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Importance of Modelling the Dependence for Risk Capital Allocation

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Abstract

Portfolio managers' first concern is the accuracy of the measurement and the allocation of the risk of the portfolio. There exists many risk measures in the literature which provide a solution to the former problem. On the other hand, risk capital allocation provides an efficient portfolio management. It distributes the diversification benefits among the sub-portfolios. It is known that one of the important steps of the risk management is the determination of the dependence structure of sub-portfolios. Copula provides a nice and easy solution to this problem. In this study it is shown that the dependence structure plays an important role for risk capital allocation and inaccurate selection of copula can create ineffective allocations.

Keywords: *risk capital allocation, copula, risk measures, geometric Brownian motion*

Öz

Risk Sermayesi Dağıtımında Bağımlılık Modellemesinin Önemi

Portföy yöneticilerinin en önemli kaygılarından biri portföy risklerinin ölçümünün ve bu risklerin dağıtımının doğruluğudur. Literatürde birçok risk ölçümü ilk soruna çözüm üretmektedir. Diğer taraftan risk sermayesi dağıtımı, çeşitlendirmeden kaynaklanan faydaları portföye dağıtarak etkin bir portföy yönetimi sağlar. Bilindiği gibi risk yönetiminde en önemli adımlardan biri portföyün bağımlılık modellemesidir. Copula bağımlılık modellemesinde etkin ve kullanışlı bir yöntemdir. Bu çalışmada bağımlılık modellemesinin risk sermayesi dağıtımında çok önemli bir rol oynadığı ve uygun olmayan copula seçiminin etkin olmayan risk sermayesi dağıtımına yol açabileceği gösterilmiştir.

Anahtar sözcükler*: risk sermayesi dağıtımı, copula, risk ölçümleri, geometrik Brownian hareketi*

1. Introduction

Risk capital allocation is of special interest to the financial companies (which is referred by the term portfolios in this study). An important issue is the solvency of these companies. Risk capital is held to assure risk managers that the company stays solvent even if claims are larger than expected. Risk capital is basically determined by a risk measure. Many risk measures exist in the literature, commonly used ones are Value at Risk and expected shortfall.

On the other hand, risk capital allocation is mainly used for certain type of decisions as e.g. risk management, pricing, financial decisions concerning the investment problem [3]. The risk capital allocation in financial firms has already been discussed by several authors (see [1, 2, 7, 11, 12, 15, 16]). An important property of the allocation method is the coherency. The concept of coherent allocation of risk capital has been introduced by [2] which define some set of properties to be fulfilled by an allocation method.

Perhaps the most important point in risk management is the determination of the dependency structure of portfolios. This structure is directly affects the total return on portfolios and should be studied in detail. In a well diversified portfolio one can overcome bad market conditions and still can make profit. For the determination of the dependence between sub-portfolios, well known Markowitz model uses variance-covariance matrix. However, this setup can neglect the non-linear dependency structure between the sub-portfolios. Therefore, we need to use an approach that considers these types of structures. Copula methods are easy and tractable for these purposes.

The contribution of this study is the investigation of the effects of different dependency structures on a stock portfolio and their effects on the risk contributions of sub-portfolios. The layout of this study is as follows. Section 2 introduces the definitions of risk measures. Section 3 defines the risk capital allocation and a particular allocation method. Section 4 describes the copula approach and some particular copulas. Section 5 presents a case study which simulates a stock portfolio with different dependency structures and the comparison of allocations on these portfolios. The final section concludes.

2. Risk Measures

Given a probability space (Ω, F, P), we will consider the vector space $L^p(\Omega, F, P)$, for $1 \leq p \leq \infty$. We treat $L^p(P)$'s elements as random variables. We have $\| X \|_p = (E_P \mid X \mid^p)^{1/p}$. A risk measure, ρ, is a mapping from a set of random variables $L^p(P)$, $1 \leq p \leq \infty$ to the real line R, i.e.

$$\rho: L^p(P) \rightarrow R$$

$$X \rightarrow \rho(X)$$

An unacceptable risk can be transformed into an acceptable risk by adding other instruments to the position and the cost of this instrument (minimum cash) measures the risk of the position. In this study we only employed popular risk measures namely, Value at Risk and expected shortfall.

