J Inno Sci Eng, 2025, 9(2):216-231

DOI:https://doi.org/ 10.38088/jise.1638824

Research Article

Soft Intersection Bi-interior Ideals of Semigroups

Aslıhan Sezgin[1*], Aleyna İlgin[1]

[1] *Amasya University, Department of Mathematics and Science Education, 05100, Amasya, Türkiye*

ARTICLE INFO

Received Date: 12/02/2025
Accepted Date: 23/05/2025

Cite this paper as:
Sezgin, A., & İlgin, A. (2025). Soft Intersection B i-interior Ideals of Semigroups. *Journal of Innovative Science and Engineering.* 9(2), 216-231.

*Corresponding author: Aslıhan Sezgin
E-mail:aslihan.sezgin@amasya.edu.tr

Keywords:
Soft set
Semigroup
Bi-interior ideals
Soft intersection bi-interior ideals
Soft simple* semigroup

ABSTRACT

Generalizing the ideals of an algebraic structure has shown to be both beneficial and interesting for mathematicians. In this context, the idea of the bi-interior ideal was introduced as a generalization of the bi-ideal and interior ideal of a semigroup. By introducing "soft intersection (S-int) bi-interior (BI) ideals of semigroups", we introduce a framework integrating semigroup theory with soft set theory in this study. Finding the relationships between S-int BI-ideals and other specific kinds of S-int ideals of a semigroup is the main aim of this study. Our results show that an S-int BI-ideal is an S-int subsemigroup of a soft simple* semigroup, and that an S-int left (right/two-sided) ideal, bi-ideal, interior ideal and quasi-ideal is an S-int BI-ideal; in other words, the S-int BI-ideal is a generalization of the S-int left (right/two-sided) ideal, bi-ideal, interior ideal and quasi-ideal, however, we provide counterexamples demonstrating that the converses do not always hold. We demonstrate that the semigroup should be a soft simple* semigroup in order to satisfy the converses. Our key theorem, which states that if a nonempty subset of a semigroup is a BI-ideal, then its soft characteristic function is an S-int BI-ideal, and vice versa, enables us to bridge the gap between semigroup theory and soft set theory. Using this theorem, we show how this idea relates to the existing algebraic structures in classical semigroup theory. Furthermore, we present conceptual characterizations and analysis of the new concept in terms of soft set operations supporting our assertions with illuminating examples.

1. Introduction

Semigroups are essential to many areas of mathematics as they provide the abstract algebraic foundation for "memoryless" systems that restart with each iteration. Semigroups, which were formally studied for the first time in the early 1900s, are crucial models for linear time-invariant systems in practical mathematics. It is crucial to theoretical computer science to investigate finite semigroups since they are inseparable from finite automata. Furthermore, semigroups and Markov processes are connected in probability theory. Since the idea of ideals is fundamental to comprehending mathematical structures and their uses, many mathematicians have concentrated a significant portion of their study on generalizing ideals in algebraic structures. Specifically, the generalization of ideals in algebraic structures is necessary for additional study of algebraic structures. Several mathematicians have shown important results and characterizations of algebraic structures by using the concept and properties of the generalization of ideals in algebraic structures. Dedekind developed the idea of ideals for the theory of algebraic numbers, and Noether expanded it to include associative rings. The concept of an ideal is extended by the concept of a one-sided ideal of any algebraic structure, and the concepts of a one-sided ideal and a two-sided ideal remain central to ring theory.

Good and Hughes [1] proposed the notion of bi-ideals for semigroups in 1952. The concept of quasi-ideals was initially introduced by Steinfeld [2] for semigroups and subsequently for rings. Bi-ideals are generalizations of quasi-ideals, while quasi-ideals are generalizations of left and right ideals. Lajos [3] established the concept of the interior ideal, and Szasz [4,5] expanded on it. Interior ideals are generalizations of ideals. Several new various kinds of ideals of semigroup, which are generalizations of the existing ones, such as bi-interior ideals, bi-quasi-interior ideals, bi-quasi ideals, quasi-interior ideals, and weak-interior ideals have been defined by Rao [6–9], respectively. Additionally, Baupradist et al. [10] introduced the concept of essential ideals of semigroups. The idea of "almost" ideals were proposed as a more generalized concept of the many types of ideals, and their characteristics as well as their relationships with one another were thoroughly investigated. In this context, the notion of almost ideals was first put up in [11]. Besides that, the various types of almost ideals of semigroups were introduced in [12-18]. Furthermore, in [13, 15–20] several fuzzy almost ideal types for semigroups were examined.

In 1999, the "Soft Set Theory" was first presented by Molodtsov [21] in order to comprehend and provide appropriate solutions for problems involving uncertainty. Since then, a great deal of substantial study has been done on soft set notions, particularly soft set operations. Maji et al. [22] defined certain operations on soft sets and introduced some concepts related to soft sets. Pei and Miao [23] and Ali et al. [24] introduced several operations of soft sets. Sezgin and Atagün [25] studied on soft set operations. We refer to [26–38] for more information on soft set operations, which have developed in popularity since its start. Çağman and Enginoğlu [39] revised the notions and operations of soft sets. Çağman et al. [40] developed the concept of $\mathbb{S}$-int groups, which led to the investigation of various soft algebraic systems. By applying soft sets to semigroup theory, Sezer et al. [41,42] defined $\mathbb{S}$-int semigroups, left (right/two-sided) ideals, interior ideals, quasi-ideals, and (generalized) bi-ideals of semigroups, and they thoroughly investigated their fundamental properties. Sezgin and Orbay [43] defined and classified some types of semigroups in terms of their $\mathbb{S}$-int substructures of semigroups. Different kinds of $\mathbb{S}$-int almost ideals were introduced and investigated in [44–55] as a generalization of $\mathbb{S}$-int ideals. In [56–73], the soft forms of different algebraic structures have been examined.

As a generalization of bi-ideals and interior ideals of semigroups, Rao [6] established the concept of bi-interior ideals and examined their characteristics. The concept of bi-interior ideals has also been studied by Rao and Venkateswarlu [74] for Γ-semirings, and Rao [75] for Γ-semigroup. By introducing "$\mathbb{S}$-int bi-interior ideals of semigroups", we introduce a framework integrating semigroup theory with soft set theory in this study. We obtain the relationships between $\mathbb{S}$-int bi-interior ideals and various kinds $\mathbb{S}$-int ideals of semigroups. Our results show every $\mathbb{S}$-int bi-interior ideal of a soft simple* semigroup is an $\mathbb{S}$-int subsemigroup and $\mathbb{S}$-int bi-interior ideal is a generalization of $\mathbb{S}$-int left ideal, right ideal, ideal, bi-ideal, interior ideal and quasi-ideal, however, we provide counterexamples demonstrating that the converses do not always hold. We demonstrate that the semigroup should be a soft simple* semigroup in order to satisfy the converses. Our significant theorem, which states that if a nonempty subset of a semigroup is a bi-interior ideal, then its soft characteristic function is an $\mathbb{S}$-int bi-interior ideal, and vice versa, enables us to bridge the gap between semigroup theory and soft set theory. Using this theorem, we show how this idea relates to the existing algebraic structures in classical semigroup theory. Furthermore, we present conceptual characterizations and analysis of the new concept in terms of regarding soft set operations and notions supporting our assertions with particular, illuminating examples. There are four sections in this study. An introduction to the subject is given in Section 1. The basic concepts of semigroup and soft set ideals, together with relevant definitions and implications, are presented in Section 2. In section 3, the concept of $\mathbb{S}$-int bi-interior ideal of semigroups was introduced and using specific examples to examine their characteristics and how they relate to other kinds of $\mathbb{S}$-int ideals. Our findings are outlined in Section 4 along with some directions for future study.

2. Preliminaries

Throughout this paper, S denotes a semigroup. $\emptyset \neq \mathcal{J} \subseteq S$ is called a subsemigroup of S if $\mathcal{J}\mathcal{J} \subseteq \mathcal{J}$, a left (right) ideal of S if $S\mathcal{J} \subseteq \mathcal{J}$ $(\mathcal{J}S \subseteq \mathcal{J})$, an ideal of S if is both left ideal of S and right ideal of S, a bi-ideal of S if $\mathcal{J}\mathcal{J} \subseteq \mathcal{J}$ and $\mathcal{J}S\mathcal{J} \subseteq \mathcal{J}$, an interior ideal of S if $\mathcal{J}\mathcal{J} \subseteq \mathcal{J}$ and $S\mathcal{J}S \subseteq \mathcal{J}$, a bi-interior ideal of S (ƁI-ideal) if $S\mathcal{J}S \cap \mathcal{J}S\mathcal{J} \subseteq \mathcal{J}$.

Definition 2.1 [21, 39]. Let E be the parameter set, U be the universal set, $P(U)$ be the power set of U, and $\mathrm{K} \subseteq E$. The soft set (SS) f_{K} over U is a function such that $f_{\mathrm{K}}: E \rightarrow P(U)$, where for all $\mathrm{v} \notin \mathrm{K}$, $f_{\mathrm{K}}(\mathrm{v}) = \emptyset$. That is,

$$f_{\mathrm{K}} = \{(\mathrm{v}, f_{\mathrm{K}}(\mathrm{v})) : \mathrm{v} \in E, f_{\mathrm{K}}(\mathrm{v}) \in P(U)\} \tag{1}$$

The set of all SSs over U is designated by $S_E(U)$ throughout this paper.

Definition 2.2 [39]. Let $f_{\mathrm{B}} \in S_E(U)$. If $f_{\mathrm{B}}(\upsilon) = \emptyset$ for all $\upsilon \in E$, then f_{B} is called a null SS and indicated by $\emptyset_E$.

Definition 2.3 [39]. Let $f_{\mathrm{B}}, f_{\mathcal{T}} \in S_E(U)$. If $f_{\mathrm{B}}(\omega) \subseteq f_{\mathcal{T}}(\omega)$, for all $\omega \in E$, then f_{B} is called a soft subset of $f_{\mathcal{T}}$ and indicated by $f_{\mathrm{B}} \tilde{\subseteq} f_{\mathcal{T}}$. If $f_{\mathrm{B}}(\omega) = f_{\mathcal{T}}(\omega)$, for all $\omega \in E$, then f_{B} is called soft equal to $f_{\mathcal{T}}$ and denoted by $f_{\mathrm{B}} = f_{\mathcal{T}}$.

Definition 2.4 [39]. Let $f_{\mathrm{A}}, f_{\mathrm{B}} \in S_E(U)$. The union (intersection) of f_{A} and f_{B} is the SS $f_{\mathrm{A}} \tilde{\cup} f_{\mathrm{B}}$ $(f_{\mathrm{A}} \tilde{\cap} f_{\mathrm{B}})$ where $(f_{\mathrm{A}} \tilde{\cup} f_{\mathrm{B}})(\hbar) = f_{\mathrm{A}}(\hbar) \cup f_{\mathrm{B}}(\hbar)$ $\left((f_{\mathrm{A}} \tilde{\cap} f_{\mathrm{B}})(\hbar) = f_{\mathrm{A}}(\hbar) \cap f_{\mathrm{B}}(\hbar)\right)$, for all $\hbar \in E$, respectively.

Definition 2.5 [40]. Let $f_{\mathcal{R}} \in S_E(U)$ and $\alpha \subseteq U$. The upper α-inclusion of $f_{\mathcal{R}}$, denoted by $\mathcal{U}(f_{\mathcal{R}}; \alpha)$, is defined as

$$\mathcal{U}(f_{\mathcal{R}}; \alpha) = \{\mathrm{e} \in \mathcal{R} \mid f_{\mathcal{R}}(\mathrm{e}) \supseteq \alpha\} \tag{2}$$

Definition 2.6 [41]. Let $\hbar_S, \wp_S \in S_S(U)$. $\mathbb{S}$-int product $\hbar_S \circ \wp_S$ is defined by

$$\mathcal{U}(\hbar_S \circ \wp_S)(x) = \begin{cases} \bigcup_{x=yz} \{\hbar_S(y) \cap \wp_S(z)\}, & if\ \exists y, z \in S\ such\ that\ x = yz \\ \emptyset, & otherwise \end{cases} \tag{3}$$

Theorem 2.7 [41]. Let $d_S, \eta_S, \upsilon_S \in S_S(U)$. Then,

i. $(d_S \circ \eta_S) \circ \upsilon_S = d_S \circ (\eta_S \circ \upsilon_S)$
ii. $d_S \circ \eta_S \neq \eta_S \circ d_S$, generally
iii. $d_S \circ (\eta_S \tilde{\cup} \upsilon_S) = (d_S \circ \eta_S) \tilde{\cup} (d_S \circ \upsilon_S)$ and $(d_S \tilde{\cup} \eta_S) \circ \upsilon_S = (d_S \circ \upsilon_S) \tilde{\cup} (\eta_S \circ \upsilon_S)$
iv. $d_S \circ (\eta_S \tilde{\cap} \upsilon_S) = (d_S \circ \eta_S) \tilde{\cap} (d_S \circ \upsilon_S)$ and $(d_S \tilde{\cap} \eta_S) \circ \upsilon_S = (d_S \circ \upsilon_S) \tilde{\cap} (\eta_S \circ \upsilon_S)$
v. If $d_S \tilde{\subseteq} \eta_S$ then $d_S \circ \upsilon_S \tilde{\subseteq} \eta_S \circ \upsilon_S$ and $\upsilon_S \circ d_S \tilde{\subseteq} \upsilon_S \circ \eta_S$
vi. If $\wp_S, \mu_S \in S_S(U)$ such that $\wp_S \tilde{\subseteq} d_S$ and $\mu_S \tilde{\subseteq} \eta_S$, then $\wp_S \circ \mu_S \tilde{\subseteq} d_S \circ \eta_S$.

