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Investigation of inverse problem for unknown coefficient in a time fractional diffusion problem with periodic boundary conditions

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Abstract. This paper is about the identification of time dependent unknown function in a time fractional diffusion problem with periodic boundary conditions. Nonlocal over-determined condition in integral form is taken into account in the determination of unknown function as well as the solution of the problem. The existence, uniqueness as well as continuous dependence on data for the inverse problem is presented with certain regularity and consistency conditions. Generalized Fourier method is utilized in this study. The algorithm of the method and an example are also presented to show the effectiveness and accuracy of the proposed method.

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Keywords. Inverse problem, Fourier method, time fractional diffusion problem, nonlocal condition, periodic boundary conditions.

1. Introduction

Mathematical models play a substantial role in various fields of science such as thermoelasticity, engineering, medical science, control theory, chemical diffusion, etc., [2–4, 15, 18, 20]. Therefore, numerous researchers study on the construction of mathematical models which requires determination of unknown functions. At this stage, the inverse problems are taken into account for their determination. Generally, overdetermined conditions are utilized in the identification of unknown functions. The definition of inverse problem can be articulated as determination of unknown feature of an object under consideration by employing overdetermined data obtained by using probing signals. Hence, the shape and features of the unknown object is determined by means of remote sensing and nondestructive testing which form a fundamental basis of the inverse problem. In real life, inverse problems have a great many of applications such as the identification of submarines, shoals of fish immersed in water. Moreover, in the identification of flying objects like airplanes and missiles, etc., inverse problems are used. Briefly, it is concluded that remote sensing provide a great advantage to form the overdetermined data and inverse problem as well [17]. Since analytical identification in inverse problems is a big challenge, various numerical methods such as implicit finite-difference schemes, Crank–Nicolson implicit finite-difference formula, the fourth-order implicit scheme have been developed for the approximate identification [4]. Moreover, the stability of these finite different schemes is provided. Furthermore, compare to explicit finite-difference schemes, the implicit difference schemes spend most of the CPU time. In recent years, time fractional differential problems including nonlocal boundary conditions have been under consideration to establish approximate or analytical solutions by utilizing various developed methods such as implicit finite difference schemes, semigroup method [5–8, 14, 19, 21, 22]. In real life, periodic boundary conditions have been encountered in numerous processes such as diffusion problem and heat transfer, etc., [1, 9, 11, 12]. In this

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research, Fourier method is employed for the identification of time dependent unknown function in one-dimensional fractional diffusion problem involving periodic boundary conditions, by means of nonlocal overdetermined condition. The existence, uniqueness, and continuous dependence of the solution on the data for the inverse problem is provided by imposing regularity and consistency conditions. The following time fractional diffusion problem is taken into consideration to identify the unknown time dependent function $r(\psi)$ and establish the solution $u(\chi,\psi)$ of the problem on $\Omega := \{(\chi,\psi) : 0 < \chi < \pi, 0 < \psi < T\}$ for a fixed number $T > 0$.

$$ {}_0^C D_\psi^\alpha \left(u\left(\chi,\psi\right)\right) = u_{\chi\chi}\left(\chi,\psi\right) + r\left(\psi\right) f\left(\chi,\psi\right),\ \chi > 0,\ 0 < \alpha \le 1,\ \psi > 0 \tag{1} $$

$$ u\left(0,\psi\right) = u\left(\pi,\psi\right), \tag{2} $$

$$ u_\chi\left(0,\psi\right) = u_\chi(\pi,\psi), \tag{3} $$

$$ u(\chi,0) = \varphi(\chi), \tag{4} $$

$$ E\left(\psi\right) = \int_0^\pi \chi u\left(\chi,\psi\right) d\chi. \tag{5} $$

where the functions $\varphi\left(\chi\right) on \left[0,\pi\right]$ and $f\left(\chi,\psi\right)$ on $\bar{\Omega}$ are fixed.
In the process of diffusion of material, the nonlocal condition Eq. (5) represents the law of variation $E\left(\psi\right)$ of the total amount of diffusion in [10]. It is given in integral form. We focus on the inverse problem (1)-(5) of the determining the pair of functions $\{r\left(\psi\right), u\left(\chi,\psi\right)\}$ on the class $C\left[0,T\right] \times C^{2,1}\left(\Omega\right) \cap C^{1,0}\left(\bar{\Omega}\right)$.

Notation:
$\varphi\left(\chi\right)$ is the initial temperature,
$u_0\left(\psi\right),\ u_{cm}\left(\psi\right),\ u_{sm}\left(\psi\right)$ are the Fourier coefficients,
M is an arbitrary constant,
$M_1, M_2, M_3, M_4, M_5, M_6$ are dimensionless constants,
$F\left(\psi\right)$ is a continuous function,
$K\left(\chi,\tau\right)$ is a kernel function,
$\Omega := \{(\chi,\psi) : 0 < \chi < \pi, 0 < \psi < T\}$ is the domain of χ,ψ.

