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Research Article

Investigations into Hermite-Hadamard-Fejér Inequalities within the Realm of Trigonometric Convexity

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Abstract
This study is predicated on the exploration of lemmas pertaining to the Hermite-Hadamard-Fejér type integral inequality, focusing on both trapezoidal and midpoint inequalities. It delves into the realm of trigonometrically convex functions and is structured around the foundational lemmas that govern these inequalities. Through rigorous analysis, the research has successfully derived novel theorems and garnered insightful results that enhance the understanding of trigonometric convexity. Further, the study has undertaken the application of these theorems to exemplify trigonometrically convex functions, thereby providing practical instances that underline the theoretical developments. These applications not only serve to demonstrate the utility of the newly formulated results but also contribute to the broader field of convex analysis by introducing innovative perspectives on integral inequalities. The synthesis of theory and application encapsulated in this research marks a significant stride in the advancement of mathematical inequalities and their relevance to the study of convex functions.

Keywords: Hermite-Hadamard-Fejer type inequality, h-convex functions, Trigonometrically convex function
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1. Introduction

Convex functions are utilized and extensively researched across a variety of fields, from physics to economics, mathematics to statistics, and even medicine. They are known to be among the most significant areas of study in the current century, with a vast body of literature dedicated to them. Amidst the importance of convex functions in the literature, many authors have identified and studied various types of convex functions. Research on these various types of convex functions has been and continues to be extensive and expanding.

One of the most significant classes of convex functions is the family of h-convex functions, originally explored and expanded in the foundational works of Varošanec [1] and Bombardelli and Varošanec [2]. This notion of h-convexity has proven remarkably versatile, giving rise to various specialized convexity classes such as s-convexity, (s,P)-convexity, trigonometric convexity, and exponential trigonometric convexity [3–5]. These generalized convexities have, in turn, facilitated the development of numerous refined inequalities of Hermite-Hadamard-Fejér type, as well as several new integral inequalities extending classical results [6–10]. For instance, Budak et al. [7, 11–13] obtained new trapezoid and midpoint-type inequalities for generalized quantum integrals, and additionally derived integral inequalities for conformable fractional integrals by employing the weight functions inherent in Fejér-type inequalities, whereas Çelik et al. [9] introduced generalized Milne-type inequalities under conformable fractional integrals. Demir [3] established novel Hermite-Hadamard-type inequalities for exponential trigonometric convex functions, while Demir et al. [5] derived Simpson's-type inequalities within the framework of trigonometric convexity. These advances build upon the classical insights of Hadamard [14] and Fejér [15], whose pioneering works laid the groundwork for modern research on convex functions and their associated integral inequalities. Later investigations by Dragomir and Pearce [8] offered a comprehensive survey of Hermite-Hadamard-type results, and Kadakal [16] further specialized these inequalities to trigonometrically convex functions. More recently, Turhan [4] presented novel generalizations of integral inequalities for trigonometrically-p functions, thereby highlighting the ongoing expansion and applicability of h-convexity in contemporary mathematical research.

Since the definition of convex functions inherently relies on an inequality condition, they are widely used in mathematics to find new lower or upper bounds, that is, for optimization. The well-known Hermite-Hadamard (H-H) inequality in the literature is stated for a continuous function $\xi : T \to \mathbb{R}$, for all $k,l \in T$ with $k < l$,

$$\xi\left(\frac{k+l}{2}\right) \leq \frac{1}{l-k}\int_k^l \xi(x)dx \leq \frac{\xi(k)+\xi(l)}{2}.$$

If ξ is a concave function, the inequality is reversed [14]. This inequality has been applied to many classes of convex functions; with the help of various lemmas, theorems on trapezoidal and midpoint type inequalities have been derived and results have been presented.

The introduction of the weighted version of the Hermite-Hadamard (H-H) inequality by Fejér in 1906 represents a significant evolution in the analysis of convex functions, culminating in what is now recognized as the H-H Fejér type inequality. This seminal development not only enriched the mathematical framework for examining convex functions but also facilitated the derivation of a broad spectrum of theorems and results tailored to various conditions of the weight function. Such advancements have had profound implications on both the left and right sides of different H-H inequalities, underscoring the historical importance and far-reaching impact of Fejér's work. Through this weighted form of the H-H inequality, Fejér's contribution has been pivotal in broadening the understanding and application of convex function inequalities, highlighting the intricate interplay between weight functions and the fundamental properties of these inequalities.

In this study, lemmas that yield trapezoidal and midpoint-type integral inequalities for trigonometric convex functions were investigated. While these lemmas are known for Hermite-Hadamard Fejer type integral inequalities and many studies, have produced trapezoidal type inequalities, new theorems, and results have also been obtained for midpoint type inequalities.

2. Preliminaries

In this section, we first present the foundational theorems and definitions that underpin this work. Following this, we introduce the pivotal lemmas that have not only inspired but also guided the development of the study, thereby establishing a robust conceptual framework for the ensuing analysis.

Theorem 2.1. *[15] Assume $\xi : [k,l] \to \mathbb{R}$ is a convex mapping. Then, the following inequality is satisfied:*

$$\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)\,dx \leq \frac{1}{l-k}\int_k^l \xi(x)w(x)\,dx \leq \frac{\xi(k)+\xi(l)}{2}\int_k^l w(x)\,dx$$

where the function $w : [k,l] \to \mathbb{R}$ is nonnegative, integrable, and exhibits symmetry about $x = \frac{k+l}{2}$.

The domain of convex analysis has been significantly expanded with the introduction of h-convexity, as delineated by Varosanec. This sophisticated class of convexity, which is predicated upon a modulating function h that is both non-negative and distinct from zero, provides a more encompassing approach than classical convexity.

