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On a Simple Unreferenced Distribution with Support $[-1, 1]$

Christophe Chesneau*

Abstract

The notion of continuous distribution is central to the theory of probability and statistics. In this article, we contribute to the field by studying a simple, but not yet referenced, one-parameter distribution with support $[-1, 1]$. It can be seen as a modified version of the logistic distribution on the interval $[-1, 1]$, and also as a ratio-exponential-type generalization of the uniform distribution over $[-1, 1]$. For a given value of the parameter, the corresponding probability density function reduces to the famous logistic function (or sigmoid), which is widely used in statistics, machine learning, and neural networks. From a more general point of view, this probability density function has the original property of always being monotonic with S or Ƨ forms. Some additional properties are examined, including the quantile, moment and distribution generation properties. Negation and opposite versions of the distribution are considered, for the first time in the context of distributions with support $[-1, 1]$. The article concludes with a brief numerical study, part of which is devoted to the maximum likelihood estimation of the unique parameter involved. It also includes the analysis of score-type data. To ensure transparency, reproducibility and applicability in concrete real-world scenarios, the main codes written in R are included in Appendix.

Keywords: *Probability, probability density function, logistic function, logistic distribution, uniform distribution, statistical modeling, score-type data, negation distributions, opposite distributions*

* *Corresponding author*

1. Introduction

The field of probability is rich in continuous distributions, each with different properties that make it suitable for modeling a wide range of real-world phenomena. These distributions differ primarily in their support, i.e., the set of possible values that the associated random variables can take, the overall form (or shape) of their probability density functions, the number of parameters they involve, and their practical applications. Details on this topic can be found in Balakrishnan and Nevzorov (2004); Cordeiro, Silva, and Nascimento (2020); Johnson, Kotz, and Balakrishnan (1995); Kleiber and Kotz (2003). The construction of adaptive continuous distributions has always been an active area of research, driven by the goal of identifying the most appropriate distributions for complex and challenging scenarios. Examples of applications include modeling extreme events in finance and insurance, capturing heavy-tailed behavior in risk analysis, improving reliability assessments in engineering, refining survival models in medicine, and enhancing machine learning algorithms with flexible probabilistic models. We refer to the books Bishop (2016); Embrechts, Kluppelberg, and Mikosch (1997); Klein and Moeschberger (2003); McNeil, Frey, and Embrechts (2015); Murphy (2012); Resnick (2007); Taleb (2020).

As a logical consequence of a thousand years of research, the development of new continuous distributions that successfully balance mathematical simplicity with innovative features has become increasingly rare. This challenge is particularly pronounced for distributions defined on a bounded support, such as the interval $[-1, 1]$, which imposes additional mathematical constraints. To give a brief state of the art, a list of twelve known or

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intuitive distributions with support $[-1, 1]$ is presented in Table 1. They are defined by their probability density functions, denoted $f(x)$.

Table 1. List of selected known or intuitive distributions with support $[-1, 1]$

Distribution	Probability density function
Uniform	$f(x) = \frac{1}{2}, \quad x \in [-1, 1]$
Triangular	$f(x) = 1 - \lvert x\rvert, \quad x \in [-1, 1]$
Epanechnikov	$f(x) = \frac{3}{4}(1 - x^2), \quad x \in [-1, 1]$
Linear	$f(x) = \frac{1}{2}(1 + x), \quad x \in [-1, 1]$
Quartic	$f(x) = \frac{15}{16}(1 - x^2)^2, \quad x \in [-1, 1]$
Logarithmic	$f(x) = \frac{1}{2[2\log(2) - 1]} \log(1 + \lvert x\rvert), \quad x \in [-1, 1]$
Exponential	$f(x) = \frac{1}{2(1 - e^{-1})} e^{-\lvert x\rvert}, \quad x \in [-1, 1]$
Wigner Semicircle	$f(x) = \frac{2}{\pi}\sqrt{1 - x^2}, \quad x \in [-1, 1]$
Arcsine	$f(x) = \frac{1}{\pi\sqrt{1 - x^2}}, \quad x \in [-1, 1]$
Cauchy truncated	$f(x) = \frac{2}{\pi(1 + x^2)}, \quad x \in [-1, 1]$
Raised cosine	$f(x) = \frac{1}{2} + \frac{1}{\pi}\cos(\pi x), \quad x \in [-1, 1]$
Modified cosine	$f(x) = \frac{\pi}{4}\cos\left(\frac{\pi x}{2}\right), \quad x \in [-1, 1]$

The definition of the support implies that $f(x) = 0$ for any $x \notin [-1, 1]$. The theoretical properties associated with these distributions are readily available. We refer again to Balakrishnan and Nevzorov (2004); Cordeiro et al. (2020); Johnson et al. (1995); Kleiber and Kotz (2003). From a practical point of view, they are mainly designed to analyze data with values in $[-1, 1]$. Such data are common in fields such as finance (as scaled returns), psychology (as standardized test scores), engineering (as signal processing), and environmental science (as relative pollution levels). There has been a resurgence of interest in these distributions in recent years, mainly in extensions and modifications of the raised cosine distribution. The latest developments can be found in Ahsanullah and Shakil (2018); Ahsanullah, Shakil, and Kibria (2019); Chesneau (2024a, 2024b); El-Shehawy and Rizk (2024); Kyurkchiev and Kyurkchiev (2016); Watagoda, Rupasinghe Arachchige Don, and Sanqui (2019).