The rest of the paper is designed as follows: fundamental definitions and conceptions are provided in section 2. The existence and uniqueness of the solution for the inverse problem are given in section 3. In section 4, the stability of the solution is presented. Finally, an example is demonstrated to exhibit the effectiveness and accuracy of the method.

2. Preliminary Results

This section is devoted to basic definitions and concepts utilized in this study [13,16].

Definition 1. *Let $v\left(\zeta,\theta\right)$ be a real valued function. Its Riemann-Liouville time fractional integral of order $\alpha > 0$ is denoted by $I_\psi^\alpha v\left(\zeta,\theta\right)$ and is defined as:*

$$ I_\theta^\alpha v\left(\zeta,\theta\right) = \frac{1}{\Gamma(\alpha)} \int_0^\theta \frac{v\left(\zeta,s\right)}{\left(\theta-s\right)^{1-\alpha}} ds. $$

Definition 2. *The Caputo time fractional derivative of $v\left(\zeta,\theta\right)$ is defined as:*

$$ D_\theta^\alpha v\left(\zeta,\theta\right) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^\theta \frac{\frac{\partial^n}{\partial s^n} v\left(\zeta,s\right)}{\left(\theta-s\right)^{1+\alpha-m}} ds, n-1 < \alpha < n, \\ \dfrac{\partial^n}{\partial\theta^n} v\left(\zeta,\theta\right), \alpha = n. \end{cases} $$

Definition 3. *The Mittag-Leffler function of two parameters $E_{\alpha,\beta}\left(\zeta\right)$ is defined as*

$$ E_{\alpha,\beta}\left(\zeta\right) = \sum_{j=0}^\infty \frac{\zeta^j}{\Gamma(\alpha j + \beta)} $$

where $Re\left(\alpha\right) > 0, \zeta, \beta \in \mathbb{C}$.

3. Existence and Uniqueness for the Solution of Inverse Problem

The solution of the problem (1)–(5) will be constructed in the following form:
$u(\chi,\psi)=\frac{u_0(\psi)}{2}+\sum_{m=1}^{\infty}u_{cm}(\psi)\cos(2m\chi)+\sum_{m=1}^{\infty}u_{sm}(\psi)\sin(2m\chi)$.
Utilization of the Fourier method leads us to determine the Fourier coefficients as follows:

$$u_0(\psi)=\varphi_0+\frac{2}{\pi}\int_0^{\psi}\int_0^{\pi}r(\tau)f(\xi,\tau)\,d\xi d\tau,$$

$$u_{cm}(\psi)=\varphi_{cm}(\psi)E_{\alpha,1}\left(-(2m)^2\psi^{\alpha}\right)+\frac{2}{\pi}\int_0^{\psi}\int_0^{\pi}r(\tau)E_{\alpha,1}\left(-(2m)^2(\psi-\tau)^{\alpha}\right)f(\xi,\tau)\cos(2m\xi)d\xi d\tau,$$

$$u_{sm}(\psi)=\varphi_{cm}(\psi)E_{\alpha,1}\left(-(2m)^2\psi^{\alpha}\right)+\frac{2}{\pi}\int_0^{\psi}\int_0^{\pi}r(\tau)E_{\alpha,1}\left(-(2m)^2(\psi-\tau)^{\alpha}\right)f(\xi,\tau)\sin(2m\xi)d\xi d\tau,$$

where

$$\varphi_0=\frac{2}{\pi}\int_0^{\pi}\varphi(\chi)\,d\chi,$$

$$\varphi_{cm}=\frac{2}{\pi}\int_0^{\pi}\varphi(\chi)\cos(2m\chi)d\chi,$$

$$\varphi_{sm}=\frac{2}{\pi}\int_0^{\pi}\varphi(\chi)\sin(2m\chi)d\chi,$$

Under the assumption of fixed $r(\psi)\in C[0,T]$, the solution of the problem (1)–(4) is established in the following form:

$$u(\chi,\psi)=\frac{1}{2}\left(\varphi_0+\frac{2}{\pi}\int_0^{\psi}\int_0^{\pi}r(\tau)f_0(\tau)\,d\xi d\tau\right)+\sum_{m=1}^{\infty}\left(\varphi_{cm}E_{\alpha,1}\left(-(2m)^2\psi^{\alpha}\right)\right.$$
$$\left.+\frac{2}{\pi}\int_0^{\psi}\int_0^{\pi}r(\tau)E_{\alpha,1}\left(-(2m)^2(\psi-\tau)^{\alpha}\right)f_{cm}(\tau)\,d\xi d\tau\right)\cos(2m\chi)+\sum_{m=1}^{\infty}\left(\varphi_{sm}E_{\alpha,1}\left(-(2m)^2\psi^{\alpha}\right)+\right.$$
$$\left.\frac{2}{\pi}\int_0^{\psi}\int_0^{\pi}r(\tau)E_{\alpha,1}\left(-(2m)^2(\psi-\tau)^{\alpha}\right)f_{sm}(\tau)\,d\xi d\tau\right)\sin(2m\chi) \tag{6}$$

where

$$f_0(\psi)=\frac{2}{\pi}\int_0^{\pi}f(\chi,\psi)\,d\chi,$$

$$f_{cm}(\psi)=\frac{2}{\pi}\int_0^{\pi}f(\chi,\psi)\cos(2m\chi)d\chi,$$

$$f_{sm}(\psi)=\frac{2}{\pi}\int_0^{\pi}f(\chi,\psi)\sin(2m\chi)d\chi.$$

Theorem 1. *The problem (1)–(5) has a unique solution under the following conditions:*