Definition 2.2. *[1] Let G and T be two intervals, and let $h : G \to \mathbb{R}$ be a non-negative function such that $h \neq 0$. A function $\xi : T \to \mathbb{R}$ is said to be an h-convex function if, for all $k,l \in T$ and for any $\kappa \in (0,1)$, the following inequality holds:*

$$\xi(\kappa k + (1-\kappa)l) \leq h(\kappa)\xi(k) + h(1-\kappa)\xi(l).$$

*Conversely, if the inequality holds in the opposite direction, then ξ is termed an *h-concave function*. Functions belonging to this class of convexity are referred to as members of the class $SX(h,K)$.*

This significant convex class has led to the emergence of numerous convexity classes. Among these, trigonometric convex functions expressed by H. Kadakal and the related H-H inequality and theorems pertaining to this convexity class are provided as follows:

Definition 2.3. *[16] Let $\xi : T \to \mathbb{R}$ be a non-negative function, where $k,l \in T$ and $\omega \in [0,1]$. In this case, if the following inequality is satisfied, the function ξ is referred to as a trigonometrically convex function:*

$$\xi(\omega k + (1-\omega)l) \leq \left(\sin\left(\frac{\pi\omega}{2}\right)\right)\xi(k) + \left(\cos\left(\frac{\pi\omega}{2}\right)\right)\xi(l).$$

The class of trigonometric convex functions is denoted by $TC(T)$. In the definition expressed, if $h(\omega) = \sin\left(\frac{\omega}{2}\right)$ is taken, then every trigonometric convex function becomes an h-convex function.

Theorem 2.4. *[16] Let T be an interval with $k,l \in T$ such that $k < l$. If the function $\xi : [k,l] \to \mathbb{R}$ is a trigonometrically convex function and $\xi \in L[k,l]$, then the following inequality is obtained:*

$$\xi\left(\frac{k+l}{2}\right) \leq \frac{\sqrt{2}}{l-k}\int_k^l \xi(\S)d\S.$$

Theorem 2.5. *[16] Let T be an interval with $k,l \in T$ such that $k < l$. If the function $\xi : [k,l] \to \mathbb{R}$ is trigonometrically convex and $\xi \in L[k,l]$, then the following inequality is obtained:*

$$\frac{1}{l-k}\int_k^l \xi(x)dx \leq \frac{2}{\pi}\left[\xi(k) + \xi(l)\right].$$

Theorem 2.6. *[16] Let T be an interval, $k,l \in T$ such that $k < l$, and $\xi : T \to \mathbb{R}$ be a continuously differentiable function. If $\xi' \in L[k,l]$ and $|\xi'|$ is a trigonometrically convex function, then the following inequality is obtained:*

$$\left|\frac{\xi(k)+\xi(l)}{2} - \frac{1}{l-k}\int_k^l \xi(x)dx\right| \leq \frac{2}{\pi}(l-k)\left[1 - \frac{4}{\pi}(\sqrt{2}-1)\right]\left[\frac{|\xi'(k)| + |\xi'(l)|}{2}\right].$$

Theorem 2.7. *[16] Let T be an interval, $k,l \in T$ such that $k < l$, and $\xi : T \to \mathbb{R}$ be a continuously differentiable function. Let $\eta > 1$ and $\frac{1}{\eta} + \frac{1}{\theta} = 1$, for $|\xi'|^{\eta}$ being a trigonometrically convex function over the interval $[k,l]$, the following inequality is obtained:*

$$\left|\frac{\xi(k)+\xi(l)}{2} - \frac{1}{l-k}\int_k^l \xi(x)dx\right| \leq \frac{l-k}{2}\left(\frac{1}{\theta+1}\right)^{\frac{1}{\theta}} 2^{\frac{2}{\eta}}\pi^{\frac{-1}{\eta}}\left(\frac{k+l}{2}\right)^{\frac{1}{\eta}}\left[\frac{|\xi'(k)|^{\eta} + |\xi'(l)|^{\eta}}{2}\right]^{\frac{1}{\eta}}.$$

Theorem 2.8. *[16] Let T be an interval, $k,l \in T$ such that $k < l$, and $\xi : T \to \mathbb{R}$ be a continuously differentiable function. For $\eta \geq 1$, with $|\xi'|^{\eta}$ being a trigonometrically convex function over the interval $[k,l]$, the following inequality is obtained:*

$$\left|\frac{\xi(k)+\xi(l)}{2} - \frac{1}{l-k}\int_k^l \xi(x)dx\right| \leq \frac{l-k}{2}\left(\frac{1}{2}\right)^{1-\frac{3}{\eta}}\left[\frac{1}{\pi} - \frac{4(\sqrt{2}-1)}{\pi^2}\right]^{\frac{1}{\eta}}\left(\frac{k+l}{2}\right)^{\frac{1}{\eta}}\left[\frac{|\xi'(k)|^{\eta} + |\xi'(l)|^{\eta}}{2}\right]^{\frac{1}{\eta}}.$$

In this research, M. Z. Sarikaya introduced two significant lemmas that serve as the foundational basis for the investigation of H-H Fejér type inequalities and their applications to trapezoidal and midpoint-type inequalities. These lemmas, detailed below, are considered the cornerstone of the research:

Lemma 2.9. *[6] Let $\xi : K^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on K°, $k,l \in K^\circ$ with $k<l$, and $w:[k,l]\to[0,\infty)$ be a differentiable mapping. If $\xi' \in L[k,l]$, then the following equality holds:*

$$\frac{1}{l-k}\int_k^l \xi(x)w(x)dx - \frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx = (l-k)\int_0^1 m(t)\xi'(\kappa k+(1-\kappa)l)d\kappa \tag{2.1}$$

for each $\kappa \in [0,1]$, where

$$m(\kappa)=\begin{cases}\int_0^\kappa w(ks+(1-s)l)ds, & \kappa\in[0,\frac{1}{2}]\\ -\int_\kappa^1 w(ks+(1-s)l)ds, & \kappa\in[\frac{1}{2},1].\end{cases}$$

Lemma 2.10. *[6] Let $\xi : K^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on K°, $k,l \in K^\circ$ with $k<l$, and $w:[k,l]\to[0,\infty)$ be a differentiable mapping. If $\xi' \in L[k,l]$, then the following equality holds:*

$$\frac{1}{l-k}\left[\frac{\xi(k)+\xi(l)}{2}+\int_k^l w(x)dx\right]-\frac{1}{l-k}\int_k^l \xi(x)w(x)dx=\frac{(l-k)}{2}\int_0^1 p(\kappa)\xi'(\kappa k+(1-\kappa)l)d\kappa$$

for each $\kappa \in [0,1]$, where

$$p(\kappa)=\int_\kappa^1 w(as+(1-s)b)ds-\int_0^\kappa w(as+(1-s)b)ds.$$

3. Main Results

Theorem 3.1. *Let $\xi : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, where $k,l \in I^\circ$ with $k<l$, and let $w:[k,l]\to\mathbb{R}$ be a differentiable function that is symmetric about $\frac{k+l}{2}$. Given that ξ' is trigonometrically convex over the interval $[k,l]$, the following inequality holds:*

$$\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right| \le \frac{2\sqrt{2}}{\pi}\left[\int_{\frac{k+l}{2}}^l w(x)\sin\left(\frac{2x-k-l}{4(l-k)}\pi\right)dx\right]\left[\left|\xi'(k)\right|+\left|\xi'(l)\right|\right]. \tag{3.1}$$