New distributions with support $[-1, 1]$ are always welcome, as they may exhibit unique behavior that cannot be captured by existing distributions. Exploring them opens up theoretical and practical possibilities. With this in mind, in this article, we present and study an unreferenced distribution with support $[-1, 1]$. It is defined by a simple and original one-parameter probability density function that mixes simple ratio and exponential functions. We call it the ratio-exponential (RE) distribution. The RE distribution can be interpreted as a modified and restricted version of the logistic distribution over the interval $[-1, 1]$, with the property of being unconstrained by the values of the unique parameter involved. It can also be seen as a ratio-exponential generalization of the uniform distribution on the same support. In terms of form properties, the corresponding probability density function exhibits monotonic patterns that are close to those observed for a cumulative distribution function and its vertically symmetric image counterpart. In particular, when the value of the parameter is -1, this function corresponds to the famous logistic function (or sigmoid) restricted to the interval $[-1, 1]$, which is central to many applications, including machine learning (as an activation function in neural networks), statistics (in logistic regression models), and physics (for modeling growth and diffusion processes). See Bishop (2016); Han and Moraga (1995); Hosmer and Lemeshow (2000); Murphy (2012). The monotonic nature of this probability density function and the flexibility of the parameter values make the RE distribution ideal for analyzing data with smooth increasing or decreasing patterns.

In addition, the RE distribution is mathematically simple and transparent. Several of its properties can be described analytically, including the expressions of key functions and measures related to the quantile function and moments. We can also use the RE distribution to create new distributions through a simple composition method. This is illustrated in the context of distributions with support $[0, 1]$ (also called unit distributions). Such distributions are of particular interest because they are constructed to analyze proportions, probabilities, and bounded values, which are fundamental in various domains, including economics (as market shares and growth rates), biology (as population proportions and genetic probabilities), and machine learning (as classification probabilities and performance metrics). A well-designed distribution on this interval must balance mathematical tractability, interpretability, and flexibility to accommodate different data behaviors, such as skewness or flatness. Recent progress on this topic can be found in Altun, Hamedani, and Fazli (2024); Chesneau (2023); Korkmaz (2020); Korkmaz and Korkmaz (2021); Mazucheli, Menezes, and Chakraborty (2019); Mazucheli, Menezes, and Dey (2019); Sarhan and Sobh (2025). We also determine the "negation counterpart" of the RE distribution, using the methodology introduced in Chesneau (2024c). Inspired by the negation scheme, we also develop the notion of opposite distribution, which can be adapted to any distributions with support $[-1, 1]$. These theoretical aspects are complemented by numerical studies focusing on the maximum likelihood estimation of the single parameter characterizing the distribution. An example of the analysis of score-type data analysis is also proposed as an application. For all graphical and numerical work, including data analysis, the free scientific software R is used. See R Core Team (2021).

The other sections are as follows: Section 2 introduces the RE distribution, focusing on its probability density function. The cumulative distribution and quantile functions are developed in Section 3. Section 4 is devoted to some moment properties. Section 5 deals with the creation of new distributions based on the RE distribution. Numerical studies are given in Section 6. A conclusion is proposed in Section 7. An Appendix contains the R codes of the main statistical analysis.

2. Definition and Main Characteristics

This section introduces the RE distribution, as well as its main characteristics.

2.1 The RE distribution

At the heart of the definition of RE distribution is the proposition below.

Proposition 2.1. *Let $\alpha \in \mathbb{R}$. Let us consider the following function:*

$$f(x; \alpha) = \frac{1}{1 + e^{\alpha x}}, \quad x \in [-1, 1], \tag{2.1}$$

completed by $f(x; \alpha) = 0$ for any $x \notin [-1, 1]$. Then $f(x; \alpha)$ is a valid probability density function.

Proof. We need to check that $f(x; \alpha)$ satisfies the necessary assumptions to be a probability density function. Since $e^{\alpha x} \geq 0$, it is clear that $f(x; \alpha) \geq 0$ for any $x \in \mathbb{R}$. Let us now verify the integral unitary property, i.e., $\int_{-\infty}^{+\infty} f(x; \alpha)dx = 1$, distinguishing the cases $\alpha = 0$ and $\alpha \in \mathbb{R}\backslash\{0\}$. For $\alpha = 0$, we have $f(x; \alpha) = 1/(1+e^0) = 1/2$ for any $x \in [-1, 1]$, so that

$$\int_{-\infty}^{+\infty} f(x; \alpha)dx = \int_{-1}^{1} \frac{1}{2}dx = \frac{1}{2}\int_{-1}^{1} dx = 1.$$

For $\alpha \in \mathbb{R}\backslash\{0\}$, we have

$$\int_{-\infty}^{+\infty} f(x;\alpha)dx = \int_{-1}^{1} \frac{1}{1+e^{\alpha x}}dx = \int_{-1}^{1} \frac{e^{-\alpha x}}{1+e^{-\alpha x}}dx = -\frac{1}{\alpha}\left[\log(1+e^{-\alpha x})\right]_{x=-1}^{x=1}$$
$$= -\frac{1}{\alpha}\left[\log(1+e^{-\alpha}) - \log(1+e^{\alpha})\right] = -\frac{1}{\alpha}\left\{\log[e^{-\alpha}(1+e^{\alpha})] - \log(1+e^{\alpha})\right\}$$
$$= -\frac{1}{\alpha}\left[-\alpha + \log(1+e^{\alpha}) - \log(1+e^{\alpha})\right] = -\frac{1}{\alpha}(-\alpha) = 1.$$

As a result, $f(x;\alpha)$ is a valid probability density function. This concludes the proof. □

We therefore call the RE distribution (of parameter α) the distribution associated with the probability density function given in Equation (2.1). Here and in the following, $f(x;\alpha)$ refers to the expression in Equation (2.1). From a probabilistic point of view, a random variable X with the RE distribution satisfies

$$\mathbb{P}(X \in \mathcal{A}) = \int_{\mathcal{A}} f(x;\alpha)dx = \int_{\mathcal{A}\cap[-1,1]} \frac{1}{1+e^{\alpha x}}dx,$$

where $\mathbb{P}$ denotes the probability operator and $\mathcal{A}$ is an arbitrary subset of $\mathbb{R}$.