Definition 2.8 [41]. Let $\emptyset \neq ₹ \subseteq S$. The soft characteristic function of ₹, denoted by $S_{₹}$, is defined as

$$S_{₹}(x) = \begin{cases} U, & if\ x \in ₹ \\ \emptyset, & if\ x \in S \backslash ₹ \end{cases} \tag{4}$$

Theorem 2.9 [41, 44]. Let $\mathcal{A}, \mathcal{V} \subseteq S$. Then,

i. $\mathcal{A} \subseteq \mathcal{V}$ iff $S_{\mathcal{A}} \tilde{\subseteq} S_{\mathcal{V}}$
ii. $S_{\mathcal{A}} \tilde{\cap} S_{\mathcal{V}} = S_{\mathcal{A} \cap \mathcal{V}}$ and $S_{\mathcal{A}} \tilde{\cup} S_{\mathcal{V}} = S_{\mathcal{A} \cup \mathcal{V}}$
iii. $S_{\mathcal{A}} \circ S_{\mathcal{V}} = S_{\mathcal{A}\mathcal{V}}$

Definition 2.10 [41, 42]. $\vartheta_S \in S_S(U)$ is called

i. an $\mathbb{S}$-int subsemigroup ($\mathcal{SS}$) of S over U if $\vartheta_S(ab) \supseteq \vartheta_S(a) \cap \vartheta_S(b)$ for all $a, b \in S$,
ii. an $\mathbb{S}$-int left (right) ideal ($\mathcal{L}$($\mathcal{R}$)-ideal) of S over U if $\vartheta_S(\mathrm{n}\upsilon) \supseteq \vartheta_S(\upsilon)$ $(\vartheta_S(\mathrm{n}\upsilon) \supseteq \vartheta_S(\mathrm{n}))$ for all $\mathrm{n}, \upsilon \in S$, and is called an $\mathbb{S}$-int two-sided ideal ($\mathbb{S}$-int ideal) of S over U if it is both $\mathbb{S}$-int $\mathcal{L}$-ideal of S over U and $\mathbb{S}$-int $\mathcal{R}$-ideal of S over U,
iii. an $\mathbb{S}$-int bi-ideal ($\mathcal{B}$-ideal) of S over U if ϑ_S is an $\mathbb{S}$-int $\mathcal{SS}$ of S over U and $\vartheta_S(\mathrm{jn\rho}) \supseteq \vartheta_S(\mathrm{j}) \cap \vartheta_S(\rho)$ for all $\mathrm{j}, \mathrm{n}, \rho \in S$,
iv. an $\mathbb{S}$-int interior ideal ($\mathcal{I}$-ideal) of S over U if $\vartheta_S(\mathrm{jn\rho}) \supseteq \vartheta_S(\mathrm{n})$ for all $\mathrm{j}, \mathrm{n}, \rho \in S$.

Note that in [41], the definition of "$\mathbb{S}$-int $\mathcal{SS}$ of S" is given as "$\mathbb{S}$-int semigroup of S"; however in this paper, we prefer to use "$\mathbb{S}$-int $\mathcal{SS}$ of S".

If $\vartheta_S(x) = U$ for all $x \in S$, then ϑ_S is an $\mathbb{S}$-int $\mathcal{SS}$ ($\mathcal{L}$-ideal, $\mathcal{R}$-ideal, ideal, $\mathcal{B}$-ideal, $\mathcal{I}$-ideal). Such a kind of $\mathbb{S}$-int $\mathcal{SS}$ ($\mathcal{L}$-ideal, $\mathcal{R}$-ideal, ideal, $\mathcal{B}$-ideal, $\mathcal{I}$-ideal) is denoted by $\tilde{\mathbb{S}}$. Here, $\tilde{\mathbb{S}} = S_S$, that is, $\tilde{\mathbb{S}}(x) = U$ for all $x \in S$ [41, 42].

Definition 2.11 [42]. $\vartheta_S \in S_S(U)$ is called an $\mathbb{S}$-int quasi-ideal ($\mathcal{Q}$-ideal) of S over U if $\left(\tilde{\mathbb{S}} \circ \vartheta_S\right) \tilde{\cap} \left(\vartheta_S \circ \tilde{\mathbb{S}}\right) \tilde{\subseteq} \vartheta_S$.

Theorem 2.12 [41]. Let $\vartheta_S \in S_S(U)$. Then,

i. $\tilde{\mathbb{S}} \circ \tilde{\mathbb{S}} \tilde{\subseteq} \tilde{\mathbb{S}}$
ii. $\tilde{\mathbb{S}} \circ \vartheta_S \tilde{\subseteq} \tilde{\mathbb{S}}$ and $\vartheta_S \circ \tilde{\mathbb{S}} \tilde{\subseteq} \tilde{\mathbb{S}}$
iii. $\vartheta_S \tilde{\cup} \tilde{\mathbb{S}} = \tilde{\mathbb{S}}$ and $\vartheta_S \tilde{\cap} \tilde{\mathbb{S}} = \vartheta_S$.

Theorem 2.13 [41, 42]. Let $\vartheta_S \in S_S(U)$. Then,

i. ϑ_S is an $\mathbb{S}$-int $\mathcal{SS}$ iff $(\vartheta_S \circ \vartheta_S) \tilde{\subseteq} \vartheta_S$,

ii. ϑ_S is an $\mathbb{S}$-int $\mathcal{L}(\mathcal{R})$-ideal iff $\left(\widetilde{\mathbb{S}} \circ \vartheta_S\right) \tilde{\subseteq} \vartheta_S$ $\left(\left(\vartheta_S \circ \widetilde{\mathbb{S}}\right) \tilde{\subseteq} \vartheta_S\right)$,

iii. ϑ_S is an $\mathbb{S}$-int ideal iff $\left(\widetilde{\mathbb{S}} \circ \vartheta_S\right) \tilde{\subseteq} \vartheta_S$ and $\left(\vartheta_S \circ \widetilde{\mathbb{S}}\right) \tilde{\subseteq} \vartheta_S$,

iv. ϑ_S is an $\mathbb{S}$-int $\mathcal{B}$-ideal iff $(\vartheta_S \circ \vartheta_S) \tilde{\subseteq} \vartheta_S$ and $\left(\vartheta_S \circ \widetilde{\mathbb{S}} \circ \vartheta_S\right) \tilde{\subseteq} \vartheta_S$,

v. ϑ_S is an $\mathbb{S}$-int $\mathcal{I}$-ideal iff $\left(\widetilde{\mathbb{S}} \circ \vartheta_S \circ \widetilde{\mathbb{S}}\right) \tilde{\subseteq} \vartheta_S$.

Theorem 2.14 [42]. Every $\mathbb{S}$-int $\mathcal{Q}$-ideal is an $\mathbb{S}$-int $\mathcal{B}$-ideal.

Now, the concept of a soft left (right) simple* semigroup will be introduced in order to characterize $\mathbb{S}$-int ideals.

Definition 2.15. Let $\mathfrak{f}_S \in S_S(U)$. Then, S is called a soft left simple* semigroup (with respect to $\mathfrak{f}_S$) if $\widetilde{\mathbb{S}} = \widetilde{\mathbb{S}} \circ \mathfrak{f}_S$, is called a soft right simple* semigroup (with respect to $\mathfrak{f}_S$) if $\widetilde{\mathbb{S}} = \mathfrak{f}_S \circ \widetilde{\mathbb{S}}$, is called a soft simple* semigroup (with respect to $\mathfrak{f}_S$) if $\widetilde{\mathbb{S}} = \widetilde{\mathbb{S}} \circ \mathfrak{f}_S = \mathfrak{f}_S \circ \widetilde{\mathbb{S}}$. If S is a soft (left/right) simple* semigroup with respect to all soft sets over U, then it is called a soft (left/right) simple* semigroup.

Soft left (right) simple* semigroup is abbreviated by soft left (right) simple*.

Example 2.16. Consider the semigroup $S = \{\text{Ѧ}, \text{O}, \text{Ѝ}\}$ defined by Table 1.:

Table 1: Cayley table of '⍟' binary operation.

⍟	Ѧ	O	Ѝ
Ѧ	Ѝ	O	Ѧ
O	O	O	O
Ѝ	Ѧ	O	Ѝ
Ѧ	Ѝ	O	Ѧ

Let $\mathfrak{f}_S$ be SS over $U = D_2 = \{\langle x, y\rangle : x^2 = y^2 = e, xy = yx\} = \{e, x, y, yx\}$ as follows:

$$\mathfrak{f}_S = \{(\text{Ѧ}, \{e\}), (\text{O}, \{x, y\}), (\text{Ѝ}, \{x, y, yx\})\} \tag{5}$$

By considering that

$$\widetilde{\mathbb{S}} = \{(\text{Ѧ}, U), (\text{O}, U), (\text{Ѝ}, U)\} \tag{6}$$

S is a soft left simple* with respect to $\mathfrak{f}_S$. In fact, since

$$\widetilde{\mathbb{S}} \circ \mathfrak{f}_S = \{(\text{Ѧ}, U), (\text{O}, U), (\text{Ѝ}, U)\} \tag{7}$$

$\widetilde{\mathbb{S}} = \widetilde{\mathbb{S}} \circ \mathfrak{f}_S$ is obtained, implying that S is a soft left simple* with respect to $\mathfrak{f}_S$. Similarly, S is a soft right simple* with respect to $\mathfrak{f}_S$. In fact, since

$$\mathfrak{f}_S \circ \widetilde{\mathbb{S}} = \{(\text{Ѧ}, U), (\text{O}, U), (\text{Ѝ}, U)\} \tag{8}$$

$\widetilde{\mathbb{S}} = \mathfrak{f}_S \circ \widetilde{\mathbb{S}}$ is obtained, implying that S is a soft right simple* with respect to $\mathfrak{f}_S$. Hence, $\widetilde{\mathbb{S}} = \widetilde{\mathbb{S}} \circ \mathfrak{f}_S = \mathfrak{f}_S \circ \widetilde{\mathbb{S}}$. Thereby, S is a soft simple* with respect to $\mathfrak{f}_S$.

Theorem 2.17. Let S be soft simple*. Then, the following conditions hold.

i. Every $\mathbb{S}$-int $\mathcal{B}$-ideal is an $\mathbb{S}$-int $\mathcal{Q}$-ideal (Here, S is enough to be soft left or right simple*).
ii. Every $\mathbb{S}$-int $\mathcal{Q}$-ideal is an $\mathbb{S}$-int ideal.
iii. Every $\mathbb{S}$-int $\mathcal{B}$-ideal is an $\mathbb{S}$-int ideal.
iv. Every $\mathbb{S}$-int $\mathcal{I}$-ideal is an $\mathbb{S}$-int ideal.

Proof:

i. The proof is presented only for soft left simple* semigroups, as the proof for soft right simple* semigroups can be shown similarly. Let S be a soft left simple* and $\mathfrak{f}_S$ be an $\mathbb{S}$-int $\mathcal{B}$-ideal. Then, $\widetilde{\mathbb{S}} = \widetilde{\mathbb{S}} \circ \mathfrak{f}_S$ and $\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S \tilde{\subseteq} \mathfrak{f}_S$. Thus,

$$\left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right) \tilde{\cap} \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \tilde{\subseteq} \mathfrak{f}_S \circ \widetilde{\mathbb{S}} = \mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S \tilde{\subseteq} \mathfrak{f}_S \tag{8}$$

is obvious. Hence, $\mathfrak{f}_S$ is an $\mathbb{S}$-int $\mathcal{Q}$-ideal.

ii. Let S be a soft simple* and $\mathfrak{f}_S$ be an $\mathbb{S}$-int $\mathcal{Q}$-ideal. Then, $\widetilde{\mathbb{S}} = \widetilde{\mathbb{S}} \circ \mathfrak{f}_S = \mathfrak{f}_S \circ \widetilde{\mathbb{S}}$ and $\left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right) \tilde{\cap} \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \tilde{\subseteq} \mathfrak{f}_S$. Since

$$\widetilde{\mathbb{S}} \circ \mathfrak{f}_S = \left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right) \tilde{\cap} \left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right) = \left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right) \tilde{\cap} \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \tilde{\subseteq} \mathfrak{f}_S \tag{9}$$

$\mathfrak{f}_S$ is an $\mathbb{S}$-int $\mathcal{L}$-ideal. Similarly, since

$$\mathfrak{f}_S \circ \widetilde{\mathbb{S}} = \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) = \left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right) \tilde{\cap} \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \tilde{\subseteq} \mathfrak{f}_S \tag{10}$$

$\mathfrak{f}_S$ is an $\mathbb{S}$-int $\mathcal{R}$-ideal. Hence, $\mathfrak{f}_S$ is an $\mathbb{S}$-int ideal.

iii. Let S be a soft simple* and $\mathfrak{f}_S$ be an $\mathbb{S}$-int $\mathcal{B}$-ideal. Then, by Theorem 2.17 (i) $\mathfrak{f}_S$ is an $\mathbb{S}$-int $\mathcal{Q}$-ideal. The rest of the proof is obvious from Theorem 2.17 (ii).

iv. Let S be a soft simple* and $\mathfrak{f}_S$ be an $\mathbb{S}$-int $\mathcal{I}$-ideal. Then, $\widetilde{\mathbb{S}} = \widetilde{\mathbb{S}} \circ \mathfrak{f}_S = \mathfrak{f}_S \circ \widetilde{\mathbb{S}}$ and $\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}} \tilde{\subseteq} \mathfrak{f}_S$. Since

$$\widetilde{\mathbb{S}} \circ \mathfrak{f}_S = \widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \mathfrak{f}_S \tilde{\subseteq} \widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}} \tilde{\subseteq} \mathfrak{f}_S \tag{11}$$

$\mathfrak{f}_S$ is an $\mathbb{S}$-int $\mathcal{L}$-ideal. Similarly, since

$$\mathfrak{f}_S \circ \widetilde{\mathbb{S}} = \mathfrak{f}_S \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}} \tilde{\subseteq} \widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}} \tilde{\subseteq} \mathfrak{f}_S \tag{12}$$

$\mathfrak{f}_S$ is an $\mathbb{S}$-int $\mathcal{R}$-ideal. Hence, $\mathfrak{f}_S$ is an $\mathbb{S}$-int ideal.

3. Soft Intersection Bi-interior Ideals of Semigroups

Definition 3.1. A soft set f_S over U is called a soft intersection ($\mathbb{S}$-int) bi-interior ideal of S over U if.

$$\left(\widetilde{\mathbb{S}} \circ f_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(f_S \circ \widetilde{\mathbb{S}} \circ f_S\right) \tilde{\subseteq} f_S \tag{13}$$

From now on, "ideal" is abbreviated by "id", and $\mathbb{S}$-int bi-interior ideal of S over U is abbreviated by $\mathbb{S}$-int $\mathbb{BI}$-id.

Example 3.2. Let $S = \{\square, \square, \text{Ꜧ}, \text{₾}\}$ be:

Table 2: Cayley table of '•' binary operation.

