0,1] \times [0,1] \to [0,1]$.

If the following properties are satisfied, then $\oplus$ is called a continuous t-norm (CTN). For $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in [0,1]$,

i. $\gamma_1 \oplus 1 = \gamma_1$,
ii. If $\gamma_1 \leq \gamma_2$ and $\gamma_3 \leq \gamma_4$, then $\gamma_1 \oplus \gamma_3 \leq \gamma_2 \oplus \gamma_4$,
iii. $\oplus$ is continuous,
iv. $\oplus$ is commutative and associative.

Definition 4. [29] Let $\circledcirc$ be a binary operation such that

$\circledcirc: [0,1] \times [0,1] \to [0,1]$.

If the following properties are satisfied, then $\circledcirc$ is called a continuous t-conorm (CTCN). For $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in [0,1]$,

i. $\gamma_1 \circledcirc 0 = \gamma_1$,
ii. If $\gamma_1 \leq \gamma_2$ and $\gamma_3 \leq \gamma_4$, then $\gamma_1 \circledcirc \gamma_3 \leq \gamma_2 \circledcirc \gamma_4$,
iii. $\circledcirc$ is continuous,
iv. $\circledcirc$ is commutative and associative.

Definition 5. [13] Let $\mathcal{X}$ be a non-empty set, $\oplus$ be a CTN and $\mathcal{R}$ be FS on $\mathcal{X}^2$x(0, ∞). A 3-tuple ($\mathcal{X}$, $\mathcal{R}$, $\oplus$) is said to be an FPMS if the following conditions are satisfied. For each $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{X}$; $\rho, p_1, p_2 > 0$,

i. $0 \leq \mathcal{R}(\gamma_1, \gamma_2, \rho) \leq 1$,
ii. $\mathcal{R}(\gamma_1, \gamma_1, \rho) \geq \mathcal{R}(\gamma_1, \gamma_2, \rho)$,
iii. $\mathcal{R}(\gamma_1, \gamma_2, \rho) = \mathcal{R}(\gamma_2, \gamma_1, \rho)$,
iv. $\mathcal{R}(\gamma_1, \gamma_2, \rho) = \mathcal{R}(\gamma_1, \gamma_1, \rho) = \mathcal{R}(\gamma_2, \gamma_2, \rho)$ if and only if $\gamma_1 = \gamma_2$,
v. $\mathcal{R}(\gamma_1, \gamma_3, p_1) \oplus \mathcal{R}(\gamma_3, \gamma_2, p_2) \leq \mathcal{R}(\gamma_1, \gamma_3, p_1 + p_2) \oplus \mathcal{R}(\gamma_3, \gamma_3, \rho)$, $\rho \geq p_1$ and $\rho \geq p_2$,
vi. $\mathcal{R}(\gamma_1, \gamma_2, .): [0, \infty) \to [0,1]$ is continuous,

The function $\mathcal{R}(\gamma_1, \gamma_2, \rho)$ denotes the degrees of nearness, between γ_1 and γ_2 with respect to ρ.

Definition 6. [13] Each FPMS $\mathcal{R}(\gamma_1,\gamma_2,\rho)$ on $\mathcal{X}^2$x(0, ∞) generates a topology τ on $\mathcal{X}$ with the family of open M-balls

$$\{B_M(\gamma_1,\varepsilon)\colon \gamma_1 \in X, 0 < \varepsilon < \mathcal{R}(\gamma_1,\gamma_1,p_1)\}$$

as a base, where for all $\gamma_1 \in \mathcal{X}$,

$$B_M(\gamma_1,\varepsilon) = \{\gamma_2 \ \in \mathcal{X}\colon \mathcal{R}(\gamma_1,\gamma_2,\rho) \oplus \mathcal{R}(\gamma_{1_1},\gamma_1,p_1) > \mathcal{R}(\gamma_{1_1},\gamma_1,p_1) - \ \varepsilon, p_1 \geq \rho\}.$$

Definition 7. [13] A sequence $\{\gamma_n\}$ in an FPMS $\mathcal{R}(\gamma_1,\gamma_2,\rho)$ converges to $\gamma_1 \in \mathcal{X}$ with respect to T if and only if

$$\lim_{n\to\infty} \mathcal{R}(\gamma_1,\gamma_n,\rho) = \ \mathcal{R}(\gamma_1,\gamma_1,\rho).$$

Definition 8. [13] A sequence $\{\gamma_n\}$ in an FPMS $\mathcal{R}(\gamma_1,\gamma_2,\rho)$ is called Cauchy if

$$\lim_{n,m\to\infty} \mathcal{R}(\gamma_m,\gamma_n,\rho)$$

exists and finite.

Lemma 1. [13] Let ($\mathcal{X}$, $\mathcal{R}$, $\oplus$) be an FPMS. Then,

i. If $\mathcal{R}(\gamma_1,\gamma_2,\rho) = 1$, then $\gamma_1 = \gamma_2$.
ii. If $\gamma_1 \neq \gamma_2$, then $\mathcal{R}(\gamma_1,\gamma_2,\rho) < 1$.
iii. If $\gamma_n \to \gamma_3$ with $\mathcal{R}(\gamma_3,\gamma_3,\rho)$ =1, then $\lim_{n\to\infty} \mathcal{R}(\gamma_2,\gamma_n,\rho) = \ \mathcal{R}(\gamma_2,\gamma_3,\rho)$ for all $\gamma_2 \in \mathcal{X}$.