As mentioned in the introduction, the RE distribution has the support $[-1,1]$ and depends on only one parameter, α, which has the property of being unbounded: we have $\alpha \in \mathbb{R}$. To the best of our knowledge, the RE distribution is the only distribution with support $[-1,1]$ with such an "unbounded" single parameter that modulates a ratio-exponential functionality in its probability density function. This, combined with a certain mathematical simplicity, makes it particularly attractive from a theoretical and practical point of view.

Furthermore, for the special case $\alpha = 0$, the associated probability density function is reduced to

$$f(x;\alpha) = \frac{1}{2}, \quad x \in [-1,1],$$

completed by $f(x;\alpha) = 0$ for any $x \notin [-1,1]$. This corresponds to the probability density function of the uniform distribution over $[-1,1]$. In this sense, the RE distribution can be seen as a generalization.

There are also strong links between the RE and logistic distributions, as described in the proposition below.

Proposition 2.2. *Some special features of the probability density function of the RE distribution are presented below.*

1. *For $\alpha \in [-1,0)$, $f(x;\alpha)$ corresponds to the cumulative distribution function associated with the logistic distribution with parameter $-\alpha$ restricted to the interval $[-1,1]$.*
2. *For any $x \in [-1,1]$ and $\alpha \in [-1,1]$, we have $f(x;\alpha) = f(\alpha;x)$.*
3. *For any $x \in [-1,0)$, the function $G(\alpha;x) = f(x;\alpha)$, $\alpha \in \mathbb{R}$, is a cumulative distribution function (with respect to α).*

Proof.

1. For any $\lambda > 0$, we recall that the cumulative distribution function associated with the logistic distribution with parameter λ is given by
$$L(x;\lambda) = \frac{1}{1+e^{-\lambda x}}, \quad x \in \mathbb{R}.$$
See, for instance, Johnson et al. (1995). Based on this definition, for $\alpha \in [-1,0)$, we find that $f(x;\alpha) = L(x;-\alpha)|_{x\in[-1,1]}$, completed with $f(x;\alpha) = 0$ for any $x \notin [-1,1]$. It thus corresponds to the cumulative distribution function associated with the logistic distribution with parameter $-\alpha$ restricted to the interval $[-1,1]$.

2. For any $x \in [-1,1]$ and $\alpha \in [-1,1]$, we clearly have
$$f(x;\alpha) = \frac{1}{1+e^{\alpha x}} = \frac{1}{1+e^{x\alpha}} = f(\alpha;x).$$

3. For any $x \in [-1, 0)$, we note that

$$G(\alpha; x) = f(x; \alpha) = \frac{1}{1 + e^{\alpha x}} = L(\alpha; -x), \quad \alpha \in \mathbb{R},$$

which is the cumulative distribution function associated with the logistic distribution with parameter $-x$. A less direct approach is to check the assumptions necessary to be a cumulative distribution function. For any $x \in [-1, 0)$, we have

$$\lim_{\alpha \to -\infty} G(x; \alpha) = \lim_{\alpha \to -\infty} \frac{1}{1 + e^{\alpha x}} = 0, \quad \lim_{\alpha \to +\infty} G(x; \alpha) = \lim_{\alpha \to +\infty} \frac{1}{1 + e^{\alpha x}} = 1$$

and

$$\frac{\partial}{\partial \alpha} G(x; \alpha) = \frac{\partial}{\partial \alpha} \left(\frac{1}{1 + e^{\alpha x}} \right) = -x \frac{e^{\alpha x}}{(1 + e^{\alpha x})^2} \geq 0,$$

meaning that $G(x; \alpha)$ is increasing with respect to α. Therefore, $G(\alpha; x)$, $\alpha \in \mathbb{R}$, is a valid cumulative distribution function.

This completes the proof. □

In addition to that, for $\alpha = -1$, we have

$$f(x; \alpha) = \frac{1}{1 + e^{-x}}, \quad x \in [-1, 1],$$

which corresponds to the famous logistic function restricted to the interval $[-1, 1]$. It has a characteristic S form, which is particularly useful in situations where the data are expected to exhibit a gradual transition between two states or categories. Applications include binary classification in machine learning, population growth models in biology, and cumulative probability modeling in economics and social sciences. We refer again to Han and Moraga (1995); Hosmer and Lemeshow (2000).

From a more theoretical point of view, we can express the probability density function of the RE distribution in terms of the hyperbolic tangent function as follows:

$$f(x; \alpha) = \frac{1}{2} \left[1 - \tanh\left(\frac{\alpha}{2} x \right) \right], \quad x \in [-1, 1],$$

where we recall that $\tanh(x) = (e^x - e^{-x})/(e^x + e^{-x})$, $x \in \mathbb{R}$. From this expression, the RE distribution appears to be an original hyperbolic distribution with support $[-1, 1]$.

All these aspects are reasons to investigate the RE distribution further. We start by examining the shapes of its probability density function in the subsection below, highlighting the role of the parameter α.

2.2 Forms of the probability density function

The probability density function of the RE distribution satisfies the following basic properties: $f(0; \alpha) = 1/2$, which is independent of α, and

$$\lim_{x \to -1} f(x; \alpha) = \lim_{x \to -1} \frac{1}{1 + e^{\alpha x}} = \frac{1}{1 + e^{-\alpha}}, \quad \lim_{x \to 1} f(x; \alpha) = \lim_{x \to 1} \frac{1}{1 + e^{\alpha x}} = \frac{1}{1 + e^{\alpha}}.$$

Furthermore, it is not symmetric for $\alpha \in \mathbb{R} \backslash \{0\}$; we can always find $x_* \in [-1, 1]$ such that

$$f(x_*; \alpha) = \frac{1}{1 + e^{\alpha x_*}} \neq \frac{1}{1 + e^{-\alpha x_*}} = f(-x_*; \alpha).$$

With these elements, and those in Proposition 2.2, we can get an idea of the possible forms of $f(x; \alpha)$. This idea is refined in the proposition below, with the study of the combined monotonicity and convexity properties.