•	$\square$	$\square$	Ꜧ	₾
$\square$	$\square$	$\square$	$\square$	₾
$\square$	$\square$	$\square$	$\square$	₾
Ꜧ	$\square$	$\square$	$\square$	₾
₾	₾	₾	₾	₾

Let ω_S and ɥ_S be SSs over $U = \mathbb{Z}$ as follows:

$$\omega_S = \{(\square, \{1,2\}), (\square, \{1,2,3\}), (\text{Ꜧ}, \{2,3\}), (\text{₾}, \mathbb{Z}^+)\} \tag{14}$$

$$\text{ɥ}_S = \{(\square, \{3,5\}), (\square, \{3,5,7,8\}), (\text{Ꜧ}, \{5,8\}), (\text{₾}, \{3,5,7\})\} \tag{15}$$

Then, ω_S is an $\mathbb{S}$-int $\mathbb{BI}$-id. In fact;

$$\left[\left(\widetilde{\mathbb{S}} \circ \omega_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\omega_S \circ \widetilde{\mathbb{S}} \circ \omega_S\right)\right](\square) = \left(\widetilde{\mathbb{S}} \circ \omega_S \circ \widetilde{\mathbb{S}}\right)(\square) \cap \left(\omega_S \circ \widetilde{\mathbb{S}} \circ \omega_S\right)(\square) = \left[\widetilde{\mathbb{S}}(\square) \cap \left(\omega_S \circ \widetilde{\mathbb{S}}\right)(\square)\right] \cap \left[\omega_S(\square) \cap \left(\widetilde{\mathbb{S}} \circ \omega_S\right)(\square)\right] = \omega_S(\square) \cap \omega_S(\square) = \omega_S(\square) \subseteq \omega_S(\square) \tag{16}$$

$$\left[\left(\widetilde{\mathbb{S}} \circ \omega_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\omega_S \circ \widetilde{\mathbb{S}} \circ \omega_S\right)\right](\square) = \left(\widetilde{\mathbb{S}} \circ \omega_S \circ \widetilde{\mathbb{S}}\right)(\square) \cap \left(\omega_S \circ \widetilde{\mathbb{S}} \circ \omega_S\right)(\square) = \left[\left[\widetilde{\mathbb{S}}(\square) \cap \left(\omega_S \circ \widetilde{\mathbb{S}}\right)(\square)\right] \cup \left[\widetilde{\mathbb{S}}(\square) \cap \left(\omega_S \circ \widetilde{\mathbb{S}}\right)(\text{Ꜧ})\right] \cup \left[\widetilde{\mathbb{S}}(\square) \cap \left(\omega_S \circ \widetilde{\mathbb{S}}\right)(\square)\right] \cup \left[\widetilde{\mathbb{S}}(\square) \cap \left(\omega_S \circ \widetilde{\mathbb{S}}\right)(\square)\right] \cup \left[\widetilde{\mathbb{S}}(\square) \cap \left(\omega_S \circ \widetilde{\mathbb{S}}\right)(\text{Ꜧ})\right] \cup \left[\widetilde{\mathbb{S}}(\text{Ꜧ}) \cap \left(\omega_S \circ \widetilde{\mathbb{S}}\right)(\square)\right] \cup \left[\widetilde{\mathbb{S}}(\text{Ꜧ}) \cap \left(\omega_S \circ \widetilde{\mathbb{S}}\right)(\square)\right] \cup \left[\widetilde{\mathbb{S}}(\text{Ꜧ}) \cap \left(\omega_S \circ \widetilde{\mathbb{S}}\right)(\text{Ꜧ})\right]\right] \cap \left[\left[\omega_S(\square) \cap \left(\widetilde{\mathbb{S}} \circ \omega_S\right)(\square)\right] \cup \left[\omega_S(\square) \cap \left(\widetilde{\mathbb{S}} \circ \omega_S\right)(\text{Ꜧ})\right] \cup \left[\omega_S(\square) \cap \left(\widetilde{\mathbb{S}} \circ \omega_S\right)(\square)\right] \cup \left[\omega_S(\square) \cap \left(\widetilde{\mathbb{S}} \circ \omega_S\right)(\square)\right] \cup \left[\omega_S(\square) \cap \left(\widetilde{\mathbb{S}} \circ \omega_S\right)(\text{Ꜧ})\right] \cup \left[\omega_S(\text{Ꜧ}) \cap \left(\widetilde{\mathbb{S}} \circ \omega_S\right)(\square)\right] \cup \left[\omega_S(\text{Ꜧ}) \cap \left(\widetilde{\mathbb{S}} \circ \omega_S\right)(\square)\right] \cup \left[\omega_S(\text{Ꜧ}) \cap \left(\widetilde{\mathbb{S}} \circ \omega_S\right)(\text{Ꜧ})\right]\right] = [\omega_S(\square) \cup \omega_S(\square) \cup \omega_S(\text{Ꜧ})] \cap [\omega_S(\square) \cup \omega_S(\square) \cup \omega_S(\text{Ꜧ})] = \omega_S(\square) \cup \omega_S(\square) \cup \omega_S(\text{Ꜧ}) \subseteq \omega_S(\square) \tag{17}$$

$$\left[\left(\widetilde{\mathbb{S}} \circ \omega_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\omega_S \circ \widetilde{\mathbb{S}} \circ \omega_S\right)\right](\text{Ꜧ}) = \emptyset \subseteq \omega_S(\text{Ꜧ}) \tag{18}$$

$$\left[\left(\widetilde{\mathbb{S}} \circ \omega_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\omega_S \circ \widetilde{\mathbb{S}} \circ \omega_S\right)\right](\text{₾}) = \left(\widetilde{\mathbb{S}} \circ \omega_S \circ \widetilde{\mathbb{S}}\right)(\text{₾}) \cap \left(\omega_S \circ \widetilde{\mathbb{S}} \circ \omega_S\right)(\text{₾}) = \left[\left[\widetilde{\mathbb{S}}(\square) \cap \left(\omega_S \circ \widetilde{\mathbb{S}}\right)(\text{₾})\right] \cup \left[\widetilde{\mathbb{S}}(\square) \cap \left(\omega_S \circ \widetilde{\mathbb{S}}\right)(\text{₾})\right] \cup \left[\widetilde{\mathbb{S}}(\text{Ꜧ}) \cap \left(\omega_S \circ \widetilde{\mathbb{S}}\right)(\text{₾})\right] \cup \left[\widetilde{\mathbb{S}}(\text{₾}) \cap \left(\omega_S \circ \widetilde{\mathbb{S}}\right)(\square)\right] \cup \left[\widetilde{\mathbb{S}}(\text{₾}) \cap \left(\omega_S \circ \widetilde{\mathbb{S}}\right)(\square)\right] \cup \left[\widetilde{\mathbb{S}}(\text{₾}) \cap \left(\omega_S \circ \widetilde{\mathbb{S}}\right)(\text{Ꜧ})\right] \cup \left[\widetilde{\mathbb{S}}(\text{₾}) \cap \left(\omega_S \circ \widetilde{\mathbb{S}}\right)(\text{₾})\right]\right] \cap \left[\left[\omega_S(\square) \cap \left(\widetilde{\mathbb{S}} \circ \omega_S\right)(\text{₾})\right] \cup \left[\omega_S(\square) \cap \left(\widetilde{\mathbb{S}} \circ \omega_S\right)(\text{₾})\right] \cup \left[\omega_S(\text{Ꜧ}) \cap \left(\widetilde{\mathbb{S}} \circ \omega_S\right)(\text{₾})\right] \cup \left[\omega_S(\text{₾}) \cap \left(\widetilde{\mathbb{S}} \circ \omega_S\right)(\square)\right] \cup \left[\omega_S(\text{₾}) \cap \left(\widetilde{\mathbb{S}} \circ \omega_S\right)(\square)\right] \cup \left[\omega_S(\text{₾}) \cap \left(\widetilde{\mathbb{S}} \circ \omega_S\right)(\text{Ꜧ})\right] \cup \left[\omega_S(\text{₾}) \cap \left(\widetilde{\mathbb{S}} \circ \omega_S\right)(\text{₾})\right]\right] = [\omega_S(\square) \cup \omega_S(\square) \cup \omega_S(\text{Ꜧ}) \cup \omega_S(\text{₾})] \cap [\omega_S(\square) \cup \omega_S(\square) \cup \omega_S(\text{Ꜧ}) \cup \omega_S(\text{₾})] = \omega_S(\square) \cup \omega_S(\square) \cup \omega_S(\text{Ꜧ}) \cup \omega_S(\text{₾}) \subseteq \omega_S(\text{₾}) \tag{19}$$

Thus, ω_S is an $\mathbb{S}$-int $\mathbb{BI}$-id. However, since

$$\left[\left(\widetilde{\mathbb{S}} \circ \text{ɥ}_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\text{ɥ}_S \circ \widetilde{\mathbb{S}} \circ \text{ɥ}_S\right)\right](\text{₾}) = \left(\widetilde{\mathbb{S}} \circ \text{ɥ}_S \circ \widetilde{\mathbb{S}}\right)(\text{₾}) \cap \left(\text{ɥ}_S \circ \widetilde{\mathbb{S}} \circ \text{ɥ}_S\right)(\text{₾}) = [\text{ɥ}_S(\square) \cup \text{ɥ}_S(\square) \cup \text{ɥ}_S(\text{Ꜧ}) \cup \text{ɥ}_S(\text{₾})] \cap [\text{ɥ}_S(\square) \cup \text{ɥ}_S(\square) \cup \text{ɥ}_S(\text{Ꜧ}) \cup \text{ɥ}_S(\text{₾})] = \text{ɥ}_S(\square) \cup \text{ɥ}_S(\square) \cup \text{ɥ}_S(\text{Ꜧ}) \cup \text{ɥ}_S(\text{₾}) \nsubseteq \text{ɥ}_S(\text{₾}) \tag{20}$$

ɥ_S is not an $\mathbb{S}$-int $\mathbb{BI}$-id.

Corollary 3.3. $\widetilde{\mathbb{S}}$ and $\emptyset_S$ are $\mathbb{S}$-int $\mathbb{BI}$-ids.

Now, we continue with the relationships between $\mathbb{S}$-int $\mathbb{BI}$-ids and other types of $\mathbb{S}$-int ids of S.

Theorem 3.4. Every $\mathbb{S}$-int $\mathbb{BI}$-id is an $\mathbb{S}$-int $\mathcal{SS}$ of a soft simple* semigroup.

Proof: Let $\mathfrak{f}_S$ be an $\mathbb{S}$-int $\mathbb{BI}$-id of a soft simple* S. Then, $\left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right) \tilde{\subseteq} \mathfrak{f}_S$ and $\widetilde{\mathbb{S}} = \widetilde{\mathbb{S}} \circ \mathfrak{f}_S = \mathfrak{f}_S \circ \widetilde{\mathbb{S}}$. Thus,

$$\mathfrak{f}_S \circ \mathfrak{f}_S = (\mathfrak{f}_S \circ \mathfrak{f}_S) \tilde{\cap} (\mathfrak{f}_S \circ \mathfrak{f}_S) \tilde{\subseteq} \left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right) \tilde{\cap} \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) = \left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \mathfrak{f}_S\right) \tilde{\cap} \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right) \tilde{\subseteq} \left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right) \tilde{\subseteq} \mathfrak{f}_S \tag{21}$$

Hence, $\mathfrak{f}_S$ is an $\mathbb{S}$-int $\mathcal{SS}$.

Theorem 3.5. Every $\mathbb{S}$-int $\mathcal{L}$-id is an $\mathbb{S}$-int $\mathbb{BI}$-id.

Proof: Let $\mathfrak{f}_S$ be an $\mathbb{S}$-int $\mathcal{L}$-id. Then, $\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \tilde{\subseteq} \mathfrak{f}_S$ and $\mathfrak{f}_S \circ \mathfrak{f}_S \tilde{\subseteq} \mathfrak{f}_S$. Thus,

$$\left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right) \tilde{\subseteq} \mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S \tilde{\subseteq} \mathfrak{f}_S \circ \mathfrak{f}_S \tilde{\subseteq} \mathfrak{f}_S \tag{22}$$

Hence, $\mathfrak{f}_S$ is an $\mathbb{S}$-int $\mathbb{BI}$-id.

We show with a counterexample that the converse of Theorem 3.5 does not hold:

Example 3.6. Consider the semigroup $S = \{\text{ꝺ}, \text{Ꝟ}\}$ defined by the following Table 3.:

Table 3: Cayley table of '◎' binary operation.

◎	ꝺ	ᵿ
ꝺ	ꝺ	ꝺ
ᵿ	ᵿ	ᵿ

Let v_S be SS over $U=\mathbb{Z}^+$ as follows:

$$v_S=\{(ꝺ,\{4\}),(ᵿ,\{1,1999\})\} \tag{23}$$

Then, v_S is an S-int ɃI-id. In fact;

$$\left[\left(\widetilde{\mathbb{S}}\circ v_S\circ\widetilde{\mathbb{S}}\right)\tilde{\cap}\left(v_S\circ\widetilde{\mathbb{S}}\circ v_S\right)\right](ꝺ)=\left(\widetilde{\mathbb{S}}\circ v_S\circ\widetilde{\mathbb{S}}\right)(ꝺ)\cap\left(v_S\circ\widetilde{\mathbb{S}}\circ v_S\right)(ꝺ)=\left(v_S(ꝺ)\cup v_S(ᵿ)\right)\cap v_S(ꝺ)=v_S(ꝺ)\subseteq v_S(ꝺ) \tag{24}$$

$$\left[\left(\widetilde{\mathbb{S}}\circ v_S\circ\widetilde{\mathbb{S}}\right)\tilde{\cap}\left(v_S\circ\widetilde{\mathbb{S}}\circ v_S\right)\right](ᵿ)=\left(\widetilde{\mathbb{S}}\circ v_S\circ\widetilde{\mathbb{S}}\right)(ᵿ)\cap\left(v_S\circ\widetilde{\mathbb{S}}\circ v_S\right)(ᵿ)=\left(v_S(ꝺ)\cup v_S(ᵿ)\right)\cap v_S(ᵿ)=v_S(ᵿ)\subseteq v_S(ᵿ) \tag{25}$$

Thus, v_S is an S-int ɃI-id. However, since

$$v_S(ꝺᵿ)=v_S(ꝺ)\not\supseteq v_S(ᵿ) \tag{26}$$

v_S is not an S-int $\mathcal{L}$-id.

Theorem 3.7 illustrates that the converse of Theorem 3.5 holds for the soft simple* semigroups.

Theorem 3.7. Let $ȹ_S\in S_S(U)$ and S be a soft simple*. Then, the following conditions are equivalent:

i. $ȹ_S$ is an S-int $\mathcal{L}$-id.
ii. $ȹ_S$ is an S-int ɃI-id.