$(S1)E(\psi)\in C^1[0,T],$

$(S2)\varphi(\chi)\in C^2([0,\pi]),\ \varphi(0)=\varphi(\pi),\ \varphi\prime(0)=\varphi\prime(\pi),and\int_0^{\pi}\chi\varphi(\chi)\,d\chi=E(0),$

$(S3)f(\chi,\psi)\in C^{2,0}\left(\bar{\Omega}\right),f(0,\psi)=f(\pi,\psi),f'(0,\psi)=f'(\pi,\psi),and\int_0^{\pi}\chi f(\chi,\psi)\,d\chi\neq 0.$

Proof. The consistency conditions are obtained as
$\varphi(0)=\varphi(\pi),f(0,\psi)=f(\pi,\psi)$
which are necessary for the solution $u(\chi,\psi)$ to be in $C^{2,1}(\Omega)\cap C^{1,0}\left(\bar{\Omega}\right)$. Moreover, series (6) and its partial derivative with respect to χ are uniformly convergent in $\bar{\Omega}$ under the smoothness conditions:
$\varphi(\chi)\in C^2([0,\pi])$ and $f(\chi,\psi)\in C^{2,0}(\Omega)$
which is the implication of absolute convergence of ψ- partial derivative series with the conditions $\varphi'\ (0)=$

$\varphi' \ (\pi)\,,\ f'\,(0,\psi) = f'\,(\pi,\psi)$ in $\bar{\Omega}$.
We differentiate equation (5) under the condition (S1) to obtain

$$ {}_0^C D_\psi^\alpha \left(E\left(\psi\right)\right) = \int_0^\pi x {}_0^C D_\psi^\alpha \left(u\left(\chi,\psi\right)\right) d\chi. \tag{7}$$

The following Volterra integral equation of the second kind

$$ r\left(\psi\right) = F\left(\psi\right) + \int_0^\pi K\left(\psi,\tau\right) r\left(\tau\right) d\tau, \psi \in [0,T]$$

is obtained from the Eqs. (6)-(7) under the consistency condition

$$\int_0^\pi \chi\varphi\left(\chi\right) d\chi = E\left(0\right).$$

Moreover, the assumption (S1)–(S3) imply the continuity of functions $F\left(\psi\right)$ on $[0,T]$ and $K\left(\psi,\tau\right)$ on $[0,T]\times[0,T]$ which are determined as

$$F\left(\psi\right) = \frac{ {}_0^C D_\psi^\alpha \left(E\left(\psi\right)\right) + \frac{\pi}{2}\sum_{m=1}^\infty \frac{(2m)^2}{m}\varphi_{sm} E_{\alpha,1}\left(-\left(2m\right)^2\psi^\alpha\right)}{f_0\left(\psi\right) + \frac{\pi}{2}\sum_{m=1}^\infty \frac{(2m)^2}{m} f_{sm}\left(\psi\right)}, \tag{8}$$

$$K\left(\psi,\tau\right) = \frac{\frac{\pi}{2}\sum_{m=1}^\infty \frac{(2m)^2}{m} f_{sm}\left(\psi\right) E_{\alpha,1}\left(-\left(2m\right)^2\left(\psi-\tau\right)^\alpha\right)}{f_0\left(\psi\right) + \frac{\pi}{2}\sum_{m=1}^\infty \frac{(2m)^2}{m} f_{sm}\left(\psi\right)}. \tag{9}$$

As a result, the unknown function $r\left(\psi\right)$ on $[0,T]$ is uniquely determined and continuous. $r\left(\psi\right)$ together with the solution $u\left(\chi,\psi\right)$ establish by Fourier series form the unique solution of the inverse problem (1)-(5) which proves the desired result. □

4. Stability of the Solution $\{r\left(\psi\right), u\left(\chi,\psi\right)\}$

The continuous dependence of solution $\{r\left(\psi\right), u\left(\chi,\psi\right)\}$ of the inverse problem (1)–(5) on the data is provided by the following theorem:

Theorem 2. *The solution of the problem (1)-(4) constantly depend on the functions f, φ and E under the assumptions (S1)-(S3)*