Proposition 2.3. *The forms of the probability density function of the RE distribution are clearly identifiable, as described below.*

- *If $\alpha < 0$, then $f(x; \alpha)$ is increasing, and convex for $x \in [-1, 0]$, then concave for $x \in [0, 1]$.*

- *If* $\alpha > 0$*, then* $f(x;\alpha)$ *is decreasing, and concave for* $x \in [-1,0]$*, then convex for* $x \in [0,1]$.

Proof. We study the signs of the following first and second derivatives of $f(x;\alpha)$ with respect to x: $f'(x;\alpha)$ and $f''(x;\alpha)$. We have

$$f'(x;\alpha) = \left(\frac{1}{1+e^{\alpha x}}\right)' = -\alpha\frac{e^{\alpha x}}{(1+e^{\alpha x})^2}$$

and

$$f''(x;\alpha) = \left[-\alpha\frac{e^{\alpha x}}{(1+e^{\alpha x})^2}\right]' = (e^{\alpha x}-1)\frac{\alpha^2 e^{\alpha x}}{(e^{\alpha x}+1)^3}.$$

For $f'(x;\alpha)$, the sign is governed by $-\alpha$ and, for $f''(x;\alpha)$, the sign is governed by $e^{\alpha x}-1$, the remaining multiplicative terms being positive.

- We first consider the case $\alpha < 0$. We have $-\alpha > 0$ so that $f'(x;\alpha) \geq 0$, meaning that $f(x;\alpha)$ is increasing. Furthermore, for $x \in [-1,0]$, we have $e^{\alpha x}-1 \geq 0$ so that $f''(x;\alpha) \geq 0$, meaning that $f(x;\alpha)$ is convex, and, for $x \in [0,1]$, we have $e^{\alpha x}-1 \leq 0$, so that $f''(x;\alpha) \leq 0$, meaning that $f(x;\alpha)$ is concave.
- We now consider the case $\alpha > 0$. We have $-\alpha < 0$ so that $f'(x;\alpha) \leq 0$, meaning that $f(x;\alpha)$ is decreasing. Furthermore, for $x \in [-1,0]$, we have $e^{\alpha x}-1 \leq 0$ so that $f''(x;\alpha) \leq 0$, meaning that $f(x;\alpha)$ is concave, and, for $x \in [0,1]$, we have $e^{\alpha x}-1 \geq 0$, so that $f''(x;\alpha) \geq 0$, meaning that $f(x;\alpha)$ is convex.

This completes the proof. □

The simplicity of the form of the probability density function of the RE distribution is clearly an advantage for further analysis.

Figure 1 illustrates Proposition 2.3 by showing some forms of $f(x;\alpha)$ for $\alpha = -10, -3, -0.5, 0.5, 3$ and 10. We use R with the basic function `curve`.

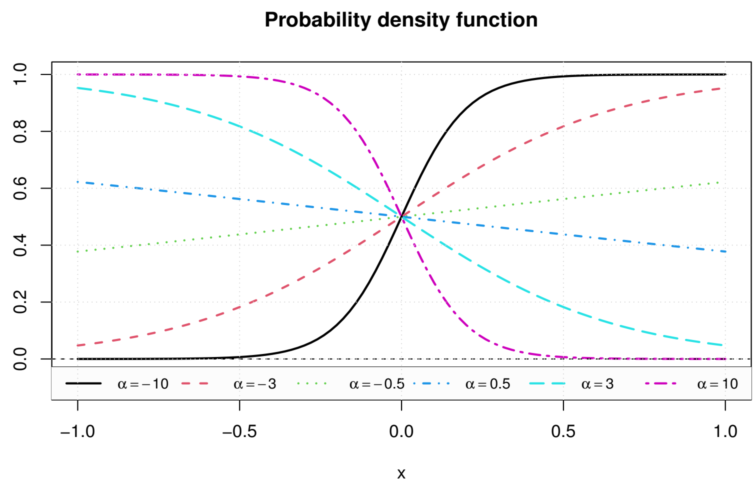


Figure 1. Sample of forms for $f(x;\alpha)$ for various values of α

It can be seen that $f(x;\alpha)$ has the S form for $\alpha = -10, -3$ and -0.5, i.e., it is is increasing and "convex then concave", and that $f(x;\alpha)$ has the Ƨ form for $\alpha = 0.5, 3$ and 10, i.e., it is is decreasing and "concave then convex", as expected. This also supports a claim made in the introduction; we see that the probability density function of the RE distribution exhibits decreasing or increasing patterns that are close to what can be observed for a cumulative distribution function and its vertically symmetric image counterpart.

The next section focuses on two other functions that are important for any distribution: the cumulative distribution and quantile functions.

3. Cumulative Distribution and Quantile Functions

The cumulative distribution and quantile functions of the RE distribution need to be examined for a full understanding, which is done in this section.

3.1 Cumulative distribution function

From a probabilistic point of view, the cumulative distribution function associated with the RE distribution is defined $F(x;\alpha) = \mathbb{P}\{X \in (-\infty, x]\}$, $x \in \mathbb{R}$, where X is a random variable with the RE distribution, so that

$$F(x;\alpha) = \int_{-\infty}^{x} f(t;\alpha)dt, \quad x \in \mathbb{R}.$$

As described in the proposition below, it has the advantage of being simple and in closed form.

Proposition 3.1. *For $\alpha \in \mathbb{R}\backslash\{0\}$, the cumulative distribution function associated with the RE distribution is given by*

$$F(x;\alpha) = \frac{1}{\alpha}\log\left(\frac{1+e^{\alpha}}{1+e^{-\alpha x}}\right), \quad x \in [-1,1], \tag{3.1}$$

completed by $F(x;\alpha) = 0$ for any $x < -1$ and $F(x;\alpha) = 1$ for any $x > 1$.