2.1. Value at Risk

Value at Risk (VaR) is based on standard portfolio theory which uses estimates of the standard deviations and correlations between the losses to different instruments. It measures the maximum potential loss of a given portfolio over a prescribed holding period at a given confidence level α where $\alpha \in (0, 1)$. For given confidence level α, VaR can be defined as,

$$VaR_\alpha(X) = -\inf\{x \in R: P(X \leq x) > \alpha\}. \tag{2.1}$$

VaR has many pros: it provides a common measure of risk across different positions, it is probabilistic and gives useful information on the probabilities associated with specified loss amounts and it can be expressed as 'lost money' [4].

However, VaR has a number of cons as well. VaR can not consider tail losses beyond the selected quantile. It can penalize the diversification in portfolios instead of rewarding it. Because of these shortcomings, a number of consistent risk measures have been introduced in the literature.

2.2. The Expected Shortfall

The expected shortfall (ES) at level α can be described as an average of VaRs at level α and higher. It is defined as in the following equation.

$$ES_\alpha(X) = -E[X \mid X \leq -VaR_\alpha(X)] \tag{2.2}$$

VaR gives only the potential loss amount in the 'bad' cases which happens with the probability α, whereas ES measures the average loss in these 'bad' cases. Moreover, expected shortfall belongs to the family of coherent risk measures (note that the coherency of risk measures are out of this papers' scope.) and it provides better approach for risk measurement. It is more sensitive to the shape of the return distribution and it counts tail of the distribution completely. For more details, see [10].

3. Risk Capital Allocation

Risk capital allocation can be used for many purposes. Firstly, by comparing different losses on capital for each portfolio, one can answer if a portfolio is worth to keep or not. Secondly, allocation provides a useful device for assessment of performance of managers, which can be linked to their compensations. Moreover, allocation can be used in pricing. The following definitions are adapted from [10].

Consider now that a portfolio has n sub-portfolios where $N=\{1,2,...,n\}$ is the set of all sub-portfolios. Each sub-portfolio's loss is represented by X_i, $i \in N$ then, aggregate loss of the portfolio can be described as in the following.

$$\sum_{i=1}^{n} X_i = X$$

In the literature, many researchers have proposed a set of axioms that any desirable allocation method is expected to satisfy [2, 8].

Let D be the set of risk capital allocation problems: pairs (N, ρ) composed of a set of n lines and a coherent risk measure ρ. Allocated capital for line i is denoted by a_i. An allocation is a functional Π: $D \rightarrow \mathbb{R}^n$ that maps each allocation problem, (N, ρ), into a unique allocation:

$$\begin{bmatrix} \Pi_1(N,\rho) \\ . \\ . \\ . \\ \Pi_n(N,\rho) \end{bmatrix} = \begin{bmatrix} a_1 \\ . \\ . \\ . \\ a_n \end{bmatrix} \tag{3.0}$$

3.1. Properties for a coherent allocation

An allocation Π is said to be coherent, if for every allocation problem (N, ρ) satisfies the following properties.

***Full Allocation*:** The allocated capitals add up to the total capital.

$$\rho(X) = \sum_{i=1}^{n} a_i$$

***No Undercut*:** The risk of any subset M of the total risk N is always lower than the sum of stand-alone risks of that subset.

$$\forall\, M \subseteq N,\ \sum_{i \in M} a_i \leq \rho\left(\sum_{i \in M} X_i\right)$$

Symmetry: For any subset $M \subseteq N \setminus \{i, j\}$, if sub-portfolios i and j make the same contribution to the risk capital of subset M, then $a_i = a_j$. This property ensures that a sub-portfolio's allocation depends only on its contribution to risk within the portfolio.

Riskless Allocation: Assume that last portfolio (line) is riskless with the initial price 1 and strictly positive price r in any state of nature at time T. Therefore, $X_n = \alpha r$ and

$$a_n = \rho(X_n) = \rho(\alpha r) = -\alpha$$

According to this axiom, a riskless portfolio should be allocated exactly its risk measure which can be negative. It is easy to see that this axiom is related to the translation invariance axiom of coherent risk measures.