Proof. Let
$F = \left\{(\varphi, E, f) : \varphi \in C^2\left([0,\pi]\right), E\left(\psi\right) \in C^1\left[0,T\right] \textit{and absolutely continuous}, f\left(\chi,\psi\right) \in C^{2,0}\left(\Omega\right)\right\}$
and its closure
$\overline{F} = \left\{\left(\overline{\varphi}, \overline{E}, \overline{f}\right) : \overline{\varphi} \in C^2\left([0,\pi]\right), \overline{E}\left(\psi\right) \in C^1\left[0,T\right] \textit{and absolutely continuous}, \overline{f}\left(\chi,\psi\right) \in C^{2,0}\left(\Omega\right)\right\}$
be two sets of the data such that

$$\begin{aligned} &\|f\|_{C^1(\Omega)} \le M_1, \left\|\overline{f}\right\|_{C^1(\Omega)} \le M_1 \\ &\|\varphi\|_{C^2[0,\pi]} \le M_2, \|\overline{\varphi}\|_{C^2[0,\pi]} \le M_2 \\ &\|E\|_{C^1[0,T]} \le M_3, \left\|\overline{E}\right\|_{C^1[0,T]} \le M_3 \\ &0 < M_4 \le \min_{(\chi,\psi)\in\overline{\Omega}} |f\left(\chi,\psi\right)|, 0 < M_4 \le \min_{(\chi,\psi)\in\overline{\Omega}} \left|\overline{f}\left(\chi,\psi\right)\right| \end{aligned}$$

where $M_i > 0, i = 1,2,3,4$ are constants.
$\|\Phi\| = \left(\|\varphi\|_{C^2[0,\pi]} + \|E\|_{C^1[0,T]} + \|f\|_{C^{2,0}\left(\overline{\Omega}\right)}\right)$. Let $(u,r)\, and\, (\overline{u},\overline{r})$ be solutions of the problems (1)-(5) corresponding to the data (φ, E, f) and $\left(\overline{\varphi}, \overline{E}, \overline{f}\right)$.

$$\begin{aligned} u(\chi,\psi) - \overline{u}(\chi,\psi) &= \frac{1}{2}\left(\phi_0 - \overline{\phi_0}\right) + \sum_{k=1}^\infty cos\left(2k\chi\right)\left[\left(\vartheta_{ck} - \overline{\vartheta_{ck}}\right) E_{\alpha,1}\left(-\left(2k\right)^2\psi^\alpha\right)\right] \\ &+ \sum_{k=1}^\infty sin\left(2k\chi\right)\left[\left(\vartheta_{sk} - \overline{\vartheta_{sk}}\right) E_{\alpha,1}\left(-\left(2k\right)^2\psi^\alpha\right)\right] \\ &+ \sum_{k=1}^\infty cos\left(2k\chi\right)\int_0^\psi \left(r(\tau) - \overline{r(\tau)}\right) f_{ck}(\tau) E_{\alpha,1}\left(-\left(2k\right)^2\left(\psi-\tau\right)^\alpha\right) d\tau \end{aligned}$$

$$+\sum_{k=1}^{\infty} cos\left(2k\chi\right)\int_{0}^{\psi}\left(f_{ck}(\tau)-\overline{f_{ck}(\tau)}\right)\overline{r(\tau)}E_{\alpha,1}\left(-\left(2k\right)^{2}\left(\psi-\tau\right)^{\alpha}\right)d\tau$$
$$+\sum_{k=1}^{\infty} sin\left(2k\chi\right)\int_{0}^{\psi}\left(r(\tau)-\overline{r(\tau)}\right)f_{sk}(\tau)E_{\alpha,1}\left(-\left(2k\right)^{2}\left(\psi-\tau\right)^{\alpha}\right)d\tau$$
$$+\sum_{k=1}^{\infty} sin\left(2k\chi\right)\int_{0}^{\psi}\left(f_{sk}(\tau)-\overline{f_{sk}(\tau)}\right)\overline{r(\tau)}E_{\alpha,1}\left(-\left(2k\right)^{2}\left(\psi-\tau\right)^{\alpha}\right)d\tau.$$

Applying Cauchy, Bessel inequality and taking maksimum of $u-\overline{u}$ lead to the following inequality for all $\alpha\in(0,1]$

$$\left\|u-\overline{u}\right\|\le\frac{1}{2}\left\|\vartheta_0-\overline{\vartheta_0}\right\|+\sum_{k=1}^{\infty}\left\|\vartheta_{ck}-\overline{\vartheta_{ck}}\right\|+\left\|\vartheta_{sk}-\overline{\vartheta_{sk}}\right\|+\left(\frac{T^{\frac{\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2}\right)}+\frac{1}{4\sqrt{3}}\right)M\left\|r-\overline{r}\right\|+$$
$$\left(\frac{T^{\frac{\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2}\right)}+\frac{1}{4\sqrt{3}}\right)M\left\|f-\overline{f}\right\|$$