Definition 9. [22] Let $\mathcal{X}$ be a non-empty set, $\oplus$ be a CTN, $\odot$ be a CTCN and $\mathcal{R}$, $\mathcal{S}$ be FS on $\mathcal{X}^2$x(0, ∞). A 5-tuple ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, $\oplus$, $\odot$) is said to be an IFMS if the following conditions are satisfied. For all $\gamma_1,\gamma_2,\gamma_3 \in \mathcal{X}$; $\rho, p_1, p_2 > 0$,

i. $0 \leq \mathcal{R}(\gamma_1,\gamma_2,\rho) \leq 1, 0 \leq \mathcal{S}(\gamma_1,\gamma_2,\rho) \leq 1$ and $0 \leq \mathcal{R}(\gamma_1,\gamma_2,\rho) + \ \mathcal{S}(\gamma_1,\gamma_2,\rho) \leq 1$,
ii. $\mathcal{R}(\gamma_1,\gamma_2,\rho) = 1$ if and only if $\gamma_1 = \gamma_2$,
iii. $\mathcal{R}(\gamma_1,\gamma_2,\rho) = \mathcal{R}(\gamma_2,\gamma_1,\rho)$,
iv. iv. $\mathcal{R}(\gamma_1,\gamma_2,p_1) \oplus \mathcal{R}(\gamma_2,\gamma_3,\rho) \leq \mathcal{R}(\gamma_1,\gamma_3,\rho + p_1)$,
v. $\mathcal{R}(\gamma_1,\gamma_2, \ .)\colon [0,\infty) \ \to [0,1]$ is continuous,
vi. vi. $\lim_{\varepsilon\to\infty} \mathcal{R}(\gamma_1,\gamma_2,\rho) = 1$,
vii. $\mathcal{S}(\gamma_1,\gamma_2,\rho) = 0$ if and only if $\gamma_1 = \gamma_2$,
viii. $\mathcal{S}(\gamma_1,\gamma_2,\rho) = \mathcal{S}(\gamma_2,\gamma_1,\rho)$,
ix. $\mathcal{S}(\gamma_1,\gamma_2,p_1) \odot \mathcal{S}(\gamma_2,\gamma_3,\rho) \geq \mathcal{S}(\gamma_1,\gamma_3,\rho + p_1)$,

The functions $\mathcal{R}(\gamma_1, \gamma_2, \rho)$ and $\mathcal{S}(\gamma_1, \gamma_2, \rho)$ denote the degrees of nearness, the degrees of non-nearness between γ_1 and γ_2 with respect to ρ; respectively.

Example 1. [22] Let d be a metric such that d: $\mathcal{X}^2 \rightarrow [0, \infty)$, $\mathcal{R}$ and $\mathcal{S}$ be two functions on $\mathcal{X}^2$x$[0, \infty)$ such that

$$\mathcal{R}(\gamma_1, \gamma_2, \rho) = \frac{\rho}{\rho + d(\gamma_1, \gamma_2)} \text{ and } \mathcal{S}(\gamma_1, \gamma_2, \rho) = \frac{d(\gamma_1, \gamma_2)}{\rho + d(\gamma_1, \gamma_2)}.$$

Also, if we take $\gamma_1 \oplus \gamma_2 = \min\{\gamma_1, \gamma_2\}$ and $\gamma_1 \circledcirc \gamma_2 = \max\{\gamma_1, \gamma_2\}$. Thus, $(\mathcal{X}, \mathcal{R}, \mathcal{S}, \oplus, \circledcirc)$ satisfies the conditions of IFMS.

Definition 10. [22] Let $(\mathcal{X}, \mathcal{R}, \mathcal{S}, \oplus, \circledcirc)$ be an IFMS and $0 < r < 1$, $\rho > 0$ and $\gamma_1 \in \mathcal{X}$. The set

$$B(\gamma_1, r, \rho) = \{ \gamma_2 \in \mathcal{X}: \mathcal{R}(\gamma_1, \gamma_2, \rho) > 1 - r, \mathcal{S}(\gamma_1, \gamma_2, \rho) < r \}$$

is called the open ball with center γ_1 and radius r with respect to ρ.

Remark 1. [22] Let $(\mathcal{X}, \mathcal{R}, \mathcal{S}, \oplus, \circledcirc)$ be an IFMS. Define

$\tau(\mathcal{R}, \mathcal{S}) = \{A \subset \mathcal{X}$: for each $\gamma_1 \in A$, there exists $\rho > 0$ and $0 < r < 1$ such that $B(\gamma_1, r, \rho) \subset A\}$.

Then, $\tau(\mathcal{R}, \mathcal{S})$ is a topology on $\mathcal{X}$.

Lemma 2. [22] Let $(\mathcal{X}, \mathcal{R}, \mathcal{S}, \oplus, \circledcirc)$ be an IFMS. Then, $\mathcal{R}(\gamma_1, \gamma_2, .)$ is non-decreasing and $\mathcal{S}(\gamma_1, \gamma_2, .)$ is non-increasing for all $\gamma_1, \gamma_2 \in \mathcal{X}$.

Theorem 1. [22] Let $(\mathcal{X}, \mathcal{R}, \mathcal{S}, \oplus, \circledcirc)$ be an IFMS. Every open ball $B(\gamma_2, r, \rho)$ in this space is an open set.

Theorem 2. [22] Let $(\mathcal{X}, \mathcal{R}, \mathcal{S}, \oplus, \circledcirc)$ be an IFMS and $\tau(\mathcal{R}, \mathcal{S})$ be a topology on $\mathcal{X}$ induced by the $(\mathcal{X}, \mathcal{R}, \mathcal{S}, \oplus, \circledcirc)$. Then for a sequence $\{\gamma_n\}$ in $(\mathcal{X}, \mathcal{R}, \mathcal{S}, \oplus, \circledcirc)$,

$$\gamma_n \rightarrow \gamma_1$$

if and only if

$$\lim_{n\to\infty} \mathcal{R}(\gamma_1, \gamma_n, \rho) = 1 \text{ and } \lim_{n\to\infty} \mathcal{S}(\gamma_1, \gamma_n, \rho) = 0.$$

Definition 11. [22] Let ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, $\oplus$, $\circledcirc$) be an IFMS and $\tau(\mathcal{R}, \mathcal{S})$ be a topology on $\mathcal{X}$ induced by the ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, $\oplus$, $\circledcirc$). Then a sequence $\{\gamma_n\}$ in ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, $\oplus$, $\circledcirc$) is said to be a Cauchy sequence with respect to $\tau(\mathcal{R}, \mathcal{S})$ if for ρ > 0, r ∈ (0, 1), there exists n_0 ∈ ℕ, n, m > n_0 such that

$$\mathcal{R}(\gamma_m, \gamma_n, \rho) \oplus \mathcal{R}(\gamma_m, \gamma_m, \rho) > 1 - r$$

and

$$\mathcal{S}(\gamma_m, \gamma_n, \rho) \circledcirc \mathcal{S}(\gamma_m, \gamma_m, \rho) < r.$$

3. Results

In this section, IFPMS are defined and their basic properties and examples are given. For IFPMS, open ball, convergent sequence, and Cauchy sequence are defined and their basic properties are achieved using the basic definitions and basic properties in Section 2. Furthermore, the relations between IFPMS and classical metric spaces, FMSs, FPMS, and IFMS are analyzed.