Proof. Considering Lemma 2.9 and taking the absolute value of both sides, given that $|\xi'|$ is a trigonometrically convex function, we proceed as follows:

$$\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right|$$

$$\le (l-k)\left|\int_0^{1/2}\left(\int_0^t w(ks+(1-s)l)\,\mathrm{ds}\right)\xi'(tk+(1-t)l)\,\mathrm{dt} - \int_{1/2}^1\left(\int_t^1 w(ks+(1-s)l)\,\mathrm{ds}\right)\xi'(tk+(1-t)l)\,\mathrm{dt}\right|$$

$$\le (l-k)\left[\int_0^{\frac{1}{2}}\left(\int_0^t w(ks+(1-s)l)\,\mathrm{ds}\right)\left|\xi'(tk+(1-t)l)\right|\mathrm{dt}+\int_{\frac{1}{2}}^1\left(\int_t^1 w(ks+(1-s)l)\,\mathrm{ds}\right)\left|\xi'(tk+(1-t)l)\right|\mathrm{dt}\right]$$

$$\le (l-k)\left\{\int_0^{\frac{1}{2}}\left(\int_0^t w(ks+(1-s)l)\,\mathrm{ds}\right)\left[\begin{array}{c}\sin\left(\frac{\pi t}{2}\right)|\xi'(k)|\\+\cos\left(\frac{\pi t}{2}\right)|\xi'(l)|\end{array}\right]dt+\int_{\frac{1}{2}}^1\left(\int_t^1 w(ks+(1-s)l)\,\mathrm{ds}\right)\left[\begin{array}{c}\sin\left(\frac{\pi t}{2}\right)|\xi'(k)|\\+\cos\left(\frac{\pi t}{2}\right)|\xi'(l)|\end{array}\right]dt\right\}.$$

The following expressions are obtained by changing the order of integration in the integrals on the right side of the obtained final inequality, in accordance with Fubini's Theorem:

$$\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right|$$
$$\leq (l-k)\left\{\int_0^{\frac{1}{2}} w(ks+(1-s)l)\left[\int_s^{\frac{1}{2}}\left(\begin{array}{c}\sin\left(\frac{\pi t}{2}\right)|\xi'(k)| \\ +\cos\left(\frac{\pi t}{2}\right)|\xi'(l)|\end{array}\right)dt\right]ds+\int_{\frac{1}{2}}^{1} w(ks+(1-s)l)\left[\int_{\frac{1}{2}}^{s}\left(\begin{array}{c}\sin\left(\frac{\pi t}{2}\right)|\xi'(k)| \\ +\cos\left(\frac{\pi t}{2}\right)|\xi'(l)|\end{array}\right)dt\right]ds\right\}$$

Upon resolving the inequalities on the right-hand side of the final inequality, the following inequality is obtained:

$$\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right|$$

$$\leq (l-k)\left\{\int_0^{\frac{1}{2}} w(ks+(1-s)l)\left[\begin{array}{c}\left(-\frac{\sqrt{2}}{\pi}+\frac{2}{\pi}\cos\left(\frac{\pi s}{2}\right)\right)|\xi'(k)| \\ +\left(\frac{\sqrt{2}}{\pi}-\frac{2}{\pi}\sin\left(\frac{\pi s}{2}\right)\right)|\xi'(l)|\end{array}\right]ds+\int_{\frac{1}{2}}^{1} w(ks+(1-s)l)\left[\begin{array}{c}\left(\frac{\sqrt{2}}{\pi}-\frac{2}{\pi}\cos\left(\frac{\pi s}{2}\right)\right)|\xi'(k)| \\ +\left(\frac{2}{\pi}\sin\left(\frac{\pi s}{2}\right)-\frac{\sqrt{2}}{\pi}\right)|\xi'(l)|\end{array}\right]ds\right\}.$$

Following the variable transformation $x=ks+(1-s)l$ within this integral, employing the theorem's hypothesis that the function $w(x)$ is symmetric with respect to $x=\frac{k+l}{2}$ yields the sought inequality. □

Corollary 3.2. *If we take $w(x)=1$ in inequality of (3.1), we get*

$$\left|\frac{1}{l-k}\int_k^l \xi(x)dx-\xi\left(\frac{k+l}{2}\right)\right|\leq\frac{(4\sqrt{2}-4)(l-k)}{\pi^2}\left[|\xi'(k)|+|\xi'(l)|\right].$$

Theorem 3.3. *Let $\xi: I^\circ\subset\mathbb{R}\to\mathbb{R}$ be a function differentiable on I°, where $k,l\in I^\circ, k<l$, and let $w:[k,l]\to\mathbb{R}$ be a differentiable function that is symmetric with respect to $\frac{k+l}{2}$. Under the condition that $q\geq 1, \frac{1}{p}+\frac{1}{q}=1$, and given that the function $|\xi'|^q$ is trigonometrically convex on the interval $[k,l]$, it follows that:*

$$\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right|$$
$$\leq (l-k)^{1-\frac{2}{p}}\left(\int_{\frac{k+l}{2}}^{l} w^p(x)\left(x-\frac{k+l}{2}\right)dx\right)^{\frac{1}{p}}\left[\begin{array}{c}\left(\frac{4\sqrt{2}-\pi\sqrt{2}}{2\pi^2}|\xi'(k)|^q+\frac{4\sqrt{2}+\pi\sqrt{2}-8}{2\pi^2}|\xi'(l)|^q\right)^{\frac{1}{q}} \\ +\frac{4\sqrt{2}+\pi\sqrt{2}-8}{2\pi^2}|\xi'(k)|^q+\left(\frac{4\sqrt{2}-\pi\sqrt{2}}{2\pi^2}|\xi'(l)|^q\right)^{\frac{1}{q}}\end{array}\right].$$

Proof. By taking the absolute value of both sides of equation (2.1) in Lemma 2.9, the following inequality is obtained:

$$\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right|$$
$$\leq (l-k)\left[\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(ks+(1-s)l)\left|\xi'(tk+(1-t)l)\right|dtds+\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w(ks+(1-s)l)\left|\xi'(tk+(1-t)l)\right|dtds\right].$$