For $\alpha = 0$, it is reduced to the cumulative distribution function associated with the uniform distribution over $[-1,1]$, i.e.,

$$F(x;\alpha) = \frac{x+1}{2}, \quad x \in [-1,1], \tag{3.2}$$

completed by $F(x;\alpha) = 0$ for any $x < -1$ and $F(x;\alpha) = 1$ for any $x > 1$.

Proof. Since the support of the RE distribution is $[-1,1]$, we immediately have $F(x;\alpha) = 0$ for any $x < -1$ and $F(x;\alpha) = 1$ for any $x > 1$. Let us distinguish the cases $\alpha \in \mathbb{R}\backslash\{0\}$ and $\alpha = 0$. For $\alpha \in \mathbb{R}\backslash\{0\}$ and any $x \in [-1,1]$, some integral developments give

$$\begin{aligned} F(x;\alpha) &= \int_{-\infty}^{x} f(t;\alpha)dt = \int_{-1}^{x} \frac{1}{1+e^{\alpha t}}dt = \int_{-1}^{x} \frac{e^{-\alpha t}}{1+e^{-\alpha t}}dt \\ &= -\frac{1}{\alpha}\left[\log(1+e^{-\alpha t})\right]_{t=-1}^{t=x} = -\frac{1}{\alpha}\left[\log(1+e^{-\alpha x}) - \log(1+e^{\alpha})\right] \\ &= \frac{1}{\alpha}\left[\log(1+e^{\alpha}) - \log(1+e^{-\alpha x})\right] = \frac{1}{\alpha}\log\left(\frac{1+e^{\alpha}}{1+e^{-\alpha x}}\right). \end{aligned}$$

For $\alpha = 0$ and any $x \in [-1,1]$, we immediately obtain

$$F(x;\alpha) = \int_{-\infty}^{x} f(t;\alpha)dt = \int_{-1}^{x} \frac{1}{2}dt = \frac{1}{2}\left[t\right]_{t=-1}^{t=x} = \frac{x+1}{2}.$$

The desired expressions are obtained, ending the proof. □

Here and in the following, $F(x;\alpha)$ refers to the expression in Equations (3.1) or (3.2), depending on $\alpha \in \mathbb{R}\backslash\{0\}$ or $\alpha = 0$, respectively.

Following the parameter configuration of Figure 1, Figure 2 illustrates this proposition by showing some forms of $F(x;\alpha)$.

The classical properties of a cumulative distribution function, i.e., the increasing and the limit properties, are clearly visible. We also distinguish convex forms for $\alpha = -10$, -3 and -0.5, and concave forms for $\alpha = 0.5$, 3 and 10, as suggested by Proposition 3.1.

Based on $f(x;\alpha)$ and $F(x;\alpha)$, we can derive several reliability functions associated with the RE distribution. Among them, there are

- the survival function of the RE distribution given by $S(x;\alpha) = 1 - F(x;\alpha)$,
- the hazard rate function of the RE distribution given by $h(x;\alpha) = f(x;\alpha)/S(x;\alpha)$,
- the cumulative hazard rate function of the RE distribution given by $H(x;\alpha) = -\log[S(x;\alpha)]$, such that $H'(x;\alpha) = h(x;\alpha)$,

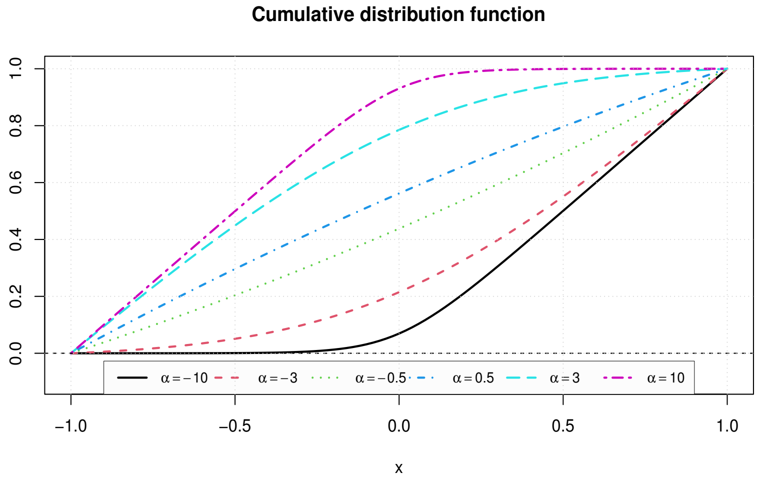


Figure 2. Sample of forms for $F(x;\alpha)$ for various values of α

- the reversed hazard rate function of the RE distribution given by $r(x;\alpha) = f(x;\alpha)/F(x;\alpha)$,
- the cumulative reversed hazard rate function of the RE distribution given by $R(x;\alpha) = -\log[F(x;\alpha)]$, such that $R'(x;\alpha) = r(x;\alpha)$.

We omit their expressions for the sake of brevity; they can easily be derived.

We end this subsection with a technical result, showing an equation that relates $F(x;\alpha)$ and $f(x;\alpha)$.

Proposition 3.2. *For $\alpha \in \mathbb{R}\backslash\{0\}$, the cumulative distribution and probability density functions of the RE distribution are related by the following special functional equation:*

$$F(x;\alpha) = x + \frac{1}{\alpha}\log(1+e^{\alpha}) + \frac{1}{\alpha}\log[f(x;\alpha)],$$

where $x \in [-1, 1]$.