Proof: (i.) implies (ii.) is obvious by Theorem 3.5. Assume that $ȹ_S$ is an S-int ɃI-id. By assumption, $\widetilde{\mathbb{S}}=\widetilde{\mathbb{S}}\circ ȹ_S=ȹ_S\circ\widetilde{\mathbb{S}}$. Thus,

$$\widetilde{\mathbb{S}}\circ ȹ_S=\left(\widetilde{\mathbb{S}}\circ ȹ_S\right)\tilde{\cap}\left(\widetilde{\mathbb{S}}\circ ȹ_S\right)=\left(\widetilde{\mathbb{S}}\circ ȹ_S\circ ȹ_S\right)\tilde{\cap}\left(ȹ_S\circ\widetilde{\mathbb{S}}\circ ȹ_S\right)\tilde{\subseteq}\left(\widetilde{\mathbb{S}}\circ ȹ_S\circ\widetilde{\mathbb{S}}\right)\tilde{\cap}\left(ȹ_S\circ\widetilde{\mathbb{S}}\circ ȹ_S\right)\tilde{\subseteq} ȹ_S \tag{27}$$

Hence, $ȹ_S$ is an S-int $\mathcal{L}$-id.

Theorem 3.8. Every S-int $\mathcal{R}$-id is an S-int ɃI-id.

Proof: Let $ʄ_S$ be an S-int $\mathcal{R}$-id. Then, $ʄ_S\circ\widetilde{\mathbb{S}}\tilde{\subseteq}ʄ_S$ and $ʄ_S\circ ʄ_S\tilde{\subseteq}ʄ_S$. Thus,

$$\left(\widetilde{\mathbb{S}}\circ ʄ_S\circ\widetilde{\mathbb{S}}\right)\tilde{\cap}\left(ʄ_S\circ\widetilde{\mathbb{S}}\circ ʄ_S\right)\tilde{\subseteq}ʄ_S\circ\widetilde{\mathbb{S}}\circ ʄ_S\tilde{\subseteq}ʄ_S\circ ʄ_S\tilde{\subseteq}ʄ_S \tag{28}$$

Hence, $ʄ_S$ is an S-int ɃI-id.

We show with a counterexample that the converse of Theorem 3.8 does not hold:

Example 3.9. Let $S=\{ɟ,𝑖\}$ be:

Table 4: Cayley table of '⬡' binary operation.

⬡	ɟ	𝑖
ɟ	ɟ	𝑖
𝑖	ɟ	𝑖

Let ϱ_S be SS over $U=\mathbb{Z}^-$ as follows:

$$\varrho_S=\{(ɟ,\{-1\}),(𝑖,\{-2\})\} \tag{29}$$

Then, ϱ_S is an S-int ɃI-id. In fact;

$$\left[\left(\widetilde{\mathbb{S}}\circ\varrho_S\circ\widetilde{\mathbb{S}}\right)\tilde{\cap}\left(\varrho_S\circ\widetilde{\mathbb{S}}\circ\varrho_S\right)\right](ɟ)=\left(\widetilde{\mathbb{S}}\circ\varrho_S\circ\widetilde{\mathbb{S}}\right)(ɟ)\cap\left(\varrho_S\circ\widetilde{\mathbb{S}}\circ\varrho_S\right)(ɟ)=\left(\varrho_S(ɟ)\cup\varrho_S(𝑖)\right)\cap\varrho_S(ɟ)=\varrho_S(ɟ)\subseteq\varrho_S(ɟ) \tag{30}$$

$$\left[\left(\widetilde{\mathbb{S}}\circ\varrho_S\circ\widetilde{\mathbb{S}}\right)\tilde{\cap}\left(\varrho_S\circ\widetilde{\mathbb{S}}\circ\varrho_S\right)\right](𝑖)=\left(\widetilde{\mathbb{S}}\circ\varrho_S\circ\widetilde{\mathbb{S}}\right)(𝑖)\cap\left(\varrho_S\circ\widetilde{\mathbb{S}}\circ\varrho_S\right)(𝑖)=\left(\varrho_S(ɟ)\cup\varrho_S(𝑖)\right)\cap\varrho_S(𝑖)=\varrho_S(𝑖)\subseteq\varrho_S(𝑖) \tag{31}$$

Thus, ϱ_S is an S-int ɃI-id. However, since

$$\varrho_S(ɟ𝑖)=\varrho_S(𝑖)\not\supseteq\varrho_S(ɟ) \tag{29}$$

ϱ_S is not an S-int $\mathcal{R}$-id.

Theorem 3.10 shows that the converse of Theorem 3.8 holds for the soft simple* semigroups.

Theorem 3.10. Let $ȹ_S\in S_S(U)$ and S be a soft simple*. Then, the following conditions are equivalent:

i. $ȹ_S$ is an S-int $\mathcal{R}$-id.
ii. $ȹ_S$ is an S-int ɃI-id.

Proof: (i.) implies (ii.) is obvious by Theorem 3.8. Assume that $ȹ_S$ is an S-int ɃI-id. By assumption, $\widetilde{\mathbb{S}}=\widetilde{\mathbb{S}}\circ ȹ_S=ȹ_S\circ\widetilde{\mathbb{S}}$. Thus,

$$ȹ_S\circ\widetilde{\mathbb{S}}=\left(ȹ_S\circ\widetilde{\mathbb{S}}\right)\tilde{\cap}\left(ȹ_S\circ\widetilde{\mathbb{S}}\right)=\left(ȹ_S\circ ȹ_S\circ\widetilde{\mathbb{S}}\right)\tilde{\cap}\left(ȹ_S\circ\widetilde{\mathbb{S}}\circ ȹ_S\right)\tilde{\subseteq}\left(\widetilde{\mathbb{S}}\circ ȹ_S\circ\widetilde{\mathbb{S}}\right)\tilde{\cap}\left(ȹ_S\circ\widetilde{\mathbb{S}}\circ ȹ_S\right)\tilde{\subseteq} ȹ_S \tag{29}$$

Hence, $ȹ_S$ is an S-int $\mathcal{R}$-id.

Theorem 3.11. Every S-int id is an S-int ɃI-id.

Proof: It follows by Theorem 3.5 and Theorem 3.8.

Note that the converse of Theorem 3.11 is not true follows from Example 3.6 and Example 3.9. Theorem 3.12 shows that the converse of Theorem 3.11 holds for soft simple* semigroups.

Theorem 3.12. Let $f_S \in S_S(U)$ and S be a soft simple*. Then, the following conditions are equivalent:

i. f_S is an $\mathbb{S}$-int id.
ii. f_S is an $\mathbb{S}$-int $\mathbb{BI}$-id.

Proof: It follows by Theorem 3.7 and Theorem 3.10.

Theorem 3.13. Every $\mathbb{S}$-int $\mathcal{B}$-id is an $\mathbb{S}$-int $\mathbb{BI}$-id.

Proof: Let $\mathfrak{f}_S$ be an $\mathbb{S}$-int $\mathcal{B}$-id. Then, $\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S \mathrel{\tilde{\subseteq}} \mathfrak{f}_S$. Thus,

$$\left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right) \mathrel{\tilde{\subseteq}} \mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S \mathrel{\tilde{\subseteq}} \mathfrak{f}_S \tag{29}$$

Hence, $\mathfrak{f}_S$ is an $\mathbb{S}$-int $\mathbb{BI}$-id.

We show with a counterexample that the converse of Theorem 3.13 does not hold:

Example 3.14. Let $S = \{\mathfrak{z}, \sigma, u, r\}$ be:

Table5: Cayley table of '◎' binary operation.

◎	$\mathfrak{z}$	σ	u	r
$\mathfrak{z}$	$\mathfrak{z}$	$\mathfrak{z}$	$\mathfrak{z}$	$\mathfrak{z}$
σ	$\mathfrak{z}$	$\mathfrak{z}$	$\mathfrak{z}$	$\mathfrak{z}$
u	$\mathfrak{z}$	$\mathfrak{z}$	σ	$\mathfrak{z}$
r	$\mathfrak{z}$	$\mathfrak{z}$	σ	σ

Suppose that $\varkappa_S$ is $\mathbb{SS}$ over $U = S_3$ as follows:

$$\varkappa_S = \left\{\begin{matrix} (\mathfrak{z}, \{(1), (12), (123), (132)\}), \\ (\sigma, \{(123)\}), (u, \{(1), (132)\}), (r, \{(12)\}) \end{matrix}\right\} \tag{30}$$

Then, $\varkappa_S$ is an $\mathbb{S}$-int $\mathbb{BI}$-id. In fact;

$$\left[\left(\widetilde{\mathbb{S}} \circ \varkappa_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\varkappa_S \circ \widetilde{\mathbb{S}} \circ \varkappa_S\right)\right](\mathfrak{z}) = \{(1), (12), (123), (132)\} \subseteq \varkappa_S(\mathfrak{z}) \tag{31}$$

$$\left[\left(\widetilde{\mathbb{S}} \circ \varkappa_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\varkappa_S \circ \widetilde{\mathbb{S}} \circ \varkappa_S\right)\right](\sigma) = \emptyset \subseteq \varkappa_S(\sigma) \tag{32}$$

$$\left[\left(\widetilde{\mathbb{S}} \circ \varkappa_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\varkappa_S \circ \widetilde{\mathbb{S}} \circ \varkappa_S\right)\right](u) = \emptyset \subseteq \varkappa_S(u) \tag{33}$$

$$\left[\left(\widetilde{\mathbb{S}} \circ \varkappa_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\varkappa_S \circ \widetilde{\mathbb{S}} \circ \varkappa_S\right)\right](r) = \emptyset \subseteq \varkappa_S(r) \tag{34}$$

Thus, $\varkappa_S$ is an $\mathbb{S}$-int $\mathbb{BI}$-id. However, since

$$\varkappa_S(uu) = \varkappa_S(\sigma) \not\supseteq \varkappa_S(u) \cap \varkappa_S(u) \tag{35}$$

$\varkappa_S$ is not an $\mathbb{S}$-int $\mathcal{SS}$. Hence, $\varkappa_S$ is not an $\mathbb{S}$-int $\mathcal{B}$-id.

Theorem 3.15 shows that the converse of Theorem 3.13 holds for the soft simple* semigroups.

Theorem 3.15. Let $\mathfrak{p}_S \in S_S(U)$ and S be a soft simple*. Then, the following conditions are equivalent:

i. $\mathfrak{p}_S$ is an $\mathbb{S}$-int $\mathcal{B}$-id.
ii. $\mathfrak{p}_S$ is an $\mathbb{S}$-int $\mathbb{BI}$-id.

Proof: (i.) implies (ii.) is obvious by Theorem 3.13. Assume that $\mathfrak{p}_S$ is an $\mathbb{S}$-int $\mathbb{BI}$-id. Then, by Theorem 3.4, $\mathfrak{p}_S$ is an $\mathbb{S}$-int $\mathcal{SS}$ and by assumption, $\widetilde{\mathbb{S}} = \widetilde{\mathbb{S}} \circ \mathfrak{p}_S = \mathfrak{p}_S \circ \widetilde{\mathbb{S}}$. Thus,

$$\mathfrak{p}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{p}_S = \left(\mathfrak{p}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{p}_S\right) \tilde{\cap} \left(\mathfrak{p}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{p}_S\right) = \left(\widetilde{\mathbb{S}} \circ \mathfrak{p}_S \circ \mathfrak{p}_S\right) \tilde{\cap} \left(\mathfrak{p}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{p}_S\right) \mathrel{\tilde{\subseteq}} \left(\widetilde{\mathbb{S}} \circ \mathfrak{p}_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\mathfrak{p}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{p}_S\right) \mathrel{\tilde{\subseteq}} \mathfrak{p}_S \tag{36}$$

Hence, $\mathfrak{p}_S$ is an $\mathbb{S}$-int $\mathcal{B}$-id.

Theorem 3.16. Every $\mathbb{S}$-int $\mathcal{J}$-id is an $\mathbb{S}$-int $\mathbb{BI}$-id.

Proof: Let $\mathfrak{f}_S$ be an $\mathbb{S}$-int $\mathcal{J}$-id. Then, $\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}} \mathrel{\tilde{\subseteq}} \mathfrak{f}_S$. Thus,

$$\left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right) \mathrel{\tilde{\subseteq}} \widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}} \mathrel{\tilde{\subseteq}} \mathfrak{f}_S \tag{37}$$

Hence, $\mathfrak{f}_S$ is an $\mathbb{S}$-int $\mathbb{BI}$-id.

We show with a counterexample that the converse of Theorem 3.16 does not hold:

Example 3.17. Let the $\mathbb{SS}$ ϱ_S in Example 3.9. It was shown in Example 3.9 that ϱ_S is an $\mathbb{S}$-int $\mathbb{BI}$-id. Since

$$\varrho_S(i\jmath i) = \varrho_S(i) \not\supseteq \varrho_S(\jmath) \tag{38}$$

ϱ_S is not an $\mathbb{S}$-int $\mathcal{J}$-id.

Theorem 3.18 shows that the converse of Theorem 3.16 holds for the soft simple* semigroups.

Theorem 3.18. Let $\mathfrak{p}_S \in S_S(U)$ and S be a soft simple*. Then, the following conditions are equivalent:

i. $\mathfrak{p}_S$ is an $\mathbb{S}$-int $\mathcal{J}$-id.
ii. $\mathfrak{p}_S$ is an $\mathbb{S}$-int $\mathbb{BI}$-id.

Proof: (i.) implies (ii.) is obvious by Theorem 3.16. Assume that $\mathfrak{p}_S$ is an $\mathbb{S}$-int $\mathbb{BI}$-id. By assumption, $\widetilde{\mathbb{S}} = \widetilde{\mathbb{S}} \circ \mathfrak{p}_S = \mathfrak{p}_S \circ \widetilde{\mathbb{S}}$. Thus,

$$\widetilde{\mathbb{S}}\circ\mathfrak{p}_S\circ\widetilde{\mathbb{S}}=\left(\widetilde{\mathbb{S}}\circ\mathfrak{p}_S\circ\widetilde{\mathbb{S}}\right)\widetilde{\cap}\left(\widetilde{\mathbb{S}}\circ\mathfrak{p}_S\circ\widetilde{\mathbb{S}}\right)=\left(\widetilde{\mathbb{S}}\circ\mathfrak{p}_S\circ\widetilde{\mathbb{S}}\right)\widetilde{\cap}\left(\mathfrak{p}_S\circ\widetilde{\mathbb{S}}\circ\widetilde{\mathbb{S}}\right)\widetilde{\subseteq}\left(\widetilde{\mathbb{S}}\circ\mathfrak{p}_S\circ\widetilde{\mathbb{S}}\right)\widetilde{\cap}\left(\mathfrak{p}_S\circ\widetilde{\mathbb{S}}\right)=\left(\widetilde{\mathbb{S}}\circ\mathfrak{p}_S\circ\widetilde{\mathbb{S}}\right)\widetilde{\cap}\left(\mathfrak{p}_S\circ\widetilde{\mathbb{S}}\circ\mathfrak{p}_S\right)\widetilde{\subseteq}\mathfrak{p}_S \quad (39)$$

Hence, $\mathfrak{p}_S$ is an $\mathbb{S}$-int $\mathcal{J}$-id.