Definition 12. Let $\mathcal{X}$ be a non-empty set, $\oplus$ be a CTN, $\circledcirc$ be a CTCN, $\mathcal{R}$ and $\mathcal{S}$ be FSs on $\mathcal{X}^2$x(0, ∞). A 5-tuple ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, $\oplus$, $\circledcirc$) is said to be an IFPMS if the following conditions are satisfied. For all $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{X}$; $\rho, p_1, p_2 > 0$,

i. $0 \le \mathcal{R}(\gamma_1, \gamma_2, \rho) \le 1, 0 \le \mathcal{S}(\gamma_1, \gamma_2, \rho) \le 1$ and $0 \le \mathcal{R}(\gamma_1, \gamma_2, \rho) + \mathcal{S}(\gamma_1, \gamma_2, \rho) \le 1$,
ii. $\mathcal{R}(\gamma_1, \gamma_1, \rho) \ge \mathcal{R}(\gamma_1, \gamma_2, \rho)$,
iii. $\mathcal{R}(\gamma_1, \gamma_2, \rho) = \mathcal{R}(\gamma_2, \gamma_1, \rho)$,
iv. $\mathcal{R}(\gamma_1, \gamma_2, \rho) = \mathcal{R}(\gamma_1, \gamma_1, \rho) = \mathcal{R}(\gamma_2, \gamma_2, \rho)$ if and only if $\gamma_1 = \gamma_2$,
v. $\mathcal{R}(\gamma_1, \gamma_3, p_1) \oplus \mathcal{R}(\gamma_3, \gamma_2, p_2) \le \mathcal{R}(\gamma_1, \gamma_3, p_1 + p_2) \oplus \mathcal{R}(\gamma_3, \gamma_3, \rho)$, $\rho \ge p_1$ and $\rho \ge p_2$,
vi. $\mathcal{R}(\gamma_1, \gamma_2, .): [0, \infty) \to [0,1]$ is continuous,
vii. $\lim_{\rho \to \infty} \mathcal{R}(\gamma_1, \gamma_1, \rho) = 1$,
viii. $\mathcal{S}(\gamma_1, \gamma_1, \rho) \le \mathcal{S}(\gamma_1, \gamma_2, \rho)$,
ix. $\mathcal{S}(\gamma_1, \gamma_2, \rho) = \mathcal{S}(\gamma_2, \gamma_1, \rho)$,
x. $\mathcal{S}(\gamma_1, \gamma_2, \rho) = \mathcal{S}(\gamma_1, \gamma_1, \rho) = \mathcal{S}(\gamma_2, \gamma_2, \rho)$ if and only if $\gamma_1 = \gamma_2$,
xi. $\mathcal{S}(\gamma_1, \gamma_3, p_1) \circledcirc \mathcal{S}(\gamma_3, \gamma_2, p_2) \ge \mathcal{S}(\gamma_1, \gamma_3, p_1 + p_2) \circledcirc \mathcal{S}(\gamma_3, \gamma_3, \rho)$, $\rho \ge p_1$ and $\rho \ge p_2$,
xii. $\mathcal{S}(\gamma_1, \gamma_2, .): [0, \infty) \to [0,1]$ is continuous,
xiii. $\lim_{\rho \to \infty} \mathcal{S}(\gamma_1, \gamma_1, \rho) = 0$,

The functions $\mathcal{R}(\gamma_1, \gamma_2, \rho)$ and $\mathcal{S}(\gamma_1, \gamma_2, \rho)$ denote the degrees of nearness, the degrees of non-nearness between γ_1 and γ_2 with respect to ρ, respectively.

Example 2. Let p be a partial metric such that p: $\mathcal{X}^2 \rightarrow [0, \infty)$, $\mathcal{R}$ and $\mathcal{S}$ be two functions on X^2x$[0, \infty)$ such that

$$\mathcal{R}(\gamma_1, \gamma_2, \rho) = \frac{\rho}{\rho + p(\gamma_1, \gamma_2)} \text{ and } \mathcal{S}(\gamma_1, \gamma_2, \rho) = \frac{p(\gamma_1, \gamma_2)}{\rho + p(\gamma_1, \gamma_2)}.$$

Also, we take a⊕b = $\min\{\gamma_1, \gamma_2\}$ and a⊚b = $\max\{\gamma_1, \gamma_2\}$. Thus, ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, ⊕, ⊚) satisfies the conditions of IFPMS.

Corollary 1. In Example 2, if we take for all $\gamma_1, \gamma_2 \in \mathcal{X}$,

$$\mathcal{R}(\gamma_1, \gamma_2, \rho) = 1 \text{ and } \mathcal{S}(\gamma_1, \gamma_2, \rho) = 0.$$

then, ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, ⊕, ⊚) satisfies the conditions of IFMS. Thus, from Definition 13 and Definition 9, every IFMS is also an IFPMS.

Corollary 2. From Example 2, we can obtain an IFPMS from every partial metric space.

Example 3. In Example 1, if we take for all $p_1 \in [0, \infty)$,

$$\mathcal{R}(\gamma_1, \gamma_2, \rho) = \frac{\rho + p_1}{\rho + p_1 + d(\gamma_1, \gamma_2)} \text{ and } \mathcal{S}(\gamma_1, \gamma_2, \rho) = \frac{d(\gamma_1, \gamma_2)}{\rho + p_1 + d(\gamma_1, \gamma_2)}$$

instead of

$$\mathcal{R}(\gamma_1, \gamma_2, \rho) = \frac{\rho}{\rho + d(\gamma_1, \gamma_2)} \text{ and } \mathcal{S}(\gamma_1, \gamma_2, \rho) = \frac{d(\gamma_1, \gamma_2)}{t + d(\gamma_1, \gamma_2)}$$

then, ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, ⊕, ⊚) satisfies the conditions of IFPMS.