Proof. Working on the expression of $F(x;\alpha)$, we get

$$\begin{aligned} F(x;\alpha) &= \frac{1}{\alpha}\log\left(\frac{1+e^{\alpha}}{1+e^{-\alpha x}}\right) = \frac{1}{\alpha}\log\left[\frac{e^{\alpha x}(1+e^{\alpha})}{1+e^{\alpha x}}\right] \\ &= \frac{1}{\alpha}\left[\log(e^{\alpha x}) + \log(1+e^{\alpha}) + \log\left(\frac{1}{1+e^{\alpha x}}\right)\right] \\ &= \frac{1}{\alpha}\{\alpha x + \log(1+e^{\alpha}) + \log[f(x;\alpha)]\} \\ &= x + \frac{1}{\alpha}\log(1+e^{\alpha}) + \frac{1}{\alpha}\log[f(x;\alpha)]. \end{aligned}$$

This completes the proof. □

This result thus establishes a direct and explicit relationship between the cumulative distribution and probability density functions of the RE distribution through a functional equation. It can be used in the context of order statistics to simplify some expressions involving these two functions. From another point of view, it gives us complementary information about the functional structure of the RE distribution.

3.2 Quantile function

A notable property of the quantile function associated with the RE distribution is its closed form. More details are given in the proposition below.

Proposition 3.3. *For $\alpha \in \mathbb{R}\backslash\{0\}$, the quantile function associated with the RE distribution is given by*

$$Q(y;\alpha) = -\frac{1}{\alpha}\log\left[(1+e^{\alpha})e^{-\alpha y} - 1\right], \quad y \in [0,1]. \tag{3.3}$$

For $\alpha = 0$, it is reduced to the quantile function associated with the uniform distribution over $[-1, 1]$, i.e.,

$$Q(y;\alpha) = 2y - 1, \quad y \in [0,1]. \tag{3.4}$$

Proof. According to the definition of a quantile function, $Q(y;\alpha)$ must satisfy the following equation: $F[Q(y;\alpha);\alpha] = y$ for any $y \in [0,1]$. Let us distinguish the cases $\alpha \in \mathbb{R}\backslash\{0\}$ and $\alpha = 0$. For $\alpha \in \mathbb{R}\backslash\{0\}$, the solution of the equation can be obtained using the following mathematical equivalences:

$$\begin{aligned}
&F[Q(y;\alpha);\alpha] = y \iff \frac{1}{\alpha}\log\left(\frac{1+e^{\alpha}}{1+e^{-\alpha Q(y;\alpha)}}\right) = y \\
&\iff \log\left(\frac{1+e^{-\alpha Q(y;\alpha)}}{1+e^{\alpha}}\right) = -\alpha y \iff \frac{1+e^{-\alpha Q(y;\alpha)}}{1+e^{\alpha}} = e^{-\alpha y} \\
&\iff 1+e^{-\alpha Q(y;\alpha)} = (1+e^{\alpha})e^{-\alpha y} \iff e^{-\alpha Q(y;\alpha)} = (1+e^{\alpha})e^{-\alpha y} - 1 \\
&\iff -\alpha Q(y;\alpha) = \log\left[(1+e^{\alpha})e^{-\alpha y} - 1\right] \iff Q(y;\alpha) = -\frac{1}{\alpha}\log\left[(1+e^{\alpha})e^{-\alpha y} - 1\right].
\end{aligned}$$

For $\alpha = 0$, the calculus is more direct. We have

$$F[Q(y;\alpha);\alpha] = y \iff \frac{Q(y;\alpha)+1}{2} = y \iff Q(y;\alpha) = 2y - 1.$$

The desired expressions are obtained, completing the proof. □

Here and in the following, $Q(y;\alpha)$ refers to the expression in Equations (3.3) or (3.4), depending on $\alpha \in \mathbb{R}\backslash\{0\}$ or $\alpha = 0$, respectively.

Based on this quantile function, we can derive several measures and functions associated with the RE distribution. Among them, there are

- the quantile density function of the RE distribution given by $q(y;\alpha) = Q'(y;\alpha)$,
- the median of the RE distribution given by $M = Q(1/2;\alpha)$,
- the interquartile range of the RE distribution given by $I = Q(3/4;\alpha) - Q(1/4;\alpha)$,
- the quartile skewness of the RE distribution given by $K = [Q(3/4;\alpha) - M]/[M - Q(1/4;\alpha)]$; if K is greater than 1, the RE distribution is positively skewed, while a result less than 1 indicates that it is negatively skewed.

The quantile function is also a key tool to generate values from the RE distribution. This will be a part of the study in Section 5. More information on the standard quantile measures can be found in Kenney and Keeping (1982); Moors (1988).

4. Moment Properties

This section is devoted to the moment properties of the RE distribution, starting with the classical moment measures.

4.1 Classical moment measures

The proposition below is a general result about the moment of a certain class of functions of a random variable with the RE distribution.

Proposition 4.1. *Let g be an even function defined on $[-1,1]$ and X be a random variable with the RE distribution. Then we have*

$$\mathbb{E}[g(X)] = \int_0^1 g(x)dx,$$

where $\mathbb{E}$ denotes the mathematical expectation.

Proof. Using the law of the unconscious statistician, we have

$$\mathbb{E}[g(X)] = \int_{-\infty}^{+\infty} g(x)f(x;\alpha)dx.$$

Based on this formula, let us distinguish the cases $\alpha = 0$ and $\alpha \in \mathbb{R}\backslash\{0\}$. For $\alpha = 0$, using the fact that g is even, which implies that $\int_{-1}^1 g(x)dx = 2\int_0^1 g(x)dx$, we obtain

$$\mathbb{E}[g(X)] = \int_{-1}^1 g(x)\frac{1}{2}dx = \frac{1}{2}\int_{-1}^1 g(x)dx = \frac{1}{2}\times 2\int_0^1 g(x)dx = \int_0^1 g(x)dx.$$

For $\alpha \in \mathbb{R}\backslash\{0\}$, with the same argument, we find that

$$\begin{aligned}\mathbb{E}[g(X)] &= \int_{-1}^1 g(x)\frac{1}{1+e^{\alpha x}}dx = \int_{-1}^1 g(x)\frac{1+e^{\alpha x}-e^{\alpha x}}{1+e^{\alpha x}}dx \\ &= \int_{-1}^1 g(x)dx - \int_{-1}^1 g(x)\frac{e^{\alpha x}}{1+e^{\alpha x}}dx \\ &= 2\int_0^1 g(x)dx - \int_{-1}^1 g(x)\frac{1}{1+e^{-\alpha x}}dx.\end{aligned}$$