3.2. Methods of Allocation

There are various allocation methods exist in the literature namely, variance-covariance method, proportional method, Merton-Perold method, The Shapley method, The Aumann-Shapley method and Euler method. Theoretical and practical aspects of these allocation methods have been analyzed in a number of papers [2, 6, 9, 10, 14, 16, 17]. However, in this particular study, we focus on the Euler method. The Euler allocation method has been suggested by several authors for different reasons:

- Tasche [1999] shows that the Euler principle is compatible with portfolio optimization / performance measurement for a positive homogeneous and differentiable risk measure.
- Denault [2001] derives the Euler allocation principle by game theory and argues that the allocation based on the Euler method is the unique fair allocation for a coherent risk measure.
- Myers and Read [2001] argues that in order to determine line by line surplus requirements effectively in an insurance company the most appropriate way is to apply the Euler principle.
- Kalkbrener [2005] argues that the Euler principle is the only allocation principle that is compatible with the diversification effects which plays an important role in the portfolio management.

For the reasons outlined above, we consider the Euler method as our only allocation method in this study.

3.2.1. Euler Method

Consider a function $\rho: L^p(P) \rightarrow R$, if ρ is positively homogeneous and differentiable at $u \in R^n$, then we have

$$\rho(X(u)) = \sum_{i=1}^{n} u_i \frac{\partial \rho(X(u))}{\partial u_i} \tag{3.1}$$

Under the Euler method the capital allocated to the sub-portfolio X_i of X is the derivative of associated risk measure ρ at X in direction of X_i. Note also that in this study all used risk measures are positive homogeneous and differentiable.

For the calculation of allocations under Euler principle we need partial derivatives of risk measures with respect to asset weights. Detailed information about partial derivatives of risk measures can be found in [9].

4. Copula

Copula methods have become very popular in modelling dependency and they provide a flexible way to express joint distributions of random variables. A d-dimensional *copula* is a distribution function, defined on $[0,1]^d$ with standard uniform marginals. It combines univariate distributions to obtain a joint distribution with a particular dependence structure. The foundation theorem for copulas, Sklar's theorem, states that for a given joint multivariate distribution function and relevant marginal distributions, there exists a copula function which relates them.

Sklar's Theorem (Bivariate Case)
For ease of notation take d=2. Let F_{XY} be a joint distribution with margins F_X and F_Y, then there exists a function $C: [0,1]^2 \rightarrow [0,1]$ such that

$$F_{XY}(x,y) = P(X \le x, Y \le y) = C(F_X(x), F_Y(y))$$

If X and Y are continuous, then C is unique. On the other hand, if C is a copula and F_X and F_Y are distribution functions, then the function F_{XY} defined by above equation is a joint distribution function with margins F_X and F_Y, see [13].

Sklar's theorem allows separating the marginal feature and the dependence structure which is represented by the copula. The function C is the cumulative distribution function of the pair (U, V) where $U= F_X(X)$ and $V= F_Y(Y)$, and

$$c(u,v) = \frac{\partial^2 C}{\partial u \partial v}(u,v)$$

is the associated probability density function. Sklar's theorem proves the existence and uniqueness of the copula. At the same time it shows how to construct the copula from the initial distribution. The copula is given by

$$C(u,v) = F_{XY}(F_X^{-1}(u),\, F_Y^{-1}(v))$$

where $F_X^{-1}(u)$ and $F_Y^{-1}(v)$ are the inverse functions with $0 \le u, v \le 1$. Using a copula to build multivariate distributions is efficient technique, because it gives flexibility of choosing different marginals and the derived multivariate distribution contains the information about the dependence structure of its components. The simplest copula types are independence, comonotony (for extreme positive dependence) and counter-comonotony (for extreme negative dependence). More advanced ones are called elliptical and Archimedean copulas. For more information about copula see [18].

For this particular study we employed Gaussian and t copula which are the members of elliptical copula family. The class of elliptical distributions provides many multivariate distributions which enables modelling of multivariate extremes and other type of non-normal dependences.