Similarly, for all $\alpha\in(0,1]$, the following inequalities for $\Phi-\overline{\Phi},K-\overline{K}$, and $r-\overline{r}$ are established as follows:

$$\left\|\Phi-\overline{\Phi}\right\|\le\frac{1}{M\Gamma\left(\frac{\alpha}{2}\right)}\left\|{}_0^CD_\psi^\alpha\left(E\left(\psi\right)\right)-\overline{{}_0^CD_\psi^\alpha\left(E\left(\psi\right)\right)}\right\|+\frac{\Gamma\left(\frac{\alpha}{2}\right)}{M}\sum_{k=1}^{\infty}\left\|\vartheta_{sk}-\overline{\vartheta_{sk}}\right\|,$$
$$\left\|K-\overline{K}\right\|\le\frac{\Gamma\left(\frac{\alpha}{2}\right)}{M}\left\|r-\overline{r}\right\|+\frac{\Gamma\left(\frac{\alpha}{2}\right)}{M}\left\|f-\overline{f}\right\|,$$
$$\left\|r-\overline{r}\right\|\le\left\|F-\overline{F}\right\|+\left\|K-\overline{K}\right\|\left\|\overline{r}\right\|+\left\|r-\overline{r}\right\|\left\|K\right\|.$$

Finally, we have the following

$$\left\|u-\overline{u}\right\|\le\left\|\vartheta-\overline{\vartheta}\right\|+\frac{1}{M}\left(\frac{T^{\frac{\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2}\right)}+\frac{1}{4\sqrt{3}}\right)\left\|{}_0^CD_\psi^\alpha\left(E\left(\psi\right)\right)-\overline{{}_0^CD_\psi^\alpha\left(E\left(\psi\right)\right)}\right\|$$
$$+\frac{\Gamma\left(\frac{\alpha}{2}\right)}{M}\left(\frac{T^{\frac{\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2}\right)}+\frac{1}{4\sqrt{3}}\right)\left\|f-\overline{f}\right\|$$
$$\left\|\vartheta-\overline{\vartheta}\right\|=\frac{1}{2}\left\|\vartheta_0-\overline{\vartheta_0}\right\|+\frac{\Gamma\left(\frac{\alpha}{2}\right)}{M}\sum_{k=1}^{\infty}\left\|\vartheta_{ck}-\overline{\vartheta_{ck}}\right\|+\left\|\vartheta_{sk}-\overline{\vartheta_{sk}}\right\|$$

which implies that $u\to\overline{u}$ as $\vartheta\to\overline{\vartheta},{}_0^CD_\psi^\alpha\left(E\left(\psi\right)\right)\to\overline{{}_0^CD_\psi^\alpha\left(E\left(\psi\right)\right)},f\to\overline{f}$. □

5. Demonstrative Example

Example 1.

$$\begin{aligned}&{}_0^CD_\psi^\alpha\left(u\left(\chi,\psi\right)\right)=u_{\chi\chi}\left(\chi,\psi\right)+r\left(\psi\right)f\left(\chi,\psi\right),\chi>0,0<\alpha\le1,\psi>0,\\&u\left(0,\psi\right)=u\left(\pi,\psi\right),\\&u_\chi\left(0,\psi\right)=u_\chi\left(\pi,\psi\right),\\&u\left(\chi,0\right)=\varphi\left(\chi\right)=1+\cos\left(2\chi\right),\\&E\left(\psi\right)=\int_0^\pi\chi u\left(\chi,\psi\right)d\chi=\frac{\pi^2}{2}e^{\psi^2}.\end{aligned}$$

where $f\left(\chi,\psi\right)=2\psi\ \left(1+\cos\left(2\chi\right)\right)+4\cos\left(2\chi\right)$. *The pair of functions for* $\alpha=1$ *is determined as* $u\left(\chi,\psi\right)=\left(1+\cos\left(2\chi\right)\right)e^{\psi^2},r\left(\psi\right)=e^{\psi^2}$. *Based on the proposed method, the approximation of unknown function* $r\left(\psi\right)$ *computed as* $r_j\left(\psi\right)=F_j\left(\psi\right)+\int_0^\pi K_j\left(\psi,\tau\right)r\left(\tau\right)d\tau,\psi\in[0,T]$ *where*

$$F_j\left(\psi\right)=\frac{{}_0^CD_\psi^\alpha\left(E\left(\psi\right)\right)+\frac{\pi}{2}\sum_{m=1}^{j}\frac{(2m)^2}{m}\varphi_{sm}E_{\alpha,1}\left(-\left(2m\right)^2\psi^\alpha\right)}{f_0\left(\psi\right)+\frac{\pi}{2}\sum_{m=1}^{j}\frac{(2m)^2}{m}f_{sm}\left(\psi\right)},$$

$$K_j(\psi,\tau)=\frac{\frac{\pi}{2}\sum_{m=1}^{j}\frac{(2m)^2}{m}f_{sm}(\psi)\,E_{\alpha,1}\left(-(2m)^2(\psi-\tau)^{\alpha}\right)}{f_0(\psi)+\frac{\pi}{2}\sum_{m=1}^{j}\frac{(2m)^2}{m}f_{sm}(\psi)}.$$

which leads to $r(\psi)$ *as* $j\to\infty$*:*
$r(\psi)=E_{\alpha,1}\left(\psi^2\right)$. *Moreover, the solution* $u(\chi,\psi)$ *is approximately computed as*