Applying the change of variables $y = -x$ in the last integral and recognizing the expression of $\mathbb{E}[g(X)]$, we get

$$\begin{aligned}\mathbb{E}[g(X)] &= 2\int_0^1 g(x)dx - \int_1^{-1} g(-y)\frac{1}{1+e^{\alpha y}}(-dy) \\ &= 2\int_0^1 g(x)dx - \int_{-1}^1 g(y)\frac{1}{1+e^{\alpha y}}dy \\ &= 2\int_0^1 g(x)dx - \mathbb{E}[g(X)],\end{aligned}$$

so that

$$2\mathbb{E}[g(X)] = 2\int_0^1 g(x)dx \;\Leftrightarrow\; \mathbb{E}[g(X)] = \int_0^1 g(x)dx.$$

This ends the proof. □

As an application of this proposition, for any non-negative integer k, considering the even function $g(x) = x^{2k}$, we have

$$\mathbb{E}(X^{2k}) = \mathbb{E}[g(X)] = \int_0^1 g(x)dx = \int_0^1 x^{2k}dx = \frac{1}{2k+1}\left[x^{2k+1}\right]_{x=0}^{x=1} = \frac{1}{2k+1}.$$

It is interesting to note that the result is independent of α, and that it is equal to that obtained when X is with the uniform distribution over $[-1,1]$. In particular, if we take $k = 1$ and 2, then we find that

$$\mathbb{E}(X^2) = \frac{1}{3}, \quad \mathbb{E}(X^4) = \frac{1}{5},$$

respectively. Here, $\mathbb{E}(X^2)$ is directly related to the variance associated with the RE distribution, providing information into the spread of X, and $\mathbb{E}(X^4)$ is related to the kurtosis, providing information about the tail heaviness and peakiness of the RE distribution. These points are discussed numerically at the end of this subsection.

In the case where g is odd or arbitrary, the determination of $\mathbb{E}[g(X)]$ is more complicated. However, a proposition about a particular moment measure involving an intermediate function is given below.

Proposition 4.2. *Let g be an odd function defined over $[-1,1]$, $\alpha \in \mathbb{R}\backslash\{0\}$ and X_α be a random variable with the RE distribution of parameter α. Then we have*

$$\mathbb{E}[g(X_{-\alpha})] = -\mathbb{E}[g(X_\alpha)].$$

Proof. Applying the law of the unconscious statistician and the change of variables $y = \alpha x$, we get

$$\mathbb{E}[g(X_\alpha)] = \int_{-\infty}^{+\infty} g(x)f(x;\alpha)dx = \int_{-1}^{1} g(x)\frac{1}{1+e^{\alpha x}}dx = \frac{1}{\alpha}\int_{-\alpha}^{\alpha} g\left(\frac{y}{\alpha}\right)\frac{1}{1+e^{y}}dy.$$

On the other hand, applying again the law of the unconscious statistician, using the change of variables $z = -\alpha x$, the fact that g is odd, which implies that $g(-z/\alpha) = -g(z/\alpha)$ and the previous moment result, we find that

$$\mathbb{E}[g(X_{-\alpha})] = \int_{-\infty}^{+\infty} g(x)f(x;-\alpha)dx = \int_{-1}^{1} g(x)\frac{1}{1+e^{-\alpha x}}dx$$
$$= \frac{1}{-\alpha}\int_{\alpha}^{-\alpha} g\left(\frac{z}{-\alpha}\right)\frac{1}{1+e^{z}}dz = \frac{1}{\alpha}\int_{-\alpha}^{\alpha} g\left(-\frac{z}{\alpha}\right)\frac{1}{1+e^{z}}dz$$
$$= -\frac{1}{\alpha}\int_{\alpha}^{\alpha} g\left(\frac{z}{\alpha}\right)\frac{1}{1+e^{z}}dz = -\mathbb{E}[g(X_\alpha)].$$

This concludes the proof. □

This result shows a kind of symmetry property for certain moment measures of the RE distribution with respect to the parameter α. However, it does not allow the direct calculation of $\mathbb{E}[g(X)]$ in the case of an odd function g, such as the mean of X, i.e., $\mathbb{E}(X)$. The lack of a closed form expression for this basic measure can be seen as a limitation of the RE distribution. However, we can always calculate it using scientific software.

For example, using R with the function `integrate`, Table 2 presents the values of the fourth first moments of X, i.e., $\mathbb{E}(X)$, $\mathbb{E}(X^2)$, $\mathbb{E}(X^3)$ and $\mathbb{E}(X^4)$, for various values of α. As expected, the values of $\mathbb{E}(X^2)$ and $\mathbb{E}(X^4)$

Table 2. Numerical values of $\mathbb{E}(X)$, $\mathbb{E}(X^2)$, $\mathbb{E}(X^3)$ and $\mathbb{E}(X^4)$, where X is a random variable with the RE distribution of parameter α, for various values of α

	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$
$\alpha = -10$	0.484	0.333	0.249	0.200
$\alpha = -5$	0.437	0.333	0.237	0.200
$\alpha = -3$	0.361	0.333	0.204	0.200
$\alpha = -0.5$	0.082	0.333	0.049	0.200
$\alpha = 0.5$	-0.082	0.333	-0.049	0.200
$\alpha = 1$	-0.159	0.333	-0.094	0.2000
$\alpha = 3$	-0.361	0.333	-0.204	0.200
$\alpha = 10$	-0.484	0.333	-0.249	0.200

are constants, with $\mathbb{E}(X^2) = 1/3$ and $\mathbb{E}(X^4) = 1/5$. Furthermore, Proposition 4.2 is also illustrated for the values $\alpha = -10$ and 10, since we have $\mathbb{E}(X_{-10}) = 0.484$, which corresponds to $-\mathbb{E}(X_{10})$, and $\mathbb{E}(X^3_{-10}) = 0.249$, which corresponds to $-\mathbb{E}(X_{10}^3)$, and the same holds for $\alpha = -5$ and 5, and $\alpha = -0.5$ and 0.5. This numerical work also proves the simplicity of computing various measures of the RE distribution.