By applying Hölder's inequality to each integral on the right-hand side of the resulting inequality, the following expression is obtained:

$$\begin{aligned}
&\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right| \\
&\leq (l-k)\left[\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w^p(ks+(1-s)l)dtds\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}}\left|\xi'(tk+(1-t)l)\right|^q dtds\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w^p(ks+(1-s)l)dtds\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s}\left|\xi'(tk+(1-t)l)\right|^q dtds\right)^{\frac{1}{q}}\right] \\
&\leq (l-k)\left[\left(\int_{\frac{k+l}{2}}^{l} w^p(x)\left(\frac{2x-l-k}{2(l-k)^2}\right)dx\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}}\left|\xi'(tk+(1-t)l)\right|^q dtds\right)^{\frac{1}{q}}\right. \\
&\left.\left(\int_k^{\frac{k+l}{2}} w^p(x)\left(\frac{k+l-2x}{2(l-k)^2}\right)dx\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s}\left|\xi'(tk+(1-t)l)\right|^q dtds\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Using the hypothesis that the function $w(x)$ is symmetric with respect to $x=\frac{k+l}{2}$ and the fact that the function $|\xi'|^q$ is trigonometrically convex, we can derive the following inequality:

$$\begin{aligned}
&\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right| \\
&\leq (l-k)^{1-\frac{2}{p}}\left(\int_{\frac{k+l}{2}}^{l} w^p(x)\left(\frac{2x-l-k}{2}\right)dx\right)^{\frac{1}{p}}\times\left[\left(\begin{array}{c}|\xi'(k)|^q\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}}\sin\left(\frac{\pi t}{2}\right)dtds \\ +|\xi'(l)|^q\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}}\cos\left(\frac{\pi t}{2}\right)dtds\end{array}\right)^{\frac{1}{q}}+\left(\begin{array}{c}|\xi'(k)|^q\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s}\sin\left(\frac{\pi t}{2}\right)dtds \\ +|\xi'(l)|^q\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s}\cos\left(\frac{\pi t}{2}\right)dtds\end{array}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Upon solving the final integrals here, the proof is completed. □

Corollary 3.4. *If $w(x)=1$ is taken in Theorem 3.3, the following inequality is obtained:*

$$\left|\frac{1}{l-k}\int_k^l \xi(x)dx-\xi\left(\frac{k+l}{2}\right)\right|\leq\frac{(l-k)}{2^{\frac{3}{p}}}\left[\begin{array}{c}\left(\frac{4\sqrt{2}-\pi\sqrt{2}}{2\pi^2}|\xi'(k)|^q+\frac{4\sqrt{2}+\pi\sqrt{2}-8}{2\pi^2}|\xi'(l)|^q\right)^{\frac{1}{q}} \\ +\frac{4\sqrt{2}+\pi\sqrt{2}-8}{2\pi^2}|\xi'(k)|^q+\left(\frac{4\sqrt{2}-\pi\sqrt{2}}{2\pi^2}|\xi'(l)|^q\right)^{\frac{1}{q}}\end{array}\right].$$

Theorem 3.5. *Let $\xi: I^{\circ}\subset\mathbb{R}\to\mathbb{R}$ be a function differentiable on I°, where $k,l\in I^{\circ}, k<l$, and let $w:[k,l]\to\mathbb{R}$ be a differentiable function that is symmetric with respect to $\frac{k+l}{2}$. Under the condition that $q>1$, and given that the function $|\xi'|^q$ is trigonometrically convex on the interval $[k,l]$, it follows that:*

$$\begin{aligned}
&\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right|\leq\frac{1}{(l-k)^{1-\frac{1}{q}}}\left(\int_{\frac{k+l}{2}}^{l} w(x)\left(x-\frac{k+l}{2}\right)dx\right)^{1-\frac{1}{q}} \\
&\left[\left(\begin{array}{c}|\xi'(k)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{-\sqrt{2}}{\pi}+\frac{2}{\pi}\cos\left(\frac{\pi(l-x)}{2(l-k)}\right)\right)dx \\ +|\xi'(l)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{\sqrt{2}}{\pi}-\frac{2}{\pi}\sin\left(\frac{\pi(l-x)}{2(l-k)}\right)\right)dx\end{array}\right)^{\frac{1}{q}}+\left(\begin{array}{c}|\xi'(k)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{-2}{\pi}\cos\left(\frac{\pi(x-k)}{2(l-k)}\right)+\frac{\sqrt{2}}{\pi}\right)dx \\ +|\xi'(l)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{2}{\pi}\sin\left(\frac{\pi(x-k)}{2(l-k)}\right)-\frac{\sqrt{2}}{\pi}\right)dx\end{array}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. In proving the theorem, after taking the absolute value of equation (2.1) in Lemma 2.9, the following inequality is obtained.

$$\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right|$$
$$\leq (l-k)\left[\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(ks+(1-s)l)\left|\xi'(tk+(1-t)l)\right|dtds+\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w(ks+(1-s)l)\left|\xi'(tk+(1-t)l)\right|dtds\right].$$

Subsequently, by applying the Power Mean inequality to each integral on the right-hand side of the resulting expression, the following inequality is derived:

$$\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right|$$
$$\leq (l-k)\left[\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(ks+(1-s)l)dtds\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(ks+(1-s)l)\left|\xi'(tk+(1-t)l)\right|^q dtds\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w(ks+(1-s)l)dtds\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w(ks+(1-s)l)\left|\xi'(tk+(1-t)l)\right|^q dtds\right)^{\frac{1}{q}}\right]$$
$$\leq (l-k)\left[\left(\int_0^{\frac{1}{2}} w(ks+(1-s)l)\left(\frac{1}{2}-s\right)ds\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(ks+(1-s)l)\left|\xi'(tk+(1-t)l)\right|^q dtds\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1} w(ks+(1-s)l)\left(s-\frac{1}{2}\right)ds\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w(ks+(1-s)l)\left|\xi'(tk+(1-t)l)\right|^q dtds\right)^{\frac{1}{q}}\right].$$