$$\varphi(\chi)=1+\cos(2\chi),$$

$$\varphi_0=\left(\frac{2}{\pi}\right)\int_0^{\pi}(1+\cos(2\chi))\,d\chi=2,$$

$$\varphi_c=\left(\frac{2}{\pi}\right)\int_0^{\pi}(1+\cos(2\chi))\cos(2k\chi)d\chi=\begin{cases}1, k=1\\0, k\neq 1\end{cases}$$

$$\varphi_s=\left(\frac{2}{\pi}\right)\int_0^{\pi}(1+\cos(2\chi))\sin(2k\chi)d\chi=0,$$

$$f(\chi,\psi)=2\psi\ \ (1+\cos(2\chi))+4\cos(2\chi),$$

$$f_0(\psi)=\left(\frac{2}{\pi}\right)\int_0^{\pi}(2\psi\ \ (1+\cos(2\chi))+4\cos(2\chi))\,d\chi=4\psi,$$

$$f_c(\psi)=\left(\frac{2}{\pi}\right)\int_0^{\pi}(2\psi\ \ (1+\cos(2\chi))+4\cos(2\chi))\cos(2k\chi)d\chi=2(2+\psi),$$

$$f_s(\psi)=\left(\frac{2}{\pi}\right)\int_0^{\pi}(2\psi\ \ (1+\cos(2\chi))+4\cos(2\chi))\sin(2k\chi)d\chi=0,$$

$$u_0(\psi)=\varphi_0+\left(\frac{2}{\pi}\right)\int_0^{\psi}\int_0^{\pi}r(\tau)\,f_0(\xi,\tau)\,d\xi d\tau$$

$$=2+2\psi^2\sum_{k=0}^{\infty}\frac{\left(\psi^2\right)^{2k}}{(1+2k)\Gamma(1+2k\alpha)}+\psi^4\sum_{k=0}^{\infty}\frac{\left(\psi^2\right)^{2k}}{(1+k)\Gamma(1+(1+2k)\alpha)},$$

$$u_{ck}(\psi)=\varphi_c(k)\,e^{-(2k)^2\psi}+\left(\frac{2}{\pi}\right)\int_0^{\psi}\int_0^{\pi}r(\tau)\,f(\xi,\tau)\cos(2k\xi)e^{-(2k)^2(\psi-\tau)}d\xi d\tau\approx\cos(2\chi)\left(1-\frac{4\psi}{\Gamma(1+\alpha)}\right.$$

$$+\frac{1}{210}\psi\left(210(4+\psi)-\frac{28\psi^3(10+3\psi)}{(\Gamma(1+\alpha))^2}+\frac{105\psi\left(-16+\psi^2\right)}{\Gamma(1+\alpha)}+\frac{8\psi^6(16+5\psi)}{(\Gamma(1+2\alpha))^2}+\right.$$

$$\left.\left.\frac{14\psi^2\left(320+40\psi+12\psi^2+5\psi^3\right)}{\Gamma(1+2\alpha)}+\frac{8\psi^4\left(56-5\psi^2\right)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right)+\frac{16\psi^2}{\Gamma(1+2\alpha)}\right)+\ldots,$$

$$u_{sk}(\psi)=\varphi_s(k)\,e^{-(2k)^2\psi}+\left(\frac{2}{\pi}\right)\int_0^{\psi}\int_0^{\pi}r(\tau)\,f(\xi,\tau)\sin(2k\xi)e^{-(2k)^2(\psi-\tau)}d\xi d\tau=\sin(2\chi)E_{\alpha,1}(-4\psi).$$

Finally, the solution $u(\chi,\psi)$ *can be established in Fourier series form as follows:*

$$u(\chi,\psi)=(cos(2\chi))^2\left(1-\frac{4\psi}{\Gamma(1+\alpha)}+\frac{1}{210}\psi\left(210(4+\psi)-\frac{28\psi^3(10+3\psi)}{(\Gamma(1+\alpha))^2}+\frac{105\psi\left(-16+\psi^2\right)}{\Gamma(1+\alpha)}\right.\right.$$
$$\left.\left.+\frac{8\psi^6(16+5\psi)}{(\Gamma(1+2\alpha))^2}+\frac{14\psi^2\left(320+40\psi+12\psi^2+5\psi^3\right)}{\Gamma(1+2\alpha)}+\frac{8\psi^4\left(56-5\psi^2\right)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right)+\frac{16\psi^2}{\Gamma(1+2\alpha)}\right)$$
$$+cos(2\chi)cos(4\chi)\left(1-\frac{4\psi}{\Gamma(1+\alpha)}+\frac{1}{210}\psi\left(210(4+\psi)-\frac{28\psi^3(10+3\psi)}{(\Gamma(1+\alpha))^2}+\frac{105\psi\left(-16+\psi^2\right)}{\Gamma(1+\alpha)}\right.\right.$$
$$\left.\left.+\frac{8\psi^6(16+5\psi)}{(\Gamma(1+2\alpha))^2}+\frac{14\psi^2\left(320+40\psi+12\psi^2+5\psi^3\right)}{\Gamma(1+2\alpha)}+\frac{8\psi^4\left(56-5\psi^2\right)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right)+\frac{16\psi^2}{\Gamma(1+2\alpha)}\right)$$
$$+cos(2\chi)sin(2\chi)\sum_{i=0}^{\infty}\frac{4^i(-\psi)^i}{\Gamma(1+i\alpha)}+cos(4\chi)sin(2\chi)\sum_{i=0}^{\infty}\frac{4^i(-\psi)^i}{\Gamma(1+i\alpha)}$$