The same work can be done for other standard moment measures, such as

- the moment skewness defined by

$$C(X) = \mathbb{E}\left\{\left[\frac{X - \mathbb{E}(X)}{\sigma(X)}\right]^3\right\},$$

  where $\sigma(X)$ is the standard deviation of X, defined by $\sigma(X) = \sqrt{\mathbb{E}\left\{[X - \mathbb{E}(X)]^2\right\}}$; if $C(X)$ is greater than 0, the RE distribution is positively skewed, while a result less than 0 indicates that it is negatively skewed,

- the moment kurtosis defined by

$$D(X) = \mathbb{E}\left\{\left[\frac{X - \mathbb{E}(X)}{\sigma(X)}\right]^4\right\},$$

  if $D(X)$ is greater than 3, the RE distribution is leptokurtic, while a result less than 3 indicates that it is platykurtic. The case where $D(X)$ is close to 3 corresponds to the mesokurtic case.

With a view to analyzing the skewness and kurtosis of the RE distribution, Table 3 presents the values of $\mathbb{E}(X)$, $\mathbb{E}(X^2)$, $\sigma(X)$, $C(X)$ and $D(X)$, for various values of α. We can see that the RE distribution is negatively

Table 3. Numerical values of $\mathbb{E}(X)$, $\mathbb{E}(X^2)$, $\sigma(X)$, $C(X)$ and $D(X)$, where X is a random variable with the RE distribution of parameter α, for various values of α

	$E(X)$	$E(X^2)$	$\sigma(X)$	$C(X)$	$D(X)$
$\alpha = -10$	0.484	0.333	0.315	-0.272	2.247
$\alpha = -5$	0.437	0.333	0.377	-0.619	2.894
$\alpha = -3$	0.361	0.333	0.451	-0.683	2.776
$\alpha = -0.5$	0.082	0.333	0.571	-0.171	1.849
$\alpha = 0.5$	-0.082	0.333	0.571	0.171	1.849
$\alpha = 1$	-0.159	0.333	0.555	0.330	1.986
$\alpha = 3$	-0.361	0.333	0.451	0.683	2.776
$\alpha = 10$	-0.484	0.333	0.315	0.272	2.247

skewed for the negative values of α because $C(X) < 0$, while it is positively skewed for the positive values of α because $C(X) > 0$. We also see that the RE distribution is mainly platykurtic; all values for $D(X)$ are less than 3. Secondarily, we see a symmetry in the kurtosis: the value of $D(X)$ is the same for $\alpha = -10$ and 10, $\alpha = -5$ and 5, and $\alpha = -3$ and 3.

4.2 Other moment measures

In this subsection, we present some moment measures of possible interest, with special transformations of a random variable X with the RE distribution.

The proposition below focuses on the moment measures associated with some exponential transformations of X.

Proposition 4.3. *Let $\alpha \in \mathbb{R}\backslash\{0\}$ and X be a random variable with the RE distribution of parameter α.*

1. *We have* $\mathbb{E}(e^{\alpha X}) = 1$.
2. *We have*
$$\mathbb{E}(e^{\alpha X/2}) = \frac{2}{\alpha}\left[\arctan(e^{\alpha/2}) - \arctan(e^{-\alpha/2})\right].$$
3. *We have*
$$\mathbb{E}(e^{-\alpha X}) = \frac{1}{\alpha}(e^{\alpha} - e^{-\alpha}) - 1.$$

Proof.

1. Using the law of the unconscious statistician and the fact that $f(x; -\alpha)$ is a probability density function, we have
$$\mathbb{E}(e^{\alpha X}) = \int_{-\infty}^{+\infty} e^{\alpha x} f(x;\alpha)dx = \int_{-1}^{1} e^{\alpha x}\frac{1}{1+e^{\alpha x}}dx = \int_{-1}^{1}\frac{1}{1+e^{-\alpha x}}dx$$
$$= \int_{-1}^{1} f(x;-\alpha)dx = 1.$$
2. Using the law of the unconscious statistician and recognizing the arctangent function as a part of the primitive, we obtain
$$\mathbb{E}(e^{\alpha X/2}) = \int_{-\infty}^{+\infty} e^{\alpha x/2} f(x;\alpha)dx = \int_{-1}^{1} e^{\alpha x/2}\frac{1}{1+e^{\alpha x}}dx = \frac{2}{\alpha}\left[\arctan(e^{\alpha x/2})\right]_{x=-1}^{x=1}$$
$$= \frac{2}{\alpha}\left[\arctan(e^{\alpha/2}) - \arctan(e^{-\alpha/2})\right].$$
3. Using the law of the unconscious statistician, a decomposition of a ratio function in simple elements and the fact that $f(x; \alpha)$ is a probability density function, we get
$$\mathbb{E}(e^{-\alpha X}) = \int_{-\infty}^{+\infty} e^{-\alpha x} f(x;\alpha)dx = \int_{-1}^{1} e^{-\alpha x}\frac{1}{1+e^{\alpha x}}dx = \int_{-1}^{1}\left(e^{-\alpha x} - \frac{1}{1+e^{\alpha x}}\right)dx$$
$$= \int_{-1}^{1} e^{-\alpha x}dx - \int_{-1}^{1}\frac{1}{1+e^{\alpha x}}dx = \frac{1}{\alpha}\left[-e^{-\alpha x}\right]_{x=-1}^{x=1} - \int_{-1}^{1} f(x;\alpha)dx$$
$$= \frac{1