Theorem 3.19. Every $\mathbb{S}$-int $\mathcal{Q}$- id is an $\mathbb{S}$-int BI-id.

Proof: Let f_S be an $\mathbb{S}$-int $\mathcal{Q}$-id Then, f_S is an $\mathbb{S}$-int $\mathcal{B}$-id by Theorem 2.14 (3). The rest is obvious by Theorem 3.13.

We show with a counterexample that the converse of Theorem 3.19 does not hold:

Example 3.20. Let the SS $\varkappa_S$ in Example 3.14. It was shown in Example 3.14 that $\varkappa_S$ is an $\mathbb{S}$-int BI-id, however, it is not an $\mathbb{S}$-int $\mathcal{B}$-id. Since $\varkappa_S$ is not an $\mathbb{S}$-int $\mathcal{B}$-id, thus $\varkappa_S$ is not an $\mathbb{S}$-int $\mathcal{Q}$-id.

Theorem 3.21 demonstrates that the converse of Theorem 3.19 holds for the soft simple* semigroups.

Theorem 3.21. Let $\mathfrak{p}_S\in S_S(U)$ and S be a soft simple*. Then, the following conditions are equivalent:

i. $\mathfrak{p}_S$ is an $\mathbb{S}$-int $\mathcal{Q}$-id.
ii. $\mathfrak{p}_S$ is an $\mathbb{S}$-int BI-id.

Proof: (1) implies (ii.) is obvious by Theorem 3.19. Assume that $\mathfrak{p}_S$ is an $\mathbb{S}$-int BI-id. By assumption, $\widetilde{\mathbb{S}}=\widetilde{\mathbb{S}}\circ\mathfrak{p}_S=\mathfrak{p}_S\circ\widetilde{\mathbb{S}}$. Thus,

$$\left(\widetilde{\mathbb{S}}\circ\mathfrak{p}_S\right)\widetilde{\cap}\left(\mathfrak{p}_S\circ\widetilde{\mathbb{S}}\right)=\left(\widetilde{\mathbb{S}}\circ\mathfrak{p}_S\circ\mathfrak{p}_S\right)\widetilde{\cap}\left(\mathfrak{p}_S\circ\widetilde{\mathbb{S}}\circ\mathfrak{p}_S\right)\widetilde{\subseteq}\left(\widetilde{\mathbb{S}}\circ\mathfrak{p}_S\circ\widetilde{\mathbb{S}}\right)\widetilde{\cap}\left(\mathfrak{p}_S\circ\widetilde{\mathbb{S}}\circ\mathfrak{p}_S\right)\widetilde{\subseteq}\mathfrak{p}_S \quad (40)$$

Hence, $\mathfrak{p}_S$ is an $\mathbb{S}$-int $\mathcal{Q}$-id.

Theorem 3.22. Let $\emptyset\neq \mathsf{M}\subseteq S$. Then, M is a BI-id iff S_{M} is an $\mathbb{S}$-int BI-id.

Proof: Let M be a BI-id. Then, $S\mathsf{M}S\cap \mathsf{M}S\mathsf{M}\subseteq \mathsf{M}$. By Theorem 2.9,

$$\left(\widetilde{\mathbb{S}}\circ S_{\mathsf{M}}\circ\widetilde{\mathbb{S}}\right)\widetilde{\cap}\left(S_{\mathsf{M}}\circ\widetilde{\mathbb{S}}\circ S_{\mathsf{M}}\right)=(S_S\circ S_{\mathsf{M}}\circ S_S)\,\widetilde{\cap}\,(S_{\mathsf{M}}\circ S_S\circ S_{\mathsf{M}})=S_{S\mathsf{M}S}\,\widetilde{\cap}\,S_{\mathsf{M}S\mathsf{M}}=S_{S\mathsf{M}S\cap\mathsf{M}S\mathsf{M}}\,\widetilde{\subseteq}\,S_{\mathsf{M}} \quad (41)$$

Hence, S_{M} is an $\mathbb{S}$-int BI-id.

Conversely, let S_{M} be an $\mathbb{S}$-int BI-id. Then, $\left(\widetilde{\mathbb{S}}\circ S_{\mathsf{M}}\circ\widetilde{\mathbb{S}}\right)\widetilde{\cap}\left(S_{\mathsf{M}}\circ\widetilde{\mathbb{S}}\circ S_{\mathsf{M}}\right)\widetilde{\subseteq}S_{\mathsf{M}}$. Let $w\in S\mathsf{M}S\cap \mathsf{M}S\mathsf{M}$. Then,

$$S_{\mathsf{M}}(w)\supseteq\left[\left(\widetilde{\mathbb{S}}\circ S_{\mathsf{M}}\circ\widetilde{\mathbb{S}}\right)\widetilde{\cap}\left(S_{\mathsf{M}}\circ\widetilde{\mathbb{S}}\circ S_{\mathsf{M}}\right)\right](w)=\left(\widetilde{\mathbb{S}}\circ S_{\mathsf{M}}\circ\widetilde{\mathbb{S}}\right)(w)\cap\left(S_{\mathsf{M}}\circ\widetilde{\mathbb{S}}\circ S_{\mathsf{M}}\right)(w)$$
$$=(S_S\circ S_{\mathsf{M}}\circ S_S)(w)\cap(S_{\mathsf{M}}\circ S_S\circ S_{\mathsf{M}})(w)=S_{S\mathsf{M}S}(w)\cap S_{\mathsf{M}S\mathsf{M}}(w)=S_{S\mathsf{M}S\cap\mathsf{M}S\mathsf{M}}(w)=U \quad (42)$$

Thus, $S_{\mathsf{M}}(w)=U$ and so $w\in\mathsf{M}$, implying that $S\mathsf{M}S\cap \mathsf{M}S\mathsf{M}\subseteq\mathsf{M}$. Therefore, M is a BI-id.

Example 3.23. Let $S=\{$Ձ, Ճ, Ք$\}$ be:

Table 6: Cayley table of '⋄' binary operation.

⋄	Ձ	Ճ	Ք
Ձ	Ք	Ք	Ք
Ճ	Ք	Ք	Ք
Ք	Ք	Ք	Ք

One can show that $A=\{$Ճ, Ք$\}$ is a BI-id. By the definition of the soft characteristic function, $S_A=\{($Ձ$,\emptyset),($Ճ$,U),($Ք$,U)\}$. Then, S_A is an $\mathbb{S}$-int BI-id. Conversely, by choosing the $\mathbb{S}$-int BI-id as $f_S=\{($Ձ$,U),($Ճ$,\emptyset),($Ք$,U)\}$, which is the soft characteristic function of $I=\{$Ձ, Ք$\}$, one can show that I is a BI-ideal.

Theorem 3.24. Let $f_S,\eta_S\in S_S(U)$. If f_S or η_S is an $\mathbb{S}$-int $\mathcal{L}$-id, then $f_S\circ\eta_S$ is an $\mathbb{S}$-int BI-id.

Proof: Let f_S be an $\mathbb{S}$-int $\mathcal{L}$-id. Then, $\widetilde{\mathbb{S}}\circ f_S\,\widetilde{\subseteq}\,f_S$ and $f_S\circ f_S\,\widetilde{\subseteq}\,f_S$. Thus,

$$\left(\widetilde{\mathbb{S}}\circ(f_S\circ\eta_S)\circ\widetilde{\mathbb{S}}\right)\widetilde{\cap}\left((f_S\circ\eta_S)\circ\widetilde{\mathbb{S}}\circ(f_S\circ\eta_S)\right)\widetilde{\subseteq}(f_S\circ\eta_S)\circ\widetilde{\mathbb{S}}\circ(f_S\circ\eta_S)=f_S\circ\eta_S\circ\left(\widetilde{\mathbb{S}}\circ f_S\right)\circ\eta_S$$
$$\widetilde{\subseteq}f_S\circ\eta_S\circ f_S\circ\eta_S\,\widetilde{\subseteq}\,f_S\circ\left(\widetilde{\mathbb{S}}\circ f_S\right)\circ\eta_S\,\widetilde{\subseteq}\,f_S\circ f_S\circ\eta_S\,\widetilde{\subseteq}\,f_S\circ\eta_S \quad (43)$$

Hence, $f_S\circ\eta_S$ is an $\mathbb{S}$-int BI-id. Also, the proof can be given similarly for η_S. Let η_S be an $\mathbb{S}$-int $\mathcal{L}$-id. Then, $\widetilde{\mathbb{S}}\circ\eta_S\,\widetilde{\subseteq}\,\eta_S$ and $\eta_S\circ\eta_S\,\widetilde{\subseteq}\,\eta_S$. Thus,

$$\left(\widetilde{\mathbb{S}}\circ(f_S\circ\eta_S)\circ\widetilde{\mathbb{S}}\right)\widetilde{\cap}\left((f_S\circ\eta_S)\circ\widetilde{\mathbb{S}}\circ(f_S\circ\eta_S)\right)\widetilde{\subseteq}(f_S\circ\eta_S)\circ\widetilde{\mathbb{S}}\circ(f_S\circ\eta_S)=f_S\circ\eta_S\circ\left(\widetilde{\mathbb{S}}\circ f_S\right)\circ\eta_S$$
$$\widetilde{\subseteq}f_S\circ\left(\widetilde{\mathbb{S}}\circ\widetilde{\mathbb{S}}\right)\circ\widetilde{\mathbb{S}}\circ\eta_S\,\widetilde{\subseteq}\,f_S\circ\left(\widetilde{\mathbb{S}}\circ\widetilde{\mathbb{S}}\right)\circ\eta_S\,\widetilde{\subseteq}\,f_S\circ\left(\widetilde{\mathbb{S}}\circ\eta_S\right)\widetilde{\subseteq}\,f_S\circ\eta_S \quad (44)$$

Hence, $f_S\circ\eta_S$ is an $\mathbb{S}$-int BI-id.

Theorem 3.25. Let $f_S,\eta_S\in S_S(U)$. If f_S or η_S is an $\mathbb{S}$-int $\mathcal{R}$-id, then $f_S\circ\eta_S$ is an $\mathbb{S}$-int BI-id.

Proof: Let η_S be an $\mathbb{S}$-int $\mathcal{R}$-id. Then, $\eta_S \circ \widetilde{\mathbb{S}} \,\widetilde{\subseteq}\, \eta_S$ and $\eta_S \circ \eta_S \,\widetilde{\subseteq}\, \eta_S$. Thus,

$$\left(\widetilde{\mathbb{S}} \circ (f_S \circ \eta_S) \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left((f_S \circ \eta_S) \circ \widetilde{\mathbb{S}} \circ (f_S \circ \eta_S)\right) \widetilde{\subseteq} (f_S \circ \eta_S) \circ \widetilde{\mathbb{S}} \circ (f_S \circ \eta_S) = f_S \circ \eta_S \circ \left(\widetilde{\mathbb{S}} \circ f_S\right) \circ \eta_S$$
$$\widetilde{\subseteq} f_S \circ \eta_S \circ f_S \circ \eta_S \widetilde{\subseteq} f_S \circ \left(\eta_S \circ \widetilde{\mathbb{S}}\right) \circ \eta_S \widetilde{\subseteq} f_S \circ \eta_S \circ \eta_S \widetilde{\subseteq} f_S \circ \eta_S \quad (45)$$

Hence, $f_S \circ \eta_S$ is an $\mathbb{S}$-int $\mathbb{BI}$-id. Also, the proof can be given similarly for f_S. Let f_S be an $\mathbb{S}$-int $\mathcal{R}$-id. Then, $f_S \circ \widetilde{\mathbb{S}} \,\widetilde{\subseteq}\, f_S$ and $f_S \circ f_S \,\widetilde{\subseteq}\, f_S$. Thus,

$$\left(\widetilde{\mathbb{S}} \circ (f_S \circ \eta_S) \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left((f_S \circ \eta_S) \circ \widetilde{\mathbb{S}} \circ (f_S \circ \eta_S)\right) \widetilde{\subseteq} (f_S \circ \eta_S) \circ \widetilde{\mathbb{S}} \circ (f_S \circ \eta_S)$$
$$\widetilde{\subseteq} f_S \circ \left(\widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}}\right) \circ \widetilde{\mathbb{S}} \circ \eta_S \widetilde{\subseteq} f_S \circ \left(\widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}}\right) \circ \eta_S \widetilde{\subseteq} \left(f_S \circ \widetilde{\mathbb{S}}\right) \circ \eta_S \widetilde{\subseteq} f_S \circ \eta_S \quad (46)$$

Hence, $f_S \circ \eta_S$ is an $\mathbb{S}$-int $\mathbb{BI}$-id.

Theorem 3.26. Let $f_S, \wp_S, h_S \in S_S(U)$. If $\wp_S$ is an $\mathbb{S}$-int id, then $f_S \circ \wp_S \circ h_S$ is an $\mathbb{S}$-int $\mathbb{BI}$-id.

Proof: Let $\wp_S$ be an $\mathbb{S}$-int id. Then, $\widetilde{\mathbb{S}} \circ \wp_S \,\widetilde{\subseteq}\, \wp_S$, $\wp_S \circ \widetilde{\mathbb{S}} \,\widetilde{\subseteq}\, \wp_S$ and $\wp_S \circ \wp_S \,\widetilde{\subseteq}\, \wp_S$. Thus,

$$\left(\widetilde{\mathbb{S}} \circ (f_S \circ \wp_S \circ h_S) \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left((f_S \circ \wp_S \circ h_S) \circ \widetilde{\mathbb{S}} \circ (f_S \circ \wp_S \circ h_S)\right) \widetilde{\subseteq} (f_S \circ \wp_S \circ h_S) \circ \widetilde{\mathbb{S}} \circ (f_S \circ \wp_S \circ h_S)$$
$$\widetilde{\subseteq} f_S \circ \left(\wp_S \circ \widetilde{\mathbb{S}}\right) \circ \left(\widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}}\right) \circ \wp_S \circ h_S \widetilde{\subseteq} f_S \circ \left(\wp_S \circ \widetilde{\mathbb{S}}\right) \circ \wp_S \circ h_S \widetilde{\subseteq} f_S \circ \wp_S \circ \wp_S \circ h_S \widetilde{\subseteq} f_S \circ \wp_S \circ h_S \quad (47)$$

Hence, $f_S \circ \wp_S \circ h_S$ is an $\mathbb{S}$-int $\mathbb{BI}$-id.