$$+\frac{1}{2}\left(2+2\psi^2\sum_{k=0}^{\infty}\frac{\left(\psi^2\right)^{2k}}{(1+2k)\Gamma\left(1+2k\alpha\right)}+\psi^4\sum_{k=0}^{\infty}\frac{\left(\psi^2\right)^{2k}}{(1+k)\Gamma\left(1+(1+2k)\alpha\right)}\right)+\dots$$

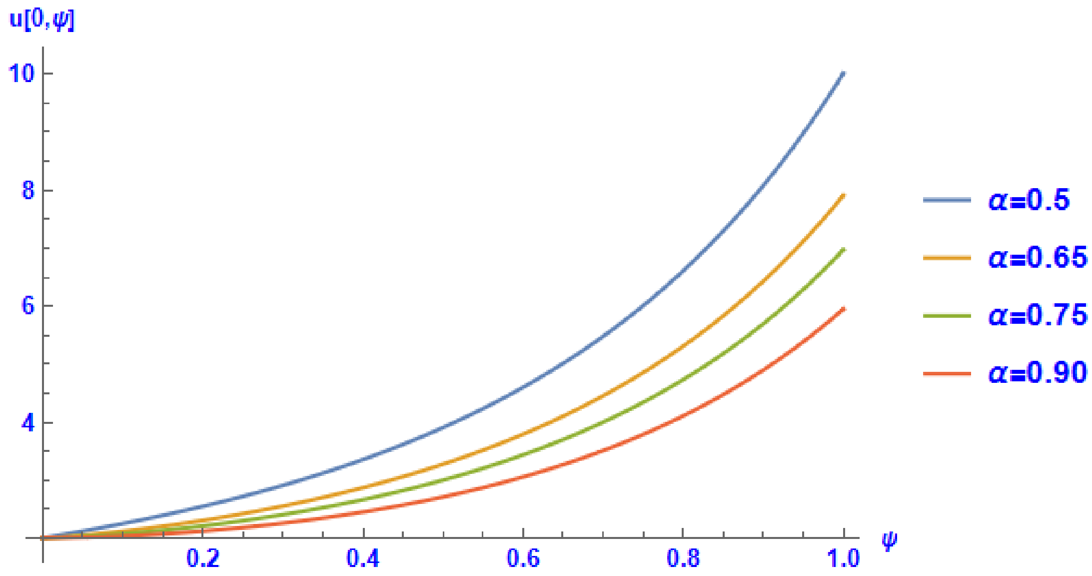

FIGURE 1. The graphs of the solution $u\left(\chi,\psi\right)$ for various values of α.

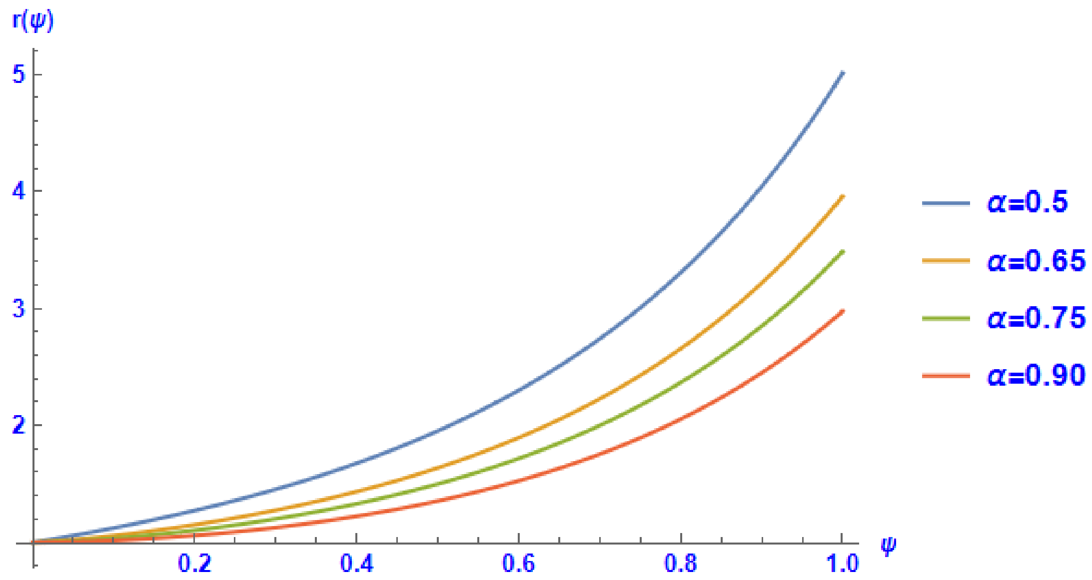

FIGURE 2. The graphs of the unknown function $r\left(\psi\right)$ for various values of α.