Since the function $|\xi'|^q$ is trigonometrically convex, the following inequality is obtained:

$$\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right|$$
$$\leq (l-k)\left[\left(\int_0^{\frac{1}{2}} w(ks+(1-s)l)\left(\frac{1}{2}-s\right)ds\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(ks+(1-s)l)\left(\begin{array}{c}\sin\left(\frac{\pi t}{2}\right)|\xi'(k)|^q\\ +\cos\left(\frac{\pi t}{2}\right)|\xi'(l)|^q\end{array}\right)dtds\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1} w(ks+(1-s)l)\left(s-\frac{1}{2}\right)ds\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w(ks+(1-s)l)\left(\begin{array}{c}\sin\left(\frac{\pi t}{2}\right)|\xi'(k)|^q\\ +\cos\left(\frac{\pi t}{2}\right)|\xi'(l)|^q\end{array}\right)dtds\right)^{\frac{1}{q}}\right].$$

If a change of variable is applied to the first integral on the right-hand side of the final inequality, the following expression is obtained:

$$\begin{aligned}
&\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right| \\
&\leq \frac{1}{(l-k)^{1-\frac{2}{q}}}\left[\left(\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{2x-k-l}{2}\right)dx\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(ks+(1-s)l)\left(\begin{array}{c}\sin\left(\frac{\pi t}{2}\right)|\xi'(k)|^q\\ +\cos\left(\frac{\pi t}{2}\right)|\xi'(l)|^q\end{array}\right)dtds\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_k^{\frac{k+l}{2}} w(x)\left(\frac{k+l-2x}{2}\right)dx\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w(ks+(1-s)l)\left(\begin{array}{c}\sin\left(\frac{\pi t}{2}\right)|\xi'(k)|^q\\ +\cos\left(\frac{\pi t}{2}\right)|\xi'(l)|^q\end{array}\right)dtds\right)^{\frac{1}{q}}\right] \\
&\leq \frac{1}{(l-k)^{1-\frac{2}{q}}}\left[\left(\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{2x-k-l}{2}\right)dx\right)^{1-\frac{1}{q}}\left(\begin{array}{c}|\xi'(k)|^q\int_0^{\frac{1}{2}} w(ks+(1-s)l)\left(\frac{-\sqrt{2}}{\pi}+\frac{2}{\pi}\cos\left(\frac{\pi s}{2}\right)\right)ds\\ +|\xi'(l)|^q\int_0^{\frac{1}{2}} w(ks+(1-s)l)\left(\frac{\sqrt{2}}{\pi}-\frac{2}{\pi}\sin\left(\frac{\pi s}{2}\right)\right)ds\end{array}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_k^{\frac{k+l}{2}} w(x)\left(\frac{k+l-2x}{2}\right)dx\right)^{1-\frac{1}{q}}\left(\begin{array}{c}|\xi'(k)|^q\int_1^{\frac{1}{2}} w(ks+(1-s)l)\left(-\frac{2}{\pi}\cos\left(\frac{\pi s}{2}\right)+\frac{\sqrt{2}}{\pi}\right)ds\\ +|\xi'(l)|^q\int_{\frac{1}{2}}^{1} w(ks+(1-s)l)\left(\frac{2}{\pi}\sin\left(\frac{\pi s}{2}\right)-\frac{\sqrt{2}}{\pi}\right)ds\end{array}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

In the final inequality, after taking the integrals on the right side and applying the variable change $x=ks+(1-s)l$, and then using the fact that the function $w(x)$ is symmetric with respect to $x=\frac{k+l}{2}$, the following inequality is obtained:

$$\begin{aligned}
&\left|\frac{1}{l-k}\int_k^l \xi(x)w(x)dx-\frac{1}{l-k}\xi\left(\frac{k+l}{2}\right)\int_k^l w(x)dx\right| \\
&\leq \frac{1}{(l-k)^{1-\frac{1}{q}}}\left[\left(\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{2x-k-l}{2}\right)dx\right)^{1-\frac{1}{q}}\left(\begin{array}{c}|\xi'(k)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{-\sqrt{2}}{\pi}+\frac{2}{\pi}\cos\left(\frac{\pi(l-x)}{2(l-k)}\right)\right)dx\\ +|\xi'(l)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{\sqrt{2}}{\pi}-\frac{2}{\pi}\sin\left(\frac{\pi(l-x)}{2(l-k)}\right)\right)dx\end{array}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_k^{\frac{k+l}{2}} w(x)\left(\frac{k+l-2x}{2}\right)dx\right)^{1-\frac{1}{q}}\left(\begin{array}{c}|\xi'(k)|^q\int_k^{\frac{k+l}{2}} w(x)\left(-\frac{2}{\pi}\cos\left(\frac{\pi(l-x)}{2(l-k)}\right)+\frac{\sqrt{2}}{\pi}\right)dx\\ +|\xi'(l)|^q\int_k^{\frac{k+l}{2}} w(x)\left(\frac{2}{\pi}\sin\left(\frac{\pi(l-x)}{2(l-k)}\right)-\frac{\sqrt{2}}{\pi}\right)dx\end{array}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{1}{(l-k)^{1-\frac{1}{q}}}\left(\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{2x-k-l}{2}\right)dx\right)^{1-\frac{1}{q}}\left[\left(\begin{array}{c}|\xi'(k)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{-\sqrt{2}}{\pi}+\frac{2}{\pi}\cos\left(\frac{\pi(l-x)}{2(l-k)}\right)\right)dx\\ +|\xi'(l)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{\sqrt{2}}{\pi}-\frac{2}{\pi}\sin\left(\frac{\pi(l-x)}{2(l-k)}\right)\right)dx\end{array}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\begin{array}{c}|\xi'(k)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(-\frac{2}{\pi}\cos\left(\frac{\pi(x-k)}{2(l-k)}\right)+\frac{\sqrt{2}}{\pi}\right)dx\\ +|\xi'(l)|^q\int_{\frac{k+l}{2}}^{l} w(x)\left(\frac{2}{\pi}\sin\left(\frac{\pi(x-k)}{2(l-k)}\right)-\frac{\sqrt{2}}{\pi}\right)dx\end{array}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Thus, the proof is completed. ☐

Corollary 3.6. *If $w(x)=1$ is assumed in Theorem 3.5, the following inequality is derived:*

$$\left|\frac{1}{l-k}\int_k^l \xi(x)dx-\xi\left(\frac{k+l}{2}\right)\right| \le \frac{(l-k)}{2^{3-\frac{3}{q}}}\left(\frac{\sqrt{2}}{\pi}\right)^{\frac{1}{q}}$$

$$\times\left[\left(\frac{4-\pi}{2\pi}|\xi(k)|^q+\frac{\pi+4-4\sqrt{2}}{2\pi}|\xi(l)|^q\right)^{\frac{1}{q}}+\left(\frac{\pi+4-4\sqrt{2}}{2\pi}|\xi(k)|^q+\frac{4-\pi}{2\pi}|\xi(l)|^q\right)^{\frac{1}{q}}\right].$$