Gaussian Copula
Gaussian copula is defined by,

$$C^{Ga}(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} exp\left(-\frac{1}{2}\frac{t^2 - 2\rho tz + z^2}{1-\rho^2}\right) dtdz \tag{4.0}$$

where $\rho \in [-1,1]$ is linear correlation coefficient of corresponding bivariate normal distribution. Gaussian copula is useful for its easy simulation method. However, it does not have tail dependence on both tails.

t-Copula
t-copula is defined by,

$$C_\nu^t(u,v) = \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left(1 + \frac{t^2 - 2\rho tz + z^2}{\nu(1-\rho^2)}\right)^{-(\nu+2)/2} dtdz \tag{4.1}$$

where ρ is linear correlation coefficient of corresponding bivariate t_{υ} distribution for $\upsilon > 2$. This copula has symmetric tail dependency and degree of tail dependency can be set by degrees of freedom parameter, for more information see [5].

5. An Application to a Stock Portfolio

We adapted the same scenario and the same data with [10] for the simulation study. In this empirical study we used a stock portfolio of five companies and monthly data, which covers the period from February 2003 to February 2012, is taken from the website of the Yahoo Finance. We assume that given total wealth is 1,000,000 TL which is equally weighted to the sub-portfolios. We consider one-period framework; therefore between time T=0 and T=1 no trading is possible. We assume 'risk' to be given by a random variable X representing a cash flow at time T=1 such as $X = X(1) - X(0)$. Individual stock returns are modelled by geometric Brownian motions (GBM), i.e. price processes X_i (i = 1,...,5) with

$$X_i(t) = X_i(0)\exp\left((\mu_i - \sigma_i^2/2)t + \sigma_i\sqrt{t}Z\right) \qquad t \geq 0 \tag{5.0}$$

where μ_i is the drift, σ_i is the diffusion coefficient, $X_i(0)$ is the price of the i-th asset at time 0 and Z is a standard normally distributed random variable.

We compute N=100,000 simulations and by using these realizations we can compute estimates using the empirical distribution given by the simulation output. For simulation we need discretization. For more details about discretization, see [14]. Note that used time-step is 1/12 (prices are monthly) for discretization. Estimated geometric Brownian motion parameters are given in Table 1. Estimated covariance matrix of hypothetical historical increments of underlying Brownian motions is given in Table 2.

Dependency structure is modelled by three different setups namely, standard covariance based setup, Gaussian copula setup and t copula setup. For copula setups model parameters are estimated with copula fitting methods. According to the estimation results which is based on the maximum likelihood method, t-copula (loglikelihood: **37.646**) fits data better than the Gaussian copula (loglikelihood: **32.703**).

Table 1: Parameters of Stock Return Distributions

Parameters	BP Ltd	GSK Ltd	PRU Ltd	TOMK Plc	TSCO Plc
Drift μ	0.093	0.012	0.112	0.016	**0.151**
Diffusion σ	0.197	0.172	**0.356**	0.319	0.207

Table 2: Correlation matrix of hypothetical historical increments of underlying Brownian motions.

	BP Ltd	GSK Ltd	PRU Ltd	TOMK Plc	TSCO Plc
BP Ltd	1.0000	0.1884	0.2279	0.1447	0.1753
GSK Ltd	0.1884	1.0000	0.2996	0.2189	0.4457
PRU Ltd	0.2279	0.2996	1.0000	0.5480	0.4292
TOMK Plc	0.1447	0.2189	0.5480	1.0000	0.3034
TSCO Plc	0.1753	0.4457	0.4292	0.3034	1.0000

According to the parameter estimates in Table 1, TSCO Plc has the highest drift and PRU Ltd has the highest variation. Table 2 shows that the correlations between different stocks distributed between 0.15

and 0.55. Table 3 shows the distributions of log-returns of the stocks. PRU Ltd shows a skewed distribution compared to the other stocks. All stocks have a long tail which indicates that the risk of the stocks might be high.