Theorem 3.27. Let f_S and η_S be $\mathbb{S}$-int $\mathbb{BI}$-ids. Then, $f_S \,\widetilde{\cap}\, \eta_S$ is an $\mathbb{S}$-int $\mathbb{BI}$-id.

Proof: Let f_S and η_S are $\mathbb{S}$-int $\mathbb{BI}$-ids. Then, $\left(\widetilde{\mathbb{S}} \circ f_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left(f_S \circ \widetilde{\mathbb{S}} \circ f_S\right) \widetilde{\subseteq} f_S$ and $\left(\widetilde{\mathbb{S}} \circ \eta_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right) \widetilde{\subseteq} \eta_S$. Thus,

$$\left(\widetilde{\mathbb{S}} \circ (f_S \widetilde{\cap} \eta_S) \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left((f_S \widetilde{\cap} \eta_S) \circ \widetilde{\mathbb{S}} \circ (f_S \widetilde{\cap} \eta_S)\right) \widetilde{\subseteq} \left(\widetilde{\mathbb{S}} \circ f_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left(f_S \circ \widetilde{\mathbb{S}} \circ f_S\right) \widetilde{\subseteq} f_S \quad (48)$$

and

$$\left(\widetilde{\mathbb{S}} \circ (f_S \widetilde{\cap} \eta_S) \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left((f_S \widetilde{\cap} \eta_S) \circ \widetilde{\mathbb{S}} \circ (f_S \widetilde{\cap} \eta_S)\right) \widetilde{\subseteq} \left(\widetilde{\mathbb{S}} \circ \eta_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right) \widetilde{\subseteq} \eta_S \quad (49)$$

Hence,

$$\left(\widetilde{\mathbb{S}} \circ (f_S \widetilde{\cap} \eta_S) \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left((f_S \widetilde{\cap} \eta_S) \circ \widetilde{\mathbb{S}} \circ (f_S \widetilde{\cap} \eta_S)\right) \widetilde{\subseteq} f_S \widetilde{\cap} \eta_S \quad (50)$$

Therefore, $f_S \,\widetilde{\cap}\, \eta_S$ is an $\mathbb{S}$-int $\mathbb{BI}$-id.

Corollary 3.28. The finite intersection of $\mathbb{S}$-int $\mathbb{BI}$-ids is an $\mathbb{S}$-int $\mathbb{BI}$-id.

Corollary 3.29. The intersection of $\mathbb{S}$-int $\mathcal{L}$ ($\mathcal{R}$/ two-sided/ $\mathcal{B}$/ $\mathcal{J}$/ $\mathcal{Q}$) id and $\mathbb{S}$-int $\mathcal{L}$ ($\mathcal{R}$/ two-sided/ $\mathcal{B}$/ $\mathcal{J}$/ $\mathcal{Q}$) id is an $\mathbb{S}$-int $\mathbb{BI}$-id.

Theorem 3.30. Let $\emptyset_S \neq f_S \in S_E(U)$. Then, every soft subset of f_S containing $\left(\widetilde{\mathbb{S}} \circ f_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left(f_S \circ \widetilde{\mathbb{S}} \circ f_S\right)$ is an $\mathbb{S}$-int $\mathbb{BI}$-id.

Proof: Let $\wp_S \,\widetilde{\subseteq}\, f_S$ containing $\left(\widetilde{\mathbb{S}} \circ f_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left(f_S \circ \widetilde{\mathbb{S}} \circ f_S\right)$. Since

$$\left(\widetilde{\mathbb{S}} \circ \wp_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left(\wp_S \circ \widetilde{\mathbb{S}} \circ \wp_S\right) \widetilde{\subseteq} \left(\widetilde{\mathbb{S}} \circ f_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left(f_S \circ \widetilde{\mathbb{S}} \circ f_S\right) \widetilde{\subseteq} \wp_S \quad (51)$$

Hence, $\wp_S$ is an $\mathbb{S}$-int $\mathbb{BI}$-id.

Theorem 3.31. Let $\emptyset_S \neq f_S \in S_S(U)$. Then, every soft subset of f_S containing $\left(\widetilde{\mathbb{S}} \circ f_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cup} \left(f_S \circ \widetilde{\mathbb{S}} \circ f_S\right)$ is an $\mathbb{S}$-int $\mathbb{BI}$-id.

Proof: Let $\wp_S \,\widetilde{\subseteq}\, f_S$ containing $\left(\widetilde{\mathbb{S}} \circ f_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cup} \left(f_S \circ \widetilde{\mathbb{S}} \circ f_S\right)$. Since

$$\left(\widetilde{\mathbb{S}} \circ \wp_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\subseteq} \left(\widetilde{\mathbb{S}} \circ f_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\subseteq} \left(\widetilde{\mathbb{S}} \circ f_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cup} \left(f_S \circ \widetilde{\mathbb{S}} \circ f_S\right) \widetilde{\subseteq} \wp_S \quad (52)$$

Thus, $\left(\widetilde{\mathbb{S}} \circ \wp_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\subseteq} \wp_S$. Furthermore, since

$$\left(\wp_S \circ \widetilde{\mathbb{S}} \circ \wp_S\right) \widetilde{\subseteq} \left(f_S \circ \widetilde{\mathbb{S}} \circ f_S\right) \widetilde{\subseteq} \left(\widetilde{\mathbb{S}} \circ f_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cup} \left(f_S \circ \widetilde{\mathbb{S}} \circ f_S\right) \widetilde{\subseteq} \wp_S \quad (53)$$

Thus, $\left(\wp_S \circ \widetilde{\mathbb{S}} \circ \wp_S\right) \widetilde{\subseteq} \wp_S$. Hence, $\left(\widetilde{\mathbb{S}} \circ \wp_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left(\wp_S \circ \widetilde{\mathbb{S}} \circ \wp_S\right) \widetilde{\subseteq} \wp_S$. Therefore, $\wp_S$ is an $\mathbb{S}$-int $\mathbb{BI}$-id.

Proposition 3.32. Let $\mathfrak{u}_S \in S_S(U)$, $\alpha \subseteq U$, $Im(\mathfrak{u}_S)$ be the image of $\mathfrak{u}_S$ such that $\tilde{\alpha} \in Im(\mathfrak{u}_S)$. If $\mathfrak{u}_S$ is an $\mathbb{S}$-int $\mathbb{BI}$-id, then $\mathcal{U}(\mathfrak{u}_S; \tilde{\alpha})$ is a $\mathbb{BI}$-id.

Proof: Since $ứ_S(\omega)=\tilde{\alpha}$ for some $\omega \in S$, $\emptyset \neq \mathcal{U}(ứ_S;\tilde{\alpha}) \subseteq S$. Let $k \in [S \cdot \mathcal{U}(ứ_S;\tilde{\alpha}) \cdot S] \cap [\mathcal{U}(ứ_S;\tilde{\alpha}) \cdot S \cdot \mathcal{U}(ứ_S;\tilde{\alpha})]$. Then, there exist $\omega, \ddot{t}, ẓ \in \mathcal{U}(ứ_S;\tilde{\alpha})$ and $a,b,c \in S$ such that $k=\omega a\ddot{t}=bẓc$. Thus, $ứ_S(\omega) \supseteq \tilde{\alpha}$, $ứ_S(\ddot{t}) \supseteq \tilde{\alpha}$ and $ứ_S(ẓ) \supseteq \tilde{\alpha}$. Since $ứ_S$ is an $\mathbb{S}$-int ƁI-id,

$$
\begin{aligned}
\left(\widetilde{\mathbb{S}} \circ ứ_S \circ \widetilde{\mathbb{S}}\right)(k) &= \cup_{k=mn}\left\{\widetilde{\mathbb{S}}(m) \cap \left(ứ_S \circ \widetilde{\mathbb{S}}\right)(n)\right\} \\
&\supseteq \widetilde{\mathbb{S}}(b) \cap \left(ứ_S \circ \widetilde{\mathbb{S}}\right)(ẓc) \\
&= U \cap \left[\bigcup_{ẓc=sr}\left\{ứ_S(s) \cap \widetilde{\mathbb{S}}(r)\right\}\right] \\
&\supseteq ứ_S(ẓ) \cap \widetilde{\mathbb{S}}(c) \\
&= ứ_S(ẓ) \cap U \\
&= ứ_S(ẓ) \\
&\supseteq \tilde{\alpha}
\end{aligned} \tag{54}
$$

and

$$
\begin{aligned}
\left(ứ_S \circ \widetilde{\mathbb{S}} \circ ứ_S\right)(k) &= \cup_{k=mn}\left\{ứ_S(m) \cap \left(\widetilde{\mathbb{S}} \circ ứ_S\right)(n)\right\} \\
&\supseteq ứ_S(\omega) \cap \left(\widetilde{\mathbb{S}} \circ ứ_S\right)(a\ddot{t}) \\
&= ứ_S(\omega) \cap \left[\bigcup_{a\ddot{t}=pq}\left\{\widetilde{\mathbb{S}}(p) \cap ứ_S(q)\right\}\right] \\
&\supseteq ứ_S(\omega) \cap \widetilde{\mathbb{S}}(a) \cap ứ_S(\ddot{t}) \\
&= ứ_S(\omega) \cap U \cap ứ_S(\ddot{t}) \\
&= ứ_S(\omega) \cap ứ_S(\ddot{t}) \\
&\supseteq \tilde{\alpha} \cap \tilde{\alpha} \\
&= \tilde{\alpha}
\end{aligned} \tag{55}
$$

Thus, $\left(\widetilde{\mathbb{S}} \circ ứ_S \circ \widetilde{\mathbb{S}}\right)(k) \cap \left(ứ_S \circ \widetilde{\mathbb{S}} \circ ứ_S\right)(k) \supseteq \tilde{\alpha}$. Since $ứ_S$ is an $\mathbb{S}$-int ƁI-id,

$$ứ_S(k) \supseteq \left(\widetilde{\mathbb{S}} \circ ứ_S \circ \widetilde{\mathbb{S}}\right)(k) \cap \left(ứ_S \circ \widetilde{\mathbb{S}} \circ ứ_S\right)(k) \supseteq \tilde{\alpha} \tag{56}$$

Thus, $k \in \mathcal{U}(ứ_S;\tilde{\alpha})$. Therefore, $[S \cdot \mathcal{U}(ứ_S;\tilde{\alpha}) \cdot S] \cap [\mathcal{U}(ứ_S;\tilde{\alpha}) \cdot S \cdot \mathcal{U}(ứ_S;\tilde{\alpha})] \subseteq \mathcal{U}(ứ_S;\tilde{\alpha})$. Hence, $\mathcal{U}(ứ_S;\tilde{\alpha})$ is a ƁI-id.

We illustrate Proposition 3.32 with Example 3.33.

Example 3.33. Let the SS ω_S in Example 3.2. By considering the image set of ω_S, that is,

$$Im(\omega_S) = \{\{1,2\},\{2,3\},\{1,2,3\},\mathbb{Z}^+\}$$

we obtain the following:

$$
\mathcal{U}(\omega_S;\tilde{\alpha}) \begin{cases} \{▯, ▯, ℭ\}, & \tilde{\alpha} = \{1,2\} \\ \{▯, Ħ, ℭ\}, & \tilde{\alpha} = \{2,3\} \\ \{▯, ℭ\}, & \tilde{\alpha} = \{1,2,3\} \\ \{ℭ\}, & \tilde{\alpha} = \mathbb{Z}^+ \end{cases}
$$

Here, $\{▯, ▯, ℭ\}$, $\{▯, Ħ, ℭ\}$, $\{▯, ℭ\}$ and $\{ℭ\}$ are all ƁI-ids. In fact, since

$$
\begin{aligned}
&(S \bullet \{▯, ▯, ℭ\} \bullet S) \cap (\{▯, ▯, ℭ\} \bullet S \bullet \{▯, ▯, ℭ\}) = \{▯, ▯, ℭ\} \subseteq \{▯, ▯, ℭ\} \\
&(S \bullet \{▯, Ħ, ℭ\} \bullet S) \cap (\{▯, Ħ, ℭ\} \bullet S \bullet \{▯, Ħ, ℭ\}) = \{▯, ℭ\} \subseteq \{▯, Ħ, ℭ\} \\
&(S \bullet \{▯, ℭ\} \bullet S) \cap (\{▯, ℭ\} \bullet S \bullet \{▯, ℭ\}) = \{▯, ℭ\} \subseteq \{▯, ℭ\} \\
&(S \bullet \{ℭ\} \bullet S) \cap (\{ℭ\} \bullet S \bullet \{ℭ\}) = \{ℭ\} \subseteq \{ℭ\}
\end{aligned} \tag{57}
$$

each $\mathcal{U}(\omega_S;\tilde{\alpha})$ is a ƁI-id.

Now, consider the SS υ_S in Example 3.2. By taking into account

$$Im(\upsilon_S) = \{\{3,5\},\{5,8\},\{3,5,7\},\{3,5,7,8\}\}$$

we obtain the following:

$$
\mathcal{U}(\upsilon_S;\tilde{\alpha}) \begin{cases} \{▯, ▯, ℭ\}, & \tilde{\alpha} = \{3,5\} \\ \{▯, Ħ\}, & \tilde{\alpha} = \{5,8\} \\ \{▯, ℭ\}, & \tilde{\alpha} = \{3,5,7\} \\ \{▯\}, & \tilde{\alpha} = \{3,5,7,8\} \end{cases}
$$

Here, $\{▯, Ħ\}$ is not a ƁI-id. In fact, since

$$(S \bullet \{▯, Ħ\} \bullet S) \cap (\{▯, Ħ\} \bullet S \bullet \{▯, Ħ\}) = \{▯, ℭ\} \nsubseteq \{▯, Ħ\} \tag{58}$$

one of the $\mathcal{U}(\upsilon_S;\tilde{\alpha})$ is not a ƁI-id. It is seen that each of $\mathcal{U}(\upsilon_S;\tilde{\alpha})$ is not a ƁI-id. On the other hand, in Example 3.2 it was shown that υ_S is not an $\mathbb{S}$-int ƁI-id.

Definition 3.34. Let f_S be an $\mathbb{S}$-int ƁI-id. Then, the ƁI-ids $\mathcal{U}(f_S;\tilde{\alpha})$ are called upper $\tilde{\alpha}$-ƁI-ids of f_S.

Theorem 3.35. For S, the following conditions are equivalent:

i. S is regular.
ii. $\mathfrak{f}_S = \left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right)$ for every $\mathbb{S}$-int ƁI-id.