Corollary 3. From Example 3, we can obtain an IFPMS from every IFMS.

Definition 13. Let ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, ⊕, ⊚) be an IFPMS and

$$0 \leq \mathcal{S}(\gamma_1, \gamma_1, \rho) < r < \mathcal{R}(\gamma_1, \gamma_1, \rho), p_1 \geq \rho > 0 \text{ and } \gamma_1 \in \mathcal{X}.$$

The set

$$\mathrm{B}(\gamma_1, \gamma_2, \rho) = \{\gamma_1 \in \mathcal{X}: \mathcal{R}(\gamma_1, \gamma_2, \rho) \oplus \mathcal{R}(\gamma_1, \gamma_1, \rho) > \mathcal{R}(\gamma_1, \gamma_1, \rho) - r,$$
$$\mathcal{S}(\gamma_1, \gamma_2, \rho) \circledcirc \mathcal{S}(\gamma_1, \gamma_1, \rho) < \mathcal{S}(\gamma_1, \gamma_1, \rho) + r\}$$

is called the open ball with center γ_2 and radius r with respect to ρ.

Corollary 4: In Definition 13, if we take for all $\gamma_1, \gamma_2 \in \mathcal{X}$,

$$\mathcal{R}(\gamma_1, \gamma_2, \rho) = 1 \text{ and } \mathcal{S}(\gamma_1, \gamma_2, \rho) = 0$$

then, ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, $\oplus$, $\circledcirc$) satisfies the conditions in Definition 10 for IFMS.

Remark 2. Let ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, $\oplus$, $\circledcirc$) be an IFPMS. Define

$\tau(\mathcal{R}, \mathcal{S})$ = {A $\subset \mathcal{X}$: for each $\gamma_1 \in A$, there exists $\rho > 0$ and

$0 \leq \mathcal{S}(\gamma_1, \gamma_1, \rho) < r < \mathcal{R}(\gamma_1, \gamma_1, \rho)$ such that $B(\gamma_1, \gamma_2, \rho) \subset A$}.

Then, $\tau(\mathcal{R}, \mathcal{S})$ is a topology on $\mathcal{X}$.

Lemma 3. Let ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, $\oplus$, $\circledcirc$) be an IFPMS. Then, $\mathcal{R}(\gamma_1, \gamma_2, .)$ is non-decreasing and $\mathcal{S}(\gamma_1, \gamma_2, .)$ is non-increasing for all $\gamma_1, \gamma_2 \in \mathcal{X}$.

Proof.

Assume that $\mathcal{R}$ is strictly decreasing. Then,

$$\mathcal{R}(\gamma_1, \gamma_2, p_1) < \mathcal{R}(\gamma_1, \gamma_2, p_2) \text{ for } 0 < p_2 < p_1 \text{ and } \mathcal{R}(\gamma_2, \gamma_2, p_1) < \mathcal{R}(\gamma_2, \gamma_2, p_1 - p_2). \quad (1)$$

From Definition 12, we get

$$\mathcal{R}(\gamma_1, \gamma_2, p_1) \oplus \mathcal{R}(\gamma_2, \gamma_2, p_1) \geq \mathcal{R}(\gamma_1, \gamma_2, p_2) \oplus \mathcal{R}(\gamma_2, \gamma_2, p_1 - p_2). \quad (2)$$

From Definition 3 and (1), we have

$$\mathcal{R}(\gamma_1, \gamma_2, p_1) \oplus \mathcal{R}(\gamma_2, \gamma_2, p_1) \leq \mathcal{R}(\gamma_2, \gamma_2, p_2) \oplus \mathcal{R}(\gamma_2, \gamma_2, p_1 - p_2). \quad (3)$$

where there is a contradiction because of (1), (2), and (3).

Hence, $\mathcal{R}(\gamma_1, \gamma_2, .)$ is non-decreasing.

Assume that $\mathcal{S}$ is strictly increasing. Then

$$\mathcal{S}(\gamma_1, \gamma_2, p_1) > \mathcal{S}(\gamma_1, \gamma_2, p_2) \text{ for } 0 < p_2 < p_1 \text{ and } \mathcal{S}(\gamma_2, \gamma_2, p_1) > \mathcal{S}(\gamma_2, \gamma_2, p_1 - p_2). \quad (4)$$

From Definition 12, we obtain

$$\mathcal{S}(\gamma_1, \gamma_2, p_1) \circledcirc \mathcal{S}(\gamma_2, \gamma_2, p_1) \leq \mathcal{S}(\gamma_1, \gamma_2, p_2) \circledcirc \mathcal{S}(\gamma_2, \gamma_2, p_1 - p_2). \quad (5)$$

From Definition 4 and (5), we obtain

$$\mathcal{S}(\gamma_1, \gamma_2, p_1) \circledcirc \mathcal{S}(\gamma_2, \gamma_2, p_1) \geq \mathcal{S}(\gamma_1, \gamma_2, p_2) \circledcirc (\gamma_2, \gamma_2, p_1 - p_2). \quad (6)$$

where there is a contradiction because of (4), (5), and (6).

Hence, $\mathcal{S}(\gamma_1, \gamma_2, .)$ is non-increasing.

Theorem 3. Let ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, $\oplus$, $\circledcirc$) be an IFPMS. Every open ball B(γ_2, r, ρ) in this space is an open set.