Table 3: Histograms of log-returns

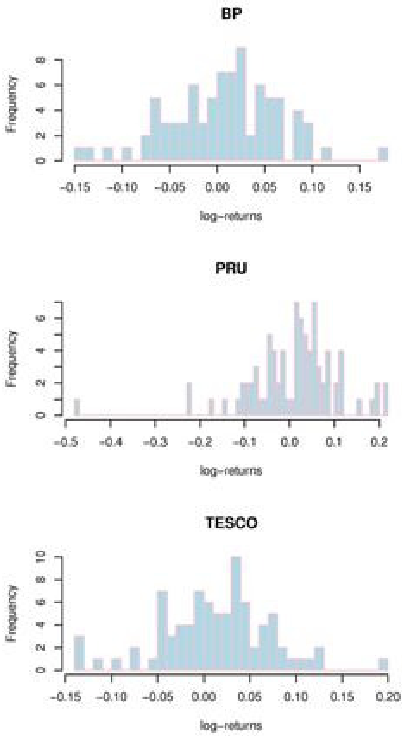
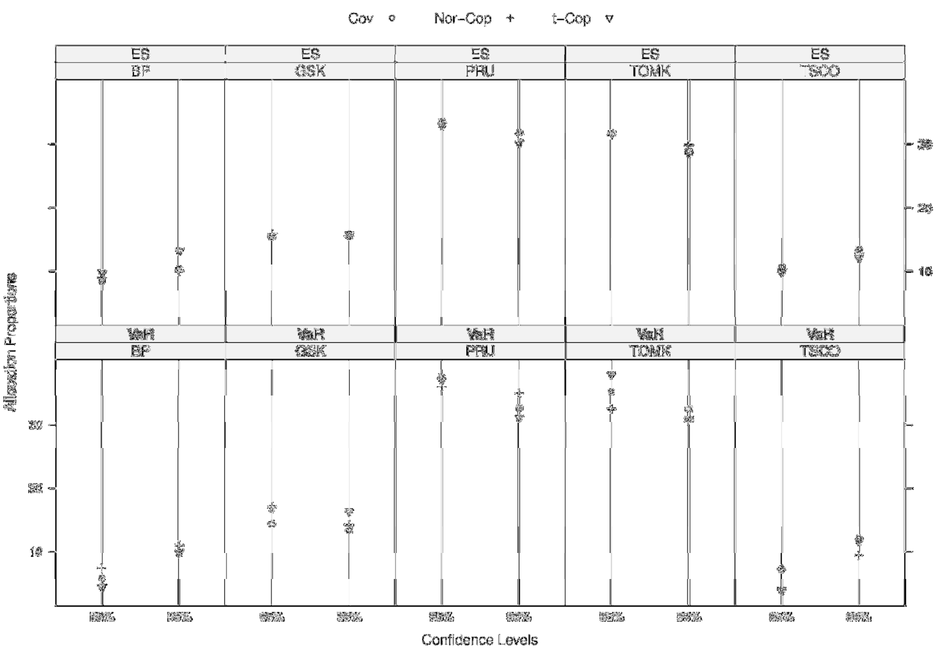


Table 4: Allocation Proportions to Sub-Portfolios for Different Dependency Setups and Confidence Levels



Allocation proportions of stock portfolio based on two most commonly used risk measures are given in Table 4. TOMK Plc. and PRU Ltd. turn out the most risky portfolios whereas BP Ltd. and TSCO Plc. seem less risky.

Allocations based on different dependency setups indicate that allocations show significant differences. By considering the fact that VaR is much sensitive to the confidence level, we can expect that these differences become more considerable for higher confidence levels.

Especially, for Value at Risk allocations can be very different for varios type of dependency structure. The main reason for this is the tail distributions of the stocks. For different confidence levels allocations vary dramatically.

On the other hand, according to the results with respect to the expected shortfall, variation in the allocations seems low. This fact results from the coherency of the expected shortfall.

Because of the skewed and long tailed shape of the log-return distributions, t-copula fits better than other choices. Therefore, allocations stem from the t-copula setup should be fair.

6. Conclusions

In this study we have analysed the effects of different dependency structures on risk capital allocations by using two commonly used risk measures namely, Value at Risk and expected shortfall. We employed the Euler method for the risk capital allocation. We found that dependency setups can change the allocations dramatically. Especially for long tailed distributions these changes can be high. Portfolio managers need to analyse the dependency on portfolios in detail and should choose an accurate models in order to get efficient and accurate results in risk management process.

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