Proof: Let (i.) hold, S be regular, $\mathfrak{f}_S$ be an $\mathbb{S}$-int ƁI-id and $x \in S$. Then, $\left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right) \tilde{\subseteq} \mathfrak{f}_S$ and there exists an element $y \in S$ such that $x = xyx$. Since

$\left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right)(x) = \cup_{x=ab}\left\{\left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right)(a) \cap \widetilde{\mathbb{S}}(b)\right\}$

$$\begin{aligned} &\supseteq \left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right)(x) \cap \widetilde{\mathbb{S}}(yx) \\ &= \left[\bigcup_{x=ab}\left\{\widetilde{\mathbb{S}}(a) \cap \mathfrak{f}_S(b)\right\}\right] \cap U \\ &\supseteq \widetilde{\mathbb{S}}(xy) \cap \mathfrak{f}_S(x) \\ &= U \cap \mathfrak{f}_S(x) \\ &= \mathfrak{f}_S(x) \end{aligned} \tag{59}$$

$\left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right)(x) = \cup_{x=ab}\left\{\left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right)(a) \cap \mathfrak{f}_S(b)\right\}$

$$\begin{aligned} &\supseteq \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right)(xy) \cap \mathfrak{f}_S(x) \\ &= \left[\bigcup_{xy=mn}\left\{\mathfrak{f}_S(m) \cap \widetilde{\mathbb{S}}(n)\right\}\right] \cap \mathfrak{f}_S(x) \\ &\supseteq \left[\mathfrak{f}_S(x) \cap \widetilde{\mathbb{S}}(yxy)\right] \cap \mathfrak{f}_S(x) \\ &= [\mathfrak{f}_S(x) \cap U] \cap \mathfrak{f}_S(x) \\ &= \mathfrak{f}_S(x) \end{aligned} \tag{60}$$

Thus, $\left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right)(x) \cap \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right)(x) \supseteq \mathfrak{f}_S(x)$ implying that $\mathfrak{f}_S \,\widetilde{\subseteq}\, \left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \,\widetilde{\cap}\, \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right)$. Therefore, $\mathfrak{f}_S = \left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \,\widetilde{\cap}\, \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right)$.

Conversely, let $\mathfrak{f}_S = \left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \,\widetilde{\cap}\, \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right)$, where $\mathfrak{f}_S$ is an $\mathbb{S}$-int $\mathbb{BI}$-id. To show that S is regular, $A = SAS \cap ASA$ for every $\mathbb{BI}$-id. $SAS \cap ASA \subseteq A$ is evident. Thus, $A \subseteq SAS \cap ASA$ should be demonstrated. Let $\omega \in A$ and A be any $\mathbb{BI}$-id. Thus, S_A is an $\mathbb{S}$-int $\mathbb{BI}$-id. Hence,

$$S_A(\omega) = \left(\widetilde{\mathbb{S}} \circ S_A \circ \widetilde{\mathbb{S}}\right)(\omega) \cap \left(S_A \circ \widetilde{\mathbb{S}} \circ S_A\right)(\omega) = (S_S \circ S_A \circ S_S)(\omega) \cap (S_A \circ S_S \circ S_A)(\omega) = S_{SAS \cap ASA}(\omega) = U \tag{61}$$

implying that $\omega \in SAS \cap ASA$. Hence, $A = SAS \cap ASA$, so S is regular.

Theorem 3.36. Let $\mathfrak{f}_S, \mathfrak{u}_S \in S_S(U)$. Then, the following conditions are equivalent:

i. S is regular.
ii. $\mathfrak{f}_S \,\widetilde{\cap}\, \mathfrak{u}_S \,\widetilde{\subseteq}\, (\mathfrak{u}_S \circ \mathfrak{f}_S \circ \mathfrak{u}_S) \,\widetilde{\cap}\, (\mathfrak{f}_S \circ \mathfrak{u}_S \circ \mathfrak{f}_S)$ for an $\mathbb{S}$-int $\mathbb{BI}$-id $\mathfrak{f}_S$ and an $\mathbb{S}$-int id $\mathfrak{u}_S$.

Proof: Let (i.) hold, S be regular, $\mathfrak{f}_S$ be an $\mathbb{S}$-int $\mathbb{BI}$-id, $\mathfrak{u}_S$ be an $\mathbb{S}$-int id and $\omega \in S$. Then, $\left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \,\widetilde{\cap}\, \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right) \,\widetilde{\subseteq}\, \mathfrak{f}_S$, for $\acute{p}, q \in S$, $\mathfrak{u}_S(\acute{p}q) \supseteq \mathfrak{u}_S(\acute{p})$ and $\mathfrak{u}_S(\acute{p}q) \supseteq \mathfrak{u}_S(q)$, and there exist an element $y \in S$ such that $\omega = \omega y \omega$. Since

$(\mathfrak{u}_S \circ \mathfrak{f}_S \circ \mathfrak{u}_S)(\omega) = \cup_{\omega=ab}\{\mathfrak{u}_S(a) \cap (\mathfrak{f}_S \circ \mathfrak{u}_S)(b)\}$

$$\begin{aligned} &\supseteq \mathfrak{u}_S(\omega y) \cap (\mathfrak{f}_S \circ \mathfrak{u}_S)(\omega) \\ &= \mathfrak{u}_S(\omega y) \cap \left[\bigcup_{\omega=ab}\{\mathfrak{f}_S(a) \cap \mathfrak{u}_S(b)\}\right] \\ &\supseteq \mathfrak{u}_S(\omega y) \cap [\mathfrak{f}_S(\omega) \cap \mathfrak{u}_S(y\omega)] \\ &\supseteq \mathfrak{u}_S(\omega) \cap [\mathfrak{f}_S(\omega) \cap \mathfrak{u}_S(\omega)] \\ &= \mathfrak{f}_S(\omega) \cap \mathfrak{u}_S(\omega) \\ &= (\mathfrak{f}_S \,\widetilde{\cap}\, \mathfrak{u}_S)(\omega) \end{aligned} \tag{62}$$

$(\mathfrak{f}_S \circ \mathfrak{u}_S \circ \mathfrak{f}_S)(\omega) = \cup_{\omega=ab}\{(\mathfrak{f}_S \circ \mathfrak{u}_S)(a) \cap \mathfrak{f}_S(b)\}$

$$\begin{aligned} &\supseteq (\mathfrak{f}_S \circ \mathfrak{u}_S)(\omega y) \cap \mathfrak{f}_S(\omega) \\ &= \left[\bigcup_{\omega y=mn}\{\mathfrak{f}_S(m) \cap \mathfrak{u}_S(n)\}\right] \cap \mathfrak{f}_S(\omega) \\ &\supseteq [\mathfrak{f}_S(\omega) \cap \mathfrak{u}_S(y\omega y)] \cap \mathfrak{f}_S(\omega) \\ &\supseteq [\mathfrak{f}_S(\omega) \cap \mathfrak{u}_S(\omega)] \cap \mathfrak{f}_S(\omega) \\ &= \mathfrak{f}_S(\omega) \cap \mathfrak{u}_S(\omega) \\ &= (\mathfrak{f}_S \,\widetilde{\cap}\, \mathfrak{u}_S)(\omega) \end{aligned} \tag{63}$$

Thus, $(\mathfrak{u}_S \circ \mathfrak{f}_S \circ \mathfrak{u}_S)(\omega) \cap (\mathfrak{f}_S \circ \mathfrak{u}_S \circ \mathfrak{f}_S)(\omega) \supseteq (\mathfrak{f}_S \,\widetilde{\cap}\, \mathfrak{u}_S)(\omega)$. Hence, $\mathfrak{f}_S \,\widetilde{\cap}\, \mathfrak{u}_S \,\widetilde{\subseteq}\, (\mathfrak{u}_S \circ \mathfrak{f}_S \circ \mathfrak{u}_S) \,\widetilde{\cap}\, (\mathfrak{f}_S \circ \mathfrak{u}_S \circ \mathfrak{f}_S)$.

Conversely, let $\mathfrak{f}_S \,\widetilde{\cap}\, \mathfrak{u}_S \,\widetilde{\subseteq}\, (\mathfrak{u}_S \circ \mathfrak{f}_S \circ \mathfrak{u}_S) \,\widetilde{\cap}\, (\mathfrak{f}_S \circ \mathfrak{u}_S \circ \mathfrak{f}_S)$ for $\mathbb{S}$-int $\mathbb{BI}$-id $\mathfrak{f}_S$ and $\mathbb{S}$-int id $\mathfrak{u}_S$. Let $\widetilde{\mathbb{S}}$ as $\mathbb{S}$-int id instead $\mathfrak{u}_S$. Then, $\mathfrak{f}_S \,\widetilde{\cap}\, \widetilde{\mathbb{S}} \,\widetilde{\subseteq}\, \left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \,\widetilde{\cap}\, \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right)$. Thus, $\mathfrak{f}_S \,\widetilde{\subseteq}\, \left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \,\widetilde{\cap}\, \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right)$. From assumption, $\left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \,\widetilde{\cap}\, \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right) \,\widetilde{\subseteq}\, \mathfrak{f}_S$. Hence, $\mathfrak{f}_S = \left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right) \,\widetilde{\cap}\, \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right)$. The rest is obvious by Theorem 3.35.

4. Conclusions

Rao [6] proposed the concept of bi-interior ideals of semigroups and examined the properties of bi-interior ideal of semigroups as a generalization of the bi-ideals and interior ideals of semigroups. In this paper, by applying the concept of bi-interior ideals of semigroups to semigroup theory and SS theory, we introduced the concept of "$\mathbb{S}$-int bi-interior ideal (abbreviated by "$\mathbb{S}$-int $\mathbb{BI}$-id" throughout the text) of semigroups". The relations between different types of $\mathbb{S}$-int ideals of a semigroup and $\mathbb{S}$-int bi-interior ideals were established. We demonstrated that an $\mathbb{S}$-int left ideal, right ideal, ideal, bi-ideal, interior ideal, and quasi-ideal is an $\mathbb{S}$-int bi-interior ideal, however, the opposite is not true with counterexamples.

For the converses, we illustrated that the semigroup should be soft simple*. Our key theorem, which states that if a nonempty subset of a semigroup is a bi-interior ideal, then its soft characteristic function is an S-int bi-interior ideal, and vice versa, enables us to bridge the gap between semigroup theory and soft set theory. Using this theorem, we show how this idea relates to the existing algebraic structures in classical semigroup theory. Furthermore, we present conceptual characterizations and analysis of the new concept in terms of regarding soft set operations supporting our assertions with illuminating examples.

In future studies, S-int bi-interior ideals can be used to characterize various semigroup types such as regular, intraregular, weakly regular, quasi-regular, duo, simple semigroups. The relation between several S-int ideals and their generalized ideals is depicted in the following Figure 1., where A → B denotes that A is B but B may not always be A.

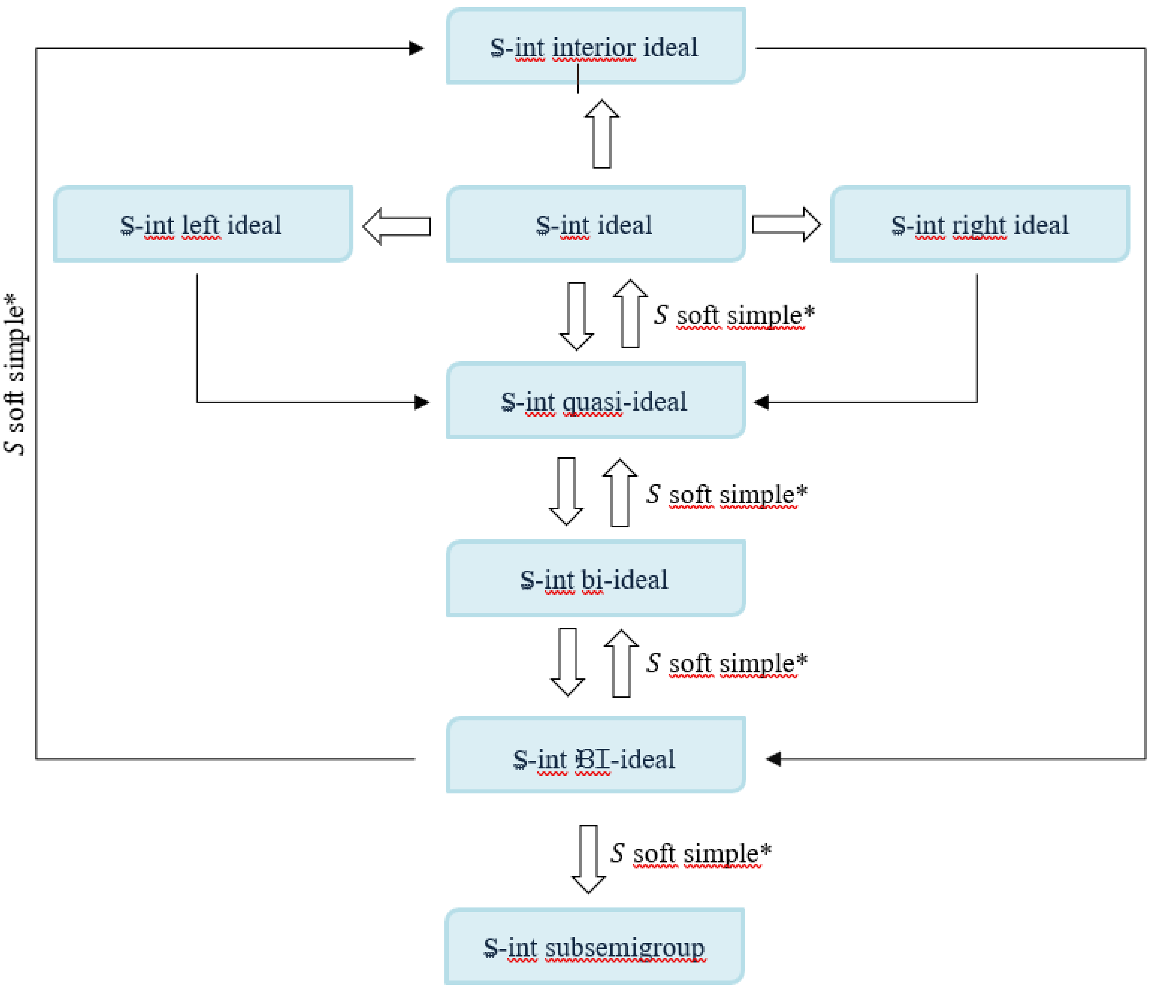

Figure 1: Diagram showing the relationships between some S-int ideals.

Article Information

Financial Disclosure: The author (s) has no received any financial support for the research, authorship or publication of this study.