Proof. We assume that

$$\gamma_1 \in \mathrm{B}(\gamma_2, r, \rho).$$

Since B(γ_2, r, ρ) is an open ball in ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, $\oplus$, $\circledcirc$), then

$$\mathcal{R}(\gamma_1, \gamma_2, \rho) \oplus \mathcal{R}(\gamma_1, \gamma_1, \rho_1) > \mathcal{R}(\gamma_1, \gamma_1, \rho_1) - r$$

and

$$\mathcal{S}(\gamma_1, \gamma_2, \rho) \circledcirc \mathcal{S}(\gamma_1, \gamma_1, \rho_1) < \mathcal{S}(\gamma_1, \gamma_1, \rho_1) + r$$

for $\rho_1 \geq \rho > 0$. From Lemma 3, since $\mathcal{R}(\gamma_1, \gamma_2, .)$ is non-decreasing, $\mathcal{S}(\gamma_1, \gamma_2, .)$ is non-increasing, there exists $\rho_2 \in (0, \rho)$ such that

$$\mathcal{R}(\gamma_1, \gamma_2, \rho_2) \oplus \mathcal{R}(\gamma_1, \gamma_1, \rho_1) > \mathcal{R}(\gamma_1, \gamma_1, \rho_1) - r$$

and

$$\mathcal{S}(\gamma_1, \gamma_2, \rho_2) \circledcirc \mathcal{S}(\gamma_1, \gamma_1, \rho_1) < \mathcal{S}(\gamma_1, \gamma_1, \rho_1) + r.$$

We assume that $r_1 = \mathcal{R}(\gamma_1, \gamma_2, \rho_2) \oplus \mathcal{R}(\gamma_1, \gamma_1, \rho_1)$. Since $r_1 > \mathcal{R}(\gamma_1, \gamma_1, \rho_1) - r$, there exist $r_2 \in (0, 1)$ such that

$$r_1 > \mathcal{R}(\gamma_1, \gamma_1, \rho_1) - r_2 > \mathcal{R}(\gamma_1, \gamma_1, \rho_1) - r.$$

Thus, there exist $r_3, r_4 \in (0, 1)$ such that

$$r_1 \oplus r_3 > \mathcal{R}(\gamma_1, \gamma_1, \rho_1) - r_2 \text{ and } (\mathcal{S}(\gamma_1, \gamma_1, \rho_1) - r_1) \circledcirc (\mathcal{S}(\gamma_1, \gamma_1, \rho_1) - r_4) \leq r_2.$$

We assume that $r_5 = \max\{r_3, r_4\}$ and B$(\gamma_1, \mathcal{R}(\gamma_1, \gamma_1, \rho_1) - r_5, \rho - \rho_2)$ is an open ball. We claim that

$$\mathrm{B}(\gamma_1, \mathcal{R}(\gamma_1, \gamma_1, \rho_1) - r_5, \rho - \rho_2) \subset \mathrm{B}(\gamma, r, \rho).$$

We assume that $\gamma_3 \in \mathrm{B}(\gamma_1, \mathcal{R}(\gamma_1, \gamma_1, \rho_1) - r_5, \rho - \rho_2)$. Hence, we obtain

$$\mathcal{R}(\gamma_3, \gamma_2, \rho - \rho_2) > r_5 \text{ and } \mathcal{S}(\gamma_3, \gamma_2, \rho_2) < 2\mathcal{R}(\gamma_1, \gamma_1, \rho - \rho_2) - r_5.$$

Thus, we get

$$\begin{aligned}\mathcal{R}(\gamma_3,\gamma_2,\rho)\oplus\mathcal{R}(\gamma_1,\gamma_1,\rho_1) &\geq \mathcal{R}(\gamma_3,\gamma_1,\rho_2)\oplus\mathcal{R}(\gamma_1,\gamma_2,\rho-\rho_2)\\ &\geq r_1\oplus r_5\\ &\geq r_1\oplus r_3\\ &\geq \mathcal{R}(\gamma_1,\gamma_1,\rho_1)-r_2\\ &> \mathcal{R}(\gamma_1,\gamma_1,\rho_1)-r\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}(\gamma_3,\gamma_2,\rho)\circledcirc\mathcal{S}(\gamma_1,\gamma_1,\rho_1) &\leq \mathcal{S}(\gamma_3,\gamma_1,\rho_2)\circledcirc\mathcal{S}(\gamma_1,\gamma_2,\rho-\rho_2)\\ &\leq (\mathcal{S}(\gamma_1,\gamma_1,\rho_1)-r_1)\circledcirc(\mathcal{S}(\gamma_1,\gamma_1,\rho_1)-r_5)\\ &\leq (\mathcal{S}(\gamma_1,\gamma_1,\rho_1)-r_1)\circledcirc(\mathcal{S}(\gamma_1,\gamma_1,\rho_1)-r_4)\\ &\leq r_2\\ &< r\\ &< \mathcal{S}(\gamma_1,\gamma_1,\rho_1)+\mathrm{r}.\end{aligned}$$

Therefore, we obtain

$$\gamma_3\in \mathrm{B}(\gamma_2,r,\rho).$$

Hence, we get

$$\mathrm{B}(\gamma_1,\mathcal{R}(\gamma_1,\gamma_1,\rho_1)-r_5,\rho-\rho_2)\subset \mathrm{B}(\gamma,r,\rho).$$

Corollary 5. Let $(\mathcal{X},\ \mathcal{R},\ \mathcal{S},\ \oplus,\ \circledcirc)$ be an IFPMS. The set

$\tau(\mathcal{R},\mathcal{S}) = \{\mathrm{A}\subset\mathcal{X}$: for each $\gamma_1\in A$, there exists $\rho>0$ and $0\leq\ \mathcal{S}(\gamma_1,\gamma_1,\rho)<\mathrm{r}<\mathcal{R}(\gamma_1,\gamma_1,\rho)$

such that $\mathrm{B}(\gamma_2,r,\rho)\subset\mathrm{A}\}$.

is a topology on $\mathcal{X}$.