In Figure 1, the graphs of the truncated solution $u\left(\chi,\psi\right)$ are presented for different values of α. Moreover, the graphs of the unknown function $r\left(\psi\right)$ are illustrated for distinct values of α in Figure 2.

6. Conclusion and Future Works

The primary objective of this research is to determine the time dependent unknown function in a time fractional diffusion problem with periodic boundary conditions through generalized Fourier method. Overdetermined condition is utilized in the identification of time dependent function. Moreover, the solution of the time fractional diffusion problem is established in the series form. Furthermore, the existence, uniqueness and continuous dependence of data are achieved under certain regularity and consistency conditions. The obtained results are confirmed by an illustrative example.
In future studies, the suggested method will be utilized to construct the unknown function and solution for various fractional problems.

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References

[1] Baglan, I., Determination of a coefficient in a quasilinear parabolic equation with periodic boundary condition, *Inverse Probl. Sci. Eng.*, 23(5) (2015), 884–900.
[2] Cannon, J. R., The solution of the heat conduction subject to the specification of energy, *Q. Appl. Math.*, 21(2) (1963), 155–160.
[3] Choi, Y. S., Chan, K. Y. , A parabolic equation with nonlocal boundary conditions arising from electrochemistry, *Nonlinear Anal.*, 18(4) (1992), 317–331.
[4] Dehghan, M., Implicit locally one-dimensional methods for two-dimensional diffusion with a non-local boundary condition, *Math. Comput. Simul.*, 49(4-5) (1999), 331–349.
[5] Durdiev, D. K., Inverse coefficient problem for the time-fractional diffusion equation, *Eurasian J. Math. Comput. Appl.*, 9(1) (2021), 44–54.
[6] Durdiev, D. K., Rahmonov, A. A., Bozorov, Z. R., A two-dimensional diffusion coefficient determination problem for the time-fractional equation, *Math. Meth. Appl. Sci.*, 44(13) (2021), 10753–10761.
[7] Durdiev, D., Rahmanov, A., A multidimensional diffusion coefficient determination problem for the time-fractional equation, *Turk. J. Math.*, 46(6) (2022), 2250–2263.
[8] Durdiev, U. D., Problem of determining the reaction coefficient in a fractional diffusion equation, *Differ. Equ.*, 57 (2021), 1195–1204.
[9] Hill, G. W., On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon, *Acta Math.*, 8 (1886), 1–36.
[10] Ionkin, N. I., The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition, *Differentsial'nye Uravneniya*, 13(2) (1977), 294–304.
[11] Kanca, F., The inverse coefficient problem of the heat equation with periodic boundary and integral overdetermination conditions, *J. Inequal. Appl.*, 108 (2013).
[12] Kanca, F., Baglan, I., An inverse coefficient problem for a quasilinear parabolic equation with nonlocal boundary conditions, *Bound. Value Probl.*, 213 (2013).
[13] Kilbas, A. A., Srivastava, H. M., Trujillo, J. J., Theory and Applications of Fractional Differential Equations, Elsevier, 2006.
[14] Miller, L., Yamamoto, M., Coefficient inverse problem for a fractional diffusion equation, *Inverse Probl.*, 29(7) (2013).
[15] Mohebbi, A., A numerical algorithm for determination of a control parameter in two-dimensional parabolic inverse problems, *Acta Math. Appl. Sin.*, 31 (2015), 213–224.
[16] Podlubny, I., Fractional Differential Equations, Academic Press, 1999.
[17] Ramm, G., Mathematical and Analytical Techniques with Application to Engineering, Springer, 2005.
[18] Sharma, P. R., Methi, G., Solution of two-dimensional parabolic equation subject to non-local boundary conditions using homotopy perturbation method, *J. Appl. Comput. Sci. Math.*, 6 (2012), 64–68.
[19] Subhonova, Z. A., Rahmonov, A. A., Problem of determining the time dependent coefficient in the fractional diffusion-wave equation, *Lobachevskii J. Math.*, 42 (2021), 3747–3760.
[20] Wang, S., Lin, Y., A finite-difference solution to an inverse problem for determining a control function in a parabolic partial differential equation, *Inverse Prob.*, 5 (1989).
[21] Xiong, X., Guo, H., Liu, X., An inverse problem for a fractional diffusion equation, *J. Comput. Appl. Math.*, 236(17) (2012), 4474–4484.
[22] Xiong, X., Zhou, Q., Hon, Y. C., An inverse problem for fractional diffusion equation in 2-dimensional case: Stability analysis and regularization, *J. Math. Anal. Appl.*, 393(1) (2012), 185–199.