Authors' Contrtibution: Concept: A.S.; Design: A.I.; Supervision: A.S.; Resources: A.I.; Data Collection: A.S., A.I.; Analysis: A.S.; Literature Search: A.S., A.I.; Writing Manuscript: A.I.; Critical Review: A.S.

Conflict of Interest/Common Interest: No conflict of interest or common interest has been declared by the authors.

Ethics Committee Approval: This study does not require ethics committee permission or any special permission.

References

[1] Good, R. A., & Hughes, D. R. (1952). Associated groups for a semigroup. *Bulletin of the American Mathematical Society*, 58(6), 624–625.

[2] Steinfeld, O. (1956). Uher die quasi ideals. *Publicationes Mathematicae Debrecen*, 4, 262–275.

[3] Lajos, S. (1976). (m;k;n)-ideals in semigroups. *Notes on Semigroups II*, Karl Marx University of Economics, Department of Mathematics, Budapest, (1), 12–19.

[4] Szasz, G. (1977). Interior ideals in semigroups. *Notes on Semigroups IV*, Karl Marx University of Economics, Department of Mathematics, Budapest, (5), 1–7.

[5] Szasz, G. (1981). Remark on interior ideals of semigroups. *Studia Scientiarum Mathematicarum Hungarica*, 16, 61–63.

[6] Rao, M. M. K. (2018). Bi-interior ideals of semigroups. *Discussiones Mathematicae – General Algebra and Applications*, 38(1), 69–78.

[7] Rao, M. M. K. (2018). A study of a generalization of bi-ideal, quasi-ideal and interior ideal of semigroup. *Mathematica Moravica*, 22(2), 103–115.

[8] Rao, M. M. K. (2020). Left bi-quasi ideals of semigroups. *Southeast Asian Bulletin of Mathematics*, 44(3), 369–376.

[9] Rao, M. M. K. (2020). Quasi-interior ideals and weak-interior ideals. *Asia Pacific Journal of Mathematics*, 7, 7–21.

[10] Baupradist, S., Chemat, B., Palanivel, K., & Chinram, R. (2021). Essential ideals and essential fuzzy ideals in semigroups. *Journal of Discrete Mathematical Sciences and Cryptography*, 24(1), 223–233.

[11] Grošek, O., & Satko, L. (1980). A new notion in the theory of semigroup. *Semigroup Forum*, 20(1), 233–240.

[12] Bogdanovic, S. (1981). Semigroups in which some bi-ideal is a group. *Zbornik Radova PMF Novi Sad*, 11, 261–266.

[13] Wattanatripop, K., Chinram, R., & Changphas, T. (2018). Quasi-A-ideals and fuzzy A-ideals in semigroups. *Journal of Discrete Mathematical Sciences and Cryptography*, 21(5), 1131–1138.

[14] Kaopusek, N., Kaewnoi, T., & Chinram, R. (2020). On almost interior ideals and weakly almost interior ideals of semigroups. *Journal of Discrete Mathematical Sciences and Cryptography*, 23(3), 773–778.

[15] Iampan, A., Chinram, R., & Petchkaew, P. (2021). A note on almost subsemigroups of semigroups. *International Journal of Mathematics and Computer Science*, 16, 1623–1629.

[16] Chinram, R., & Nakkhasen, W. (2021). Almost bi-quasi-interior ideals and fuzzy almost bi-quasi-interior ideals of semigroups. *Journal of Mathematics and Computer Science*, 26, 128–136.

[17] Gaketem, T. (2022). Almost bi-interior ideal in semigroups and their fuzzifications. *European Journal of Pure and Applied Mathematics*, 15(1), 281–289.

[18] Gaketem, T., & Chinram, R. (2023). Almost bi-quasi ideals and their fuzzifications in semigroups. *Annals of the University of Craiova – Mathematics and Computer Science Series*, 50(2), 342–352.

[19] Wattanatripop, K., Chinram, R., & Changphas, T. (2018). Fuzzy almost bi-ideals in semigroups. *International Journal of Mathematics and Computer Science*, 13(1), 51–58.

[20] Krailoet, W., Simuen, A., Chinram, R., & Petchkaew, P. (2021). A note on fuzzy almost interior ideals in semigroups. *International Journal of Mathematics and Computer Science*, 16(2), 803–808.

[21] Molodtsov, D. (1999). Soft set theory—First results. *Computers and Mathematics with Applications*, 37(4–5), 19–31.

[22] Maji, P. K., Biswas, R., & Roy, A. R. (2003). Soft set theory. *Computers and Mathematics with Applications*, 45(4–5), 555–562.

[23] Pei, D., & Miao, D. (2005). From soft sets to information systems. *2005 IEEE International Conference on Granular Computing* (Vol. 2, pp. 617–621). IEEE.

[24] Ali, M. I., Feng, F., Liu, X., Min, W. K., & Shabir, M. (2009). On some new operations in soft set theory. *Computers and Mathematics with Applications*, 57(9), 1547–1553.

[25] Sezgin, A., & Atagün, A. O. (2011). On operations of soft sets. *Computers and Mathematics with Applications*, 61(5), 1457–1467.

[26] Ali, M. I., Shabir, M., & Naz, M. (2011). Algebraic structures of soft sets associated with new operations. *Computers and Mathematics with Applications*, 61(9), 2647–2654.

[27] Sezgin, A., Ahmad, S., & Mehmood, A. (2019). A new operation on soft sets: Extended difference of soft sets. *Journal of New Theory*, 27, 33–42.

[28] Stojanović, N. S. (2021). A new operation on soft sets: Extended symmetric difference of soft sets. *Vojnotehnički Glasnik/Military Technical Courier*, 69(4), 779–791.

[29] Sezgin, A., & Atagün, A. O. (2023). A new soft set operation: Complementary soft binary piecewise plus operation. *Matrix Science Mathematics*, 7(2), 125–142.

[30] Sezgin, A., & Aybek, F. N. (2023). A new soft set operation: Complementary soft binary piecewise gamma operation. *Matrix Science Mathematics*, 7(1), 27–45.

[31] Sezgin, A., Aybek, F. N., & Atagün, A. O. (2023). A new soft set operation: Complementary soft binary piecewise intersection operation. *Black Sea Journal of Engineering and Science*, 6(4), 330–346.

[32] Sezgin, A., Aybek, F. N., & Güngör, N. B. (2023). A new soft set operation: Complementary soft binary piecewise union operation. *Acta Informatica Malaysia*, 7(1), 38–53.

[33] Sezgin, A., & Demirci, A. M. (2023). A new soft set operation: Complementary soft binary piecewise star operation. *Ikonion Journal of Mathematics*, 5(2), 24–52.

[34] Sezgin, A., & Yavuz, E. (2023). A new soft set operation: Complementary soft binary piecewise lambda operation. *Sinop University Journal of Natural Sciences*, 8(5), 101–133.

[35] Sezgin, A., & Yavuz, E. (2023). A new soft set operation: Soft binary piecewise symmetric difference operation. *Necmettin Erbakan University Journal of Science and Engineering*, 5(2), 189–208.

[36] Sezgin, A., & Sarıalioğlu, M. (2024). A new soft set operation: Complementary soft binary piecewise theta (**θ**) operation. *Kadirli Uygulamalı Bilimler Fakültesi Dergisi*, 4(2), 325–357.

[37] Sezgin, A., & Çağman, N. (2024). A new soft set operation: Complementary soft binary piecewise difference operation. *Osmaniye Korkut Ata University Journal of the Institute of Science and Technology*, 7(1), 58–94.

[38] Sezgin, A., & Çalışıcı, H. (2024). A comprehensive study on soft binary piecewise difference operation. *Eskişehir Teknik Üniversitesi Bilim ve Teknoloji Dergisi B – Teorik Bilimler*, 12(1), 32–54.

[39] Çağman, N., & Enginoğlu, S. (2010). Soft set theory and uni-int decision making. *European Journal of Operational Research*, 207(2), 848–855.

[40] Çağman, N., Çıtak, F., & Aktaş, H. (2012). Soft int-group and its applications to group theory. *Neural Computing and Applications*, 21, 151–158.

[41] Sezer, A. S., Çağman, N., Atagün, A. O., Ali, M. I., & Türkmen, E. (2015). Soft intersection semigroups, ideals and bi-ideals; A new application on semigroup theory I. *Filomat*, 29(5), 917–946.

[42] Sezer, A. S., Çağman, N., & Atagün, A. O. (2014). Soft intersection interior ideals, quasi-ideals and generalized bi-ideals: A new approach to semigroup theory II. *Journal of Multiple-Valued Logic and Soft Computing*, 23(1–2), 161–207.

[43] Sezgin, A., & Orbay, M. (2022). Analysis of semigroups with soft intersection ideals. *Acta Universitatis Sapientiae, Mathematica*, 14(1), 166–210.

[44] Sezgin, A., & İlgin, A. (2024). Soft intersection almost subsemigroups of semigroups. *International Journal of Mathematics and Physics*, 15(1), 13–20.

[45] Sezgin, A., & İlgin, A. (2024). Soft intersection almost ideals of semigroups. *Journal of Innovative Engineering and Natural Science*, 4(2), 466–481.

[46] Sezgin, A., & Onur, B. (2024). Soft intersection almost bi-ideals of semigroups. *Systemic Analytics*, 2(1), 95–105.

[47] Sezgin, A., & Baş, Z. H. (2024). Soft-int almost interior ideals for semigroups. *Information Sciences with Applications*, 4, 25–36.

[48] Sezgin, A., & Kocakaya, F. Z. (2025). Soft intersection almost quasi-ideals of semigroups.

Songklanakarin Journal of Science and Technology, (in press).

[49] Sezgin, A., & İlgin, A. (2024). Soft intersection almost weak-interior ideals of semigroups: A theoretical study. *JNSM Journal of Natural Sciences and Mathematics of UT*, 9(17–18), 372–385.

[50] Sezgin, A., & İlgin, A. (2024). Soft intersection almost bi-interior ideals of semigroups. *Journal of Natural and Applied Sciences Pakistan*, 6(1), 1619–1638.

[51] Sezgin, A., & İlgin, A. (2024). Soft intersection almost bi-quasi ideals of semigroups. *Soft Computing Fusion with Applications*, 1(1), 27–42.

[52] Sezgin, A., Kocakaya, F. Z., & İlgin, A. (2024). Soft intersection almost quasi-interior ideals of semigroups. *Eskişehir Teknik Üniversitesi Bilim ve Teknoloji Dergisi B – Teorik Bilimler*, 12(2), 81–99.

[53] Sezgin, A., Baş, Z. H., & İlgin, A. (2025). Soft intersection almost bi-quasi-interior ideals of semigroups. *Journal of Fuzzy Extension and Application*, 6(1), 43–58.

[54] Sezgin, A., Onur, B., & İlgin, A. (2024). Soft intersection almost tri-ideals of semigroups. *SciNexuses*, 1, 126–138.

[55] Sezgin, A., İlgin, A., & Atagün, A. O. (2024). Soft intersection almost tri-bi-ideals of semigroups. *Science & Technology Asia*, 29(4), 1–13.

[56] Aktaş, H., & Çağman, N. (2007). Soft sets and soft groups. *Information Sciences*, 177(13), 2726–2735.

[57] Feng, F., Jun, Y. B., & Zhao, X. (2008). Soft semirings. *Computers and Mathematics with Applications*, 56(10), 2621–2628.

[58] Acar, U., Koyuncu, F., & Tanay, B. (2010). Soft sets and soft rings. *Computers and Mathematics with Applications*, 59(11), 3458–3463.

[59] Sezgin, A., & Atagün, A. O. (2011). Soft groups and normalistic soft groups. *Computers and Mathematics with Applications*, 62(2), 685–698.

[60] Çağman, N., Çıtak, F., & Aktaş, H. (2012). Soft int-group and its applications to group theory. *Neural Computing and Applications*, 21, 151–158.

[61] Sezer, A. S. (2014). A new approach to LA-semigroup theory via the soft sets. *Journal of Intelligent & Fuzzy Systems*, 26(5), 2483–2495.

[62] Sezer, A. S. (2014). Certain characterizations of LA-semigroups by soft sets. *Journal of Intelligent & Fuzzy Systems*, 27(2), 1035–1046.

[63] Sezer, A. S., Çağman, N., & Atagün, A. O. (2015). Uni-soft substructures of groups. *Annals of Fuzzy Mathematics and Informatics*, 9(2), 235–246.

[64] Muştuoğlu, E., Sezgin, A., & Türk, Z. K. (2016). Some characterizations on soft uni-groups and normal soft uni-groups. *International Journal of Computer Applications*, 155(10), 1–8.

[65] Tunçay, M., & Sezgin, A. (2016). Soft union ring and its applications to ring theory. *International Journal of Computer Applications*, 151(9), 7–13.

[66] Sezer, A. S., & Atagün, A. O. (2016). A new kind of vector space: Soft vector space. *Southeast Asian Bulletin of Mathematics*, 40(5), 753–770.

[67] Sezgin, A., Çağman, N., & Atagün, A. O. (2017). A completely new view to soft-int rings via soft uni-int product. *Applied Soft Computing*, 54, 366–392.

[68] Mahmood, T., Rehman, Z. U., & Sezgin, A. (2018). Lattice ordered soft near rings. *Korean Journal of Mathematics*, 26(3), 503–517.

[69] Atagün, A. O., & Sezgin, A. (2018). Soft subnear-rings, soft ideals and soft N-subgroups of near-rings. *Mathematical Science Letters*, 7, 37–42.

[70] Sezgin, A. (2018). A new view on AG-groupoid theory via soft sets for uncertainty modeling. *Filomat*, 32(8), 2995–3030.

[71] Jana, C., Pal, M., Karaaslan, F., & Sezgin, A. (2019). (α, β)-Soft intersectional rings and ideals with their applications. *New Mathematics and Natural Computation*, 15(2), 333–350.

[72] Özlü, Ş., & Sezgin, A. (2020). Soft covered ideals in semigroups. *Acta Universitatis Sapientiae, Mathematica*, 12(2), 317–346.

[73] Sezgin, A., Atagün, A. O., Çağman, N., & Demir, H. (2022). On near-rings with soft union ideals and applications. *New Mathematics and Natural Computation*, 18(2), 495–511.

[74] Rao, M. M. K., & Venkateswarlu, B. (2018). Bi-interior ideals of Γ-semirings. *Discussiones Mathematicae – General Algebra and Applications*, 38(2), 239–254.

[75] Rao, M. M. K. (2022). Bi-interior ideals and fuzzy bi-interior ideals of Γ-semigroups. *Annals of Fuzzy Mathematics and Informatics*, 24(1), 85–100.