Theorem 4. Let $(\mathcal{X},\ \mathcal{R},\ \mathcal{S},\ \oplus,\ \circledcirc)$ be an IFPMS and $\tau(\mathcal{R},\ \mathcal{S})$ be a topology on $\mathcal{X}$ induced by the $(\mathcal{X},\ \mathcal{R},\ \mathcal{S},\ \oplus,\ \circledcirc)$. Then for a sequence $\{\gamma_n\}$ in $(\mathcal{X},\ \mathcal{R},\ \mathcal{S},\ \oplus,\ \circledcirc)$,

$\gamma_n \to \gamma_1$

if and only if

$$\lim_{n\to\infty} \mathcal{R}(\gamma_1, \gamma_n, \rho) = \mathcal{R}(\gamma_1, \gamma_1, \rho) \text{ and } \lim_{n\to\infty} \mathcal{S}(\gamma_1, \gamma_n, \rho) = \mathcal{S}(\gamma_1, \gamma_1, \rho).$$

Proof. We assume that for ρ > 0,

$\gamma_n \to \gamma_1.$

Then for r ∈ (0, 1), there exists $n_0 \in \mathbb{N}$, n > n_0 such that

$\gamma_n \in \mathrm{B}(\gamma_1, r, \rho).$

Hence, we get

$$\mathcal{R}(\gamma_1, \gamma_n, \rho) \oplus \mathcal{R}(\gamma_1, \gamma_1, \rho) > \mathcal{R}(\gamma_1, \gamma_1, \rho) - r$$

and

$$\mathcal{S}(\gamma_1, \gamma_n, \rho) \circledcirc \mathcal{S}(\gamma_1, \gamma_1, \rho) < \mathcal{S}(\gamma_1, \gamma_1, \rho) + r.$$

Thus, we obtain

$$\mathcal{R}(\gamma_1, \gamma_1, \rho) - \mathcal{R}(\gamma_1, \gamma_n, \rho) \oplus \mathcal{R}(\gamma_1, \gamma_1, \rho) < r$$

and

$$\mathcal{S}(\gamma_1, \gamma_n, \rho) \circledcirc \mathcal{S}(\gamma_1, \gamma_1, \rho) - \mathcal{S}(\gamma_1, \gamma_1, \rho) < r.$$

From Definition 12, clearly

$$\mathcal{R}(\gamma_1, \gamma_n, \rho) \leq \mathcal{R}(\gamma_1, \gamma_1, \rho) \leq 1$$

and from Definition 12, clearly

$$\mathcal{R}(\gamma_1, \gamma_n, \rho) \oplus \mathcal{R}(\gamma_1, \gamma_1, \rho) \leq \mathcal{R}(\gamma_1, \gamma_1, \rho).$$

Hence, we have

$$\mathcal{R}(\gamma_1, \gamma_1, \rho) - \mathcal{R}(\gamma_1, \gamma_n, \rho) \oplus \mathcal{R}(\gamma_1, \gamma_1, \rho) < \mathcal{R}(\gamma_1, \gamma_1, \rho) - \mathcal{R}(\gamma_1, \gamma_n, \rho) < r.$$

Since $\gamma_n \to \gamma_1$, we obtain

$$\mathcal{R}(\gamma_1, \gamma_1, \rho) - \mathcal{R}(\gamma_1, \gamma_n, \rho) = 0.$$

Hence,

$$\mathcal{R}(\gamma_1, \gamma_1, \rho) \rightarrow \mathcal{R}(\gamma_1, \gamma_n, \rho).$$

From Definition 12, clearly

$$\mathcal{S}(\gamma_1, \gamma_n, \rho) \leq \mathcal{S}(\gamma_1, \gamma_n, \rho),$$

$$0 \leq (\gamma_1, \gamma_1, \rho),$$

and from Definition 12, clearly

$$\mathcal{S}(\gamma_1, \gamma_n, \rho) \leq \mathcal{S}(\gamma_1, \gamma_n, \rho) \circledcirc \mathcal{S}(\gamma_1, \gamma_1, \rho).$$

Hence, we have

$$\mathcal{S}(\gamma_1, \gamma_n, \rho) \circledcirc \mathcal{S}(\gamma_1, \gamma_1, \rho) - \mathcal{S}(\gamma_1, \gamma_1, \rho) < \mathcal{S}(\gamma_1, \gamma_n, \rho) - \mathcal{S}(\gamma_1, \gamma_1, \rho) < r.$$

Since $\gamma_n \rightarrow \gamma_1$, we obtain

$$\mathcal{S}(\gamma_1, \gamma_n, \rho) - \mathcal{S}(\gamma_1, \gamma_1, \rho) = 0.$$

Hence,

$$\mathcal{S}(\gamma_1, \gamma_n, \rho) \rightarrow \mathcal{S}(\gamma_1, \gamma_1, \rho).$$

Conversely, we assume that

$$\lim_{n\to\infty} \mathcal{R}(\gamma_1, \gamma_n, \rho) = \mathcal{R}(\gamma_1, \gamma_1, \rho), \ \lim_{n\to\infty} \mathcal{S}(\gamma_1, \gamma_n, \rho) = \mathcal{S}(\gamma_1, \gamma_1, \rho)$$

and

$$\lim_{n\to\infty} \mathcal{H}(\gamma_1, \gamma_n, \rho) = \mathcal{H}(\gamma_1, \gamma_1, \rho).$$

Thus, for r ∈ (0, 1), there exists there exists $n_0 \in \mathbb{N}$, n > n_0 such that

$$\mathcal{R}(\gamma_1, \gamma_1, \rho) - \mathcal{R}(\gamma_1, \gamma_n, \rho) < r \text{ and } < \mathcal{S}(\gamma_1, \gamma_n, \rho) - \mathcal{S}(\gamma_1, \gamma_1, \rho) < r.$$

Since

$$\mathcal{R}(\gamma_1, \gamma_n, \rho) \oplus \mathcal{R}(\gamma_1, \gamma_1, \rho) \leq \mathcal{R}(\gamma_1, \gamma_1, \rho) \text{ and } (\gamma_1, \gamma_n, \rho) \leq \mathcal{S}(\gamma_1, \gamma_n, \rho) \circledcirc \mathcal{S}(\gamma_1, \gamma_1, \rho),$$

we obtain

$$\mathcal{R}(\gamma_1, \gamma_1, \rho) - \mathcal{R}(\gamma_1, \gamma_n, \rho) \oplus \mathcal{R}(\gamma_1, \gamma_1, \rho) < r$$

and

$$\mathcal{S}(\gamma_1,\gamma_n,\rho)\circledcirc\mathcal{S}(\gamma_1,\gamma_1,\rho)-\mathcal{S}(\gamma_1,\gamma_1,\rho)<r.$$