Theorem 3.7. *Let $\xi: I^\circ\subset\mathbb{R}\to\mathbb{R}$ be a function that is differentiable on I°, with $k,l\in I^\circ$ and $k<l$, and let $w:[k,l]\to\mathbb{R}$ be a differentiable function that is symmetric with respect to $\frac{k+l}{2}$. Given that $\frac{1}{p}+\frac{1}{q}=1$ and $q\ge 1$, and considering that the function $|\xi'|^q$ is trigonometrically convex over the interval $[k,l]$, the following inequality is satisfied:*

$$\left|\frac{1}{l-k}\frac{\xi(k)+\xi(l)}{2}\int_k^l w(x)dx-\frac{1}{l-k}\int_k^l \xi(x)w(x)dx\right| \le \frac{1}{\pi}\left(\int_0^1 h^p(t)dt\right)^{\frac{1}{p}}\left(\left|\xi'(k)\right|^q+\left|\xi'(l)\right|^q\right)^{\frac{1}{q}},$$

where

$$h(t)=\left|\int_{k+(l-k)t}^{l-(l-k)t} w(x)dx\right|.$$

Proof. If we start from Lemma 2.10, it is obtained

$$\left|\frac{1}{l-k}\frac{\xi(k)+\xi(l)}{2}\int_k^l w(x)dx-\frac{1}{l-k}\int_k^l \xi(x)w(x)dx\right|$$

$$\le\frac{l-k}{2}\int_0^1\left|\int_t^1 w(ks+(1-s)l)ds-\int_0^t w(ks+(1-s)l)ds\right|\left|\xi'(kt+(1-t)l)\right|dt$$

$$=\frac{1}{2}\left|\int_k^{kt+(1-t)l} w(x)dx-\int_{kt+(1-t)l}^{l} w(x)dx\right|\left|\xi'(kt+(1-t)l)\right|dt. \tag{3.2}$$

Given that the function $w(x)$ is symmetric with respect to $x=\frac{k+l}{2}$,

1. Since $\forall x\in[0,\frac{1}{2}]$, $\int\limits_{tl+(1-t)k}^{l}\frac{w(x)dx}{l-k}-\int\limits_{tk+(1-t)l}^{l}\frac{w(x)dx}{l-k}=\int\limits_{k+(l-k)t}^{l-(l-k)t}\frac{w(x)dx}{l-k}$.
2. Since $\forall x\in[\frac{1}{2},1]$, $\int\limits_{tl+(1-t)k}^{l}\frac{w(x)dx}{l-k}-\int\limits_{tk+(1-t)l}^{l}\frac{w(x)dx}{l-k}=-\int\limits_{k+(l-k)t}^{l-(l-k)t}\frac{w(x)dx}{l-k}$,

it follows. From this point, for all $t\in[0,1]$, let $h(t)=\left|\int_{k+(l-k)t}^{l-(l-k)t}w(x)dx\right|$. Substituting this expression into inequality (3.2) and then applying Hölder's inequality, given that the function $|\xi'|^q$ is trigonometrically convex,

$$\begin{aligned}
&\left|\frac{1}{l-k}\frac{\xi(k)+\xi(l)}{2}\int_k^l w(x)dx-\frac{1}{l-k}\int_k^l \xi(x)w(x)dx\right| \\
&\leq \frac{1}{2}\int_0^1 h(t)\left|\xi'(kt+(1-t)l)\right|dt \qquad (3.3)\\
&\leq \frac{1}{2}\left(\int_0^1 h^p(t)dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|\xi'(kt+(1-t)l)\right|^q dt\right)^{\frac{1}{q}}\\
&\leq \frac{1}{2}\left(\int_0^1 h^p(t)dt\right)^{\frac{1}{p}}\left(\left|\xi'(k)\right|^q\int_0^1 \sin\left(\frac{\pi t}{2}\right)dt+\left|\xi'(l)\right|^q\int_0^1 \cos\left(\frac{\pi t}{2}\right)dt\right)^{\frac{1}{q}}
\end{aligned}$$

it follows. When the simple integrals on the right side are solved, the proof is completed. □

Corollary 3.8. *If $w(x)=1$ is assumed in Theorem 3.7, the following inequality is derived:*

$$\left|\frac{\xi(k)+\xi(l)}{2}-\frac{1}{l-k}\int_k^l \xi(x)dx\right| \leq \frac{l-k}{\pi(p+1)^{\frac{1}{p}}}\left(\left|\xi'(k)\right|^q+\left|\xi'(l)\right|^q\right)^{\frac{1}{q}}.$$

Proof. Given that $\int_0^1 h^p(t)dt=\left|\int_{k+(l-k)t}^{l-(l-k)t}dx\right|^p=(l-k)^p\left|1-2t\right|^p dt=(l-k)^p\left(\frac{1}{p+1}\right)$, the proof is completed. □

Theorem 3.9. *Let $\xi: I^\circ\subset\mathbb{R}\to\mathbb{R}$ be a function that is differentiable on I°, with $k,l\in I^\circ$ and $k<l$, and let $w:[k,l]\to\mathbb{R}$ be a differentiable function that is symmetric with respect to $\frac{k+l}{2}$. Under the condition $q>1$, given that the function $|\xi'|^q$ is trigonometrically convex over the interval $[k,l]$, the following inequality is satisfied:*

$$\begin{aligned}
&\left|\frac{1}{l-k}\frac{\xi(k)+\xi(l)}{2}\int_k^l w(x)dx-\frac{1}{l-k}\int_k^l \xi(x)w(x)dx\right| \\
&\leq \frac{1}{2}\left(\int_0^1 h(t)dt\right)^{1-\frac{1}{q}}\left(\left|\xi'(k)\right|^q\int_0^1 h(t)\sin\left(\frac{\pi t}{2}\right)dt+\left|\xi'(l)\right|^q\int_0^1 h(t)\cos\left(\frac{\pi t}{2}\right)dt\right)^{\frac{1}{q}}, \qquad (3.4)
\end{aligned}$$

where

$$h(t)=\left|\int_{k+(l-k)t}^{l-(l-k)t} w(x)dx\right|$$

is defined.

Proof. By applying the Power Mean inequality together with the fact that the function $|\xi'|^q$ is trigonometrically convex in inequality (3.3), we obtain the following inequality:

$$\begin{aligned}
&\left|\frac{1}{l-k}\frac{\xi(k)+\xi(l)}{2}\int_k^l w(x)dx-\frac{1}{l-k}\int_k^l \xi(x)w(x)dx\right| \\
&\leq \frac{1}{2}\left(\int_0^1 h(t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 h(t)\left|\xi'(kt+(1-t)l)\right|^q dt\right)^{\frac{1}{q}}\\
&\leq \frac{1}{2}\left(\int_0^1 h(t)dt\right)^{1-\frac{1}{q}}\left(\left|\xi'(k)\right|^q\int_0^1 h(t)\sin\left(\frac{\pi t}{2}\right)dt+\left|\xi'(l)\right|^q\int_0^1 h(t)\cos\left(\frac{\pi t}{2}\right)dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Thus, the proof is completed. □

Corollary 3.10. *If $w(x)=1$ is assumed in Theorem 3.9, the following inequality is obtained:*

$$\left|\frac{\xi(k)+\xi(l)}{2}-\frac{1}{l-k}\int_k^l \xi(x)dx\right| \leq \frac{(l-k)}{2^{2-\frac{1}{q}}}\left(\frac{2\pi+8-8\sqrt{2}}{\pi^2}\right)^{\frac{1}{q}}\left(\left|\xi'(k)\right|^q+\left|\xi'(l)\right|^q\right)^{\frac{1}{q}}.$$

Proof. Since $w(x)=1$ implies $h(t)=(l-k)|1-2t|$, the integrals obtained in inequality (3.4) can be found using Python as follows,

$$\int_0^1 |1-2t|\sin\left(\frac{\pi t}{2}\right)dt = \frac{2\pi+8-8\sqrt{2}}{\pi^2}$$
$$\int_0^1 |1-2t|\cos\left(\frac{\pi t}{2}\right)dt = \frac{2\pi+8-8\sqrt{2}}{\pi^2}.$$

□

4. Application

Recent studies have emphasized the significant role of visualizing theoretical expressions through graphical representations. Inspired by this idea, certain results were generated in Python for specific data sets. One of the targeted outcomes was to demonstrate the impact of Hölder and Power Mean inequalities on the upper bound of an inequality, showcasing examples with variations across different values of p and q.

Example 4.1. *In Corollary 3.2, the function $\xi(x)=x^2$ was evaluated for randomly chosen values of k and l within the interval $[0,2]$ under the condition $k<l$, using a step size of 0.1. The left and right sides of the inequality were computed, and their graphs were illustrated:*

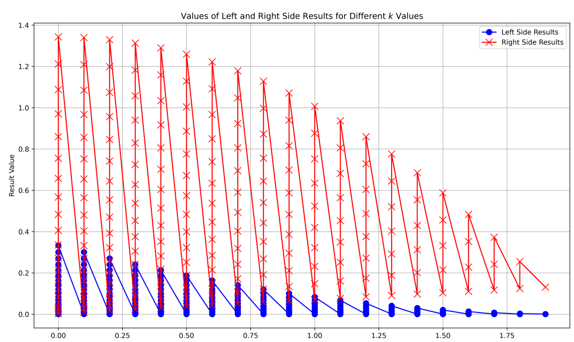

Figure 4.1. Graph of Corollary 3.2

Example 4.2. *In Corollary 3.4, for the function $\xi(x)=x^2$, with the left endpoint of the interval fixed at $k=0.5$ and randomly chosen values of l within $[0.6,2]$ under the condition $k<l$, graphs illustrating the effects on the upper bound of the Hölder inequality for various values of q and p were obtained.*

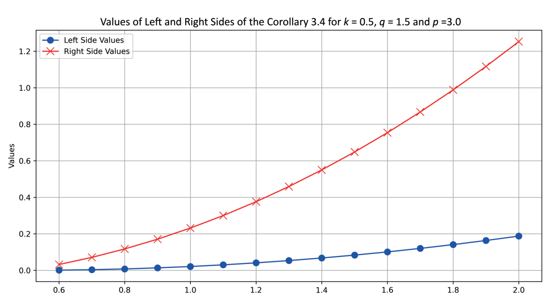

Figure 4.2. Graph of Corollary 3.4 with $p=3, q=1.5$

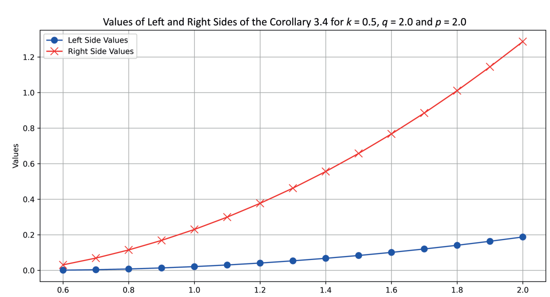

Figure 4.3. Graph of Corollary 3.4 with $p=q=2$

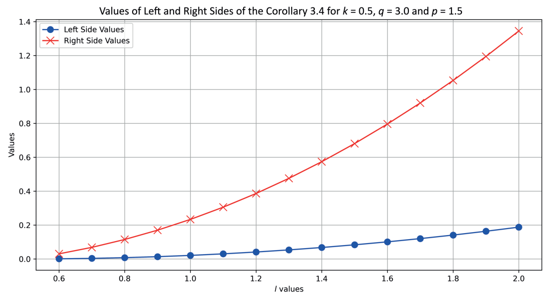

Figure 4.4. Graph of Corollary 3.4 with $p = 1.5, q = 3$

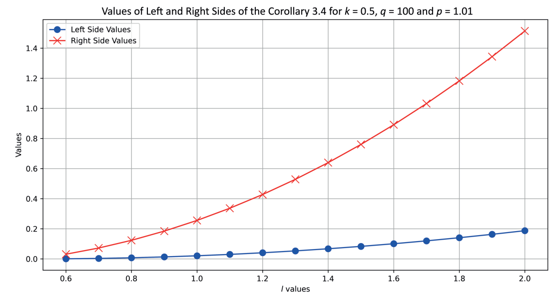

Figure 4.5. Graph of Corollary 3.4 with $p = 1.01, q = 100$

Example 4.3. *In corollary 3.6, with* $\xi(x) = x^2$ *and a fixed* $k = 0.5$*, randomly specific values of* l *were generated under the condition* $k < l$*. For these values, the fulfillment of the inequality for both the left and right sides, and for any values of* q*, was demonstrated, and the variations were obtained graphically:*