Thus, from Definition 13, we get

$$\gamma_n\in \mathrm{B}(\gamma_1,r,\rho) \text{ and } \gamma_n\to\gamma_1.$$

Lemma 4. Let ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, $\oplus$, $\circledcirc$) be an IFPMS. Then,

i. If $\mathcal{R}(\gamma_1,\gamma_2,\rho)=1$ and $\mathcal{S}(\gamma_1,\gamma_2,\rho)=0$, then $\gamma_1=\gamma_2$.

ii. If $\gamma_1\neq\gamma_2$, then $\mathcal{R}(\gamma_1,\gamma_2,\rho)<1$ and $\mathcal{S}(\gamma_1,\gamma_2,\rho)>0$

iii. If $\gamma_n\to\gamma_3$ with $\mathcal{R}(\gamma_3,\gamma_3,\rho)=1$ and $\mathcal{S}(\gamma_3,\gamma_3,\rho)=0$, then $\lim_{n\to\infty}\mathcal{R}(\gamma_n,\gamma_2,\rho)=\mathcal{R}(\gamma_3,\gamma_2,\rho)$ and $\lim_{n\to\infty}\mathcal{S}(\gamma_n,\gamma_2,\rho)=\mathcal{S}(\gamma_3,\gamma_2,\rho)$

for all $\gamma_2\in\mathcal{X}$.

Proof.

i) From Definition 12, clearly,

$$\mathcal{R}(\gamma_1,\gamma_2,\rho)\leq 1,\ \mathcal{R}(\gamma_1,\gamma_1,\rho)\geq\mathcal{R}(\gamma_1,\gamma_2,\rho) \text{ and } \mathcal{R}(\gamma_2,\gamma_2,\rho)\geq\mathcal{R}(\gamma_1,\gamma_2,\rho).$$

If $\mathcal{R}(\gamma_1,\gamma_2,\rho)=1$, then

$$\mathcal{R}(\gamma_1,\gamma_1,\rho)\geq\mathcal{R}(\gamma_1,\gamma_2,\rho)=1.$$

Hence, we get

$$\mathcal{R}(\gamma_1,\gamma_1,\rho)=1.$$

Similarly,

If $\mathcal{R}(\gamma_1,\gamma_2,\rho)=1$, then

$$\mathcal{R}(\gamma_2,\gamma_2,\rho)\geq\mathcal{R}(\gamma_1,\gamma_2,\rho)=1.$$

Thus, we obtain

$$\mathcal{R}(\gamma_2,\gamma_2,\rho)=1.$$

Therefore,

$$\mathcal{R}(\gamma_1,\gamma_2,\rho)=\mathcal{R}(\gamma_1,\gamma_1,\rho)=\mathcal{R}(\gamma_2,\gamma_2,\rho)=1.$$

From Definition 12, we obtain

$$\gamma_1 = \gamma_2.$$

Similarly, we get

$$0 \leq \mathcal{S}(\gamma_1, \gamma_2, \rho), \mathcal{S}(\gamma_2, \gamma_2, \rho) \leq \mathcal{S}(\gamma_1, \gamma_2, \rho) \text{ and } \mathcal{S}(\gamma_1, \gamma_1, \rho) \leq \mathcal{S}(\gamma_1, \gamma_2, \rho).$$

If $\mathcal{S}(\gamma_1, \gamma_2, \rho) = 0$, then

$$\mathcal{S}(\gamma_1, \gamma_1, \rho) \leq \mathcal{S}(\gamma_1, \gamma_2, \rho) = 0.$$

Thus,

$$\mathcal{S}(\gamma_1, \gamma_1, \rho) = 0.$$

If $\mathcal{S}(\gamma_1, \gamma_2, \rho) = 0$, then

$$\mathcal{S}(\gamma_2, \gamma_2, \rho) \leq \mathcal{S}(\gamma_1, \gamma_2, \rho) = 0.$$

Thus,

$$\mathcal{S}(\gamma_2, \gamma_2, \rho) = 0.$$

Therefore,

$$\mathcal{S}(\gamma_1, \gamma_2, \rho) = \mathcal{S}(\gamma_1, \gamma_1, \rho) = \mathcal{S}(\gamma_2, \gamma_2, \rho) = 0.$$

From Definition 12, we obtain

$$\gamma_1 = \gamma_2.$$

ii) Let $\gamma_1 \neq \gamma_2$. We assume that

$$\mathcal{R}(\gamma_1, \gamma_2, \rho) \geq 1 \text{ and } \mathcal{S}(\gamma_1, \gamma_2, \rho) \leq 0.$$

Thus, From Definition 12,

$$\mathcal{R}(\gamma_1, \gamma_2, \rho) = 1 \text{ and } \mathcal{S}(\gamma_1, \gamma_2, \rho) = 0 \tag{7}$$

since $\mathcal{R}(\gamma_1, \gamma_2, \rho), \mathcal{S}(\gamma_1, \gamma_2, \rho) \in [0, 1]$.

From (7) and i), we obtain $\gamma_1 = \gamma_2$. (8)

where there is a contradiction because of (7) and (8).

Hence,

$\mathcal{R}(\gamma_1, \gamma_2, \rho) < 1$ and $\mathcal{S}(\gamma_1, \gamma_2, \rho) > 0$.

iii) From Theorem 4, if $\gamma_n \to \gamma_3$, then

$$\lim_{n\to\infty} \mathcal{R}(\gamma_3, \gamma_n, \rho) = \mathcal{R}(\gamma_3, \gamma_3, \rho) \text{ and } \lim_{n\to\infty} \mathcal{S}(\gamma_3, \gamma_n, \rho) = \mathcal{S}(\gamma_3, \gamma_3, \rho).$$

Thus, since

$\mathcal{R}(\gamma_3, \gamma_3, \rho) = 1$ and $\mathcal{S}(\gamma_3, \gamma_3, \rho) = 0$,

we obtain

$$\lim_{n\to\infty} \mathcal{R}(\gamma_3, \gamma_n, \rho) = 1 \text{ and } \lim_{n\to\infty} \mathcal{S}(\gamma_3, \gamma_n, \rho) = 0. \tag{9}$$

From Definition 12, we obtain

$$\mathcal{R}(\gamma_n, \gamma_3, \rho) \oplus \mathcal{R}(\gamma_3, \gamma_n, \rho) \leq \mathcal{R}(\gamma_n, \gamma_n, 2\rho) \oplus \mathcal{R}(\gamma_3, \gamma_3, \rho). \tag{10}$$