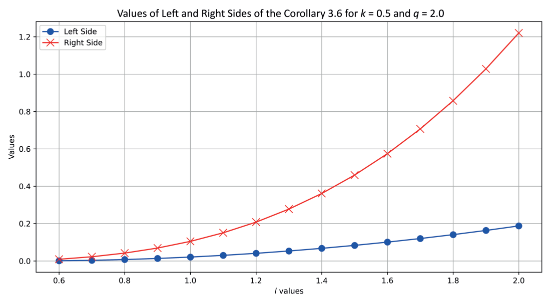

Figure 4.6. Graph of Corollary 3.6 with $q = 2$

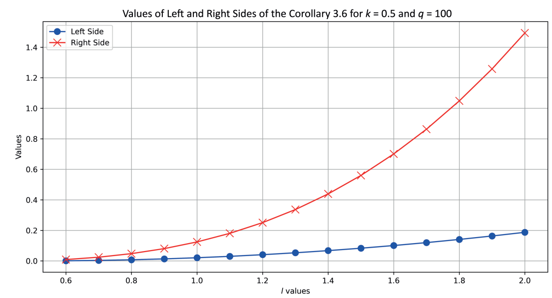

Figure 4.7. Graph of Corollary 3.6 with $q = 100$

Example 4.4. *In corollary 3.10, with* $\xi(x) = x^2$ *and a fixed* $k = 0.5$*, randomly specific values of* l *were generated under the condition* $k < l$*. For these values, it was demonstrated that the inequality was satisfied for both the left and right sides, and for any values of* q*, with the variations obtained graphically:*

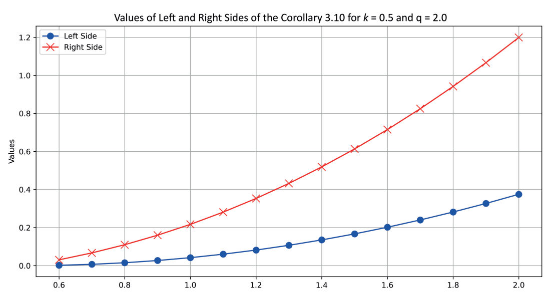

Figure 4.8. Graph of Corollary 3.10 with $q = 2$

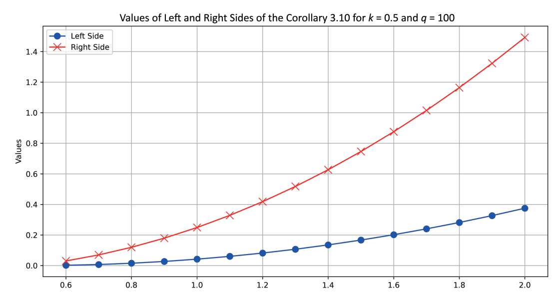

Figure 4.9. Graph of Corollary 3.10 with $q = 100$

5. Conclusion

In conclusion, the study has advanced the understanding of Hermite-Hadamard-Fejér type inequalities within the realm of trigonometrically convex functions. By employing Hölder's inequality and, consequently, the Power Mean inequality, novel upper bounds were derived and subsequently illustrated through graphical representations for various functions, thereby demonstrating their optimality across different parameter values.

Additionally, since the Fejér inequality is expressed through a weight function that can be transformed into various fractional integrals, the framework developed herein allows for the derivation of Hermite-Hadamard type inequalities for trigonometrically convex functions in the context of different fractional integrals. Essentially, this work constitutes a generalization of the classical Hermite-Hadamard midpoint and trapezoidal type inequalities, extending their applicability to fractional integral settings.

Future research could further explore these extensions by investigating broader classes of convex functions and their corresponding fractional integrals, as well as by examining potential applications in numerical analysis and optimization theory.

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References

[1] S. Varošanec, *On h-convexity*, J. Math. Anal. Appl., **326** (2007), 303–311.

[2] M. Bombardelli, S. Varošanec, *Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities*, Comput. Math. Appl., **58**(9) (2009), 1869–1877.

[3] Ş. Demir, *New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are exponential trigonometric convex*, Sigma, **41**(3) (2023), 451–456.

[4] S. Turhan, *Novel results based on generalisation of some integral inequalities for trigonometrically-p function*, Sakarya University Journal of Science, **24**(4) (2020), 665–674.

[5] Ş. Demir, S. Maden, İ. İscan, M. Kadakal, *On new Simpson's type inequalities for trigonometrically convex functions with applications*, Cumhuriyet Sci. J., **41**(4) (2020), 862–874.

[6] M. Z. Sarikaya, *On new Hermite-Hadamard-Fejér type integral inequalities*, Stud. Univ. Babes-Bolyai Math., **57**(3) (2012).

[7] H. Budak, H. Kara, T. Tunc, F. Hezenci, S. Khan, *On new trapezoid and midpoint type inequalities for generalized quantum integrals*, Filomat, **38**(7) (2024), 2323–2341.

[8] S. S. Dragomir, C. E. M. Pearce, *Selected topics on Hermite-Hadamard inequalities and its applications*, RGMIA Monograph, (2002).

[9] B. Çelik, H. Budak, E. Set, *On generalized Milne type inequalities for new conformable fractional integrals*, Filomat, **38**(5) (2024), 1807–1823.

[10] G. Zabandan, *A new refinement of the Hermite-Hadamard inequality for convex functions*, J. Inequal. Pure Appl. Math., **10**(2) (2009), Article ID 45.

[11] C. Ünal, F. Hezenci, H. Budak, *Conformable fractional Newton-type inequalities with respect to differentiable convex functions*, J. Inequal. Appl., **2023**(1) (2023), 85.

[12] F. Hezenci, H. Budak, *An extensive study on parameterized inequalities for conformable fractional integrals*, Anal. Math. Phys., **13**(5) (2023), 82.

[13] F. Hezenci, H. Budak, *Novel results on trapezoid-type inequalities for conformable fractional integrals*, Turkish J. Math., **47**(2) (2023), 425–438.

[14] J. Hadamard, *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl., **58** (1893), 171–215.

[15] L. Fejér, *Über die Fourierreihen II*, Math. Naturwiss. Anz. Ungar. Akad. Wiss., **24** (1906), 369–390.

[16] H. Kadakal, *Hermite-Hadamard type inequalities for trigonometrically convex functions*, Sci. Stud. Res. Ser. Math. Inform., **28**(2) (2018), 19–28.