Also, from (9) and (10), it is held that

$$1 \oplus 1 \leq \mathcal{R}(\gamma_n, \gamma_n, 2\rho) \oplus 1.$$

From Definition 3, clearly

$$\mathcal{R}(\gamma_n, \gamma_n, 2\rho) = 1. \tag{11}$$

Also, from Definition 12 and for $p_1 - \rho \leq \rho$ and $0 < \rho < p_1$, we have

$$\mathcal{R}(\gamma_2, \gamma_3, p_1 - \rho) \oplus \mathcal{R}(\gamma_3, \gamma_n, \rho) \leq \mathcal{R}(\gamma_2, \gamma_n, \rho) \oplus \mathcal{R}(\gamma_3, \gamma_3, \rho)$$

and

$$\mathcal{R}(\gamma_2, \gamma_n, \rho) \oplus \mathcal{R}(\gamma_n, \gamma_3, \rho) \leq \mathcal{R}(\gamma_2, \gamma_3, 2\rho) \oplus \mathcal{R}(\gamma_n, \gamma_n, 2\rho). \tag{12}$$

From (9), (11) and (12), we obtain

$$\mathcal{R}(\gamma_2, \gamma_3, p_1 - \rho) \leq \mathcal{R}(\gamma_2, \gamma_n, \rho)$$

and

$$\mathcal{R}(\gamma_2, \gamma_n, \rho) \leq \mathcal{R}(\gamma_2, \gamma_3, 2\rho) \tag{13}$$

since $\mathcal{R}(\gamma_1, \gamma_2, .): [0, \infty) \to [0,1]$ is continuous. From (13), we obtain

$$\mathcal{R}(\gamma_3, \gamma_2, \rho) \leq \lim_{n\to\infty} \mathcal{R}(\gamma_n, \gamma_2, \rho)$$

and

$$\lim_{n\to\infty} \mathcal{R}(\gamma_n, \gamma_2, \rho) \leq \mathcal{R}(\gamma_3, \gamma_2, \rho).$$

Hence, we get

$$\lim_{n\to\infty} \mathcal{R}(\gamma_n, \gamma_2, \rho) = \mathcal{R}(\gamma_3, \gamma_2, \rho).$$

Also, similarly, we have

$$\lim_{n\to\infty} \mathcal{S}(\gamma_n, \gamma_2, \rho) = \mathcal{S}(\gamma_3, \gamma_2, \rho)$$

for all $\gamma_2 \in \mathcal{X}$.

Corollary 6. In Lemma 4, if we take for all $\gamma_1, \gamma_2 \in \mathcal{X}$,

$\mathcal{R}(\gamma_1, \gamma_2, \rho) = 1$ and $\mathcal{S}(\gamma_1, \gamma_2, \rho) = 0$

then, ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, $\oplus$, $\circledcirc$) satisfies the conditions for IFMS.

Definition 14. Let ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, $\oplus$, $\circledcirc$) be an IFPMS and $\tau(\mathcal{R}, \mathcal{S})$ be a topology on $\mathcal{X}$ induced by the ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, $\oplus$, $\circledcirc$).

a) Then a sequence $\{\gamma_n\}$ in ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, $\oplus$, $\circledcirc$) is said to be a Cauchy sequence with respect to τ($\mathcal{R}$, $\mathcal{S}$) if for ρ > 0, r ∈ (0, 1), there exists n_0 ∈ ℕ; n, m > n_0 such that

$$\mathcal{R}(\gamma_m, \gamma_n, \rho) \oplus \mathcal{R}(\gamma_m, \gamma_m, \rho) > \mathcal{R}(\gamma_m, \gamma_m, \rho) - r$$

and

$$\mathcal{S}(\gamma_m, \gamma_n, \rho) \circledcirc \mathcal{S}(\gamma_m, \gamma_m, \rho) < \mathcal{S}(\gamma_m, \gamma_m, \rho) + r.$$

b) ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, $\oplus$, $\circledcirc$) is called complete if every Cauchy sequence is convergent in this space.

Corollary 7. Let ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, $\oplus$, $\circledcirc$) be an IFPMS and τ($\mathcal{R}$, $\mathcal{S}$) be a topology on $\mathcal{X}$ induced by the ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, $\oplus$, $\circledcirc$). Then a sequence $\{\gamma_n\}$ in ($\mathcal{X}$, $\mathcal{R}$, $\mathcal{S}$, $\oplus$, $\circledcirc$) is said to be a Cauchy with respect to τ($\mathcal{R}$, $\mathcal{S}$) if

$$\lim_{n,m\to\infty} \mathcal{R}(\gamma_m, \gamma_n, \rho) \text{ and } \lim_{n,m\to\infty} \mathcal{S}(\gamma_m, \gamma_n, \rho)$$

exists and finite.

4. Conclusion

In this paper, IFPMS are defined and their basic properties and examples are achieved. For IFPMS, open ball, convergent sequence, and Cauchy sequence are defined and their basic properties are obtained. Furthermore, the relations between IFPMS and classical metric spaces, FMSs, FPMS, and IFMS are analyzed.

Thanks to this paper, researchers can define some partial metric spaces based on IFS. For example, researchers can define intuitionistic fuzzy partial G-metric spaces, intuitionistic fuzzy partial m-metric spaces, and intuitionistic fuzzy partial b-metric spaces. Also, Researchers can introduce new fixed point theories for these new types of metrics. Furthermore, researchers can define intuitionistic fuzzy partial normed space and neutrosophic partial metric space by taking advantage of IFPMS.

Abbreviations

Continuous t-conorm: CTCN

Continuous t-norm: CTN

Intuitionistic Fuzzy Partial Metric Spaces: IFPMS

Intuitionistic Fuzzy Metric Spaces: IFMS

Fuzzy Partial Metric Spaces: FPMS

Fuzzy Metric Spaces: FMS

Intuitionistic Fuzzy Sets: IFS

Fuzzy Sets: FS

Fuzzy Logic: FL

Fixed Point Theorem: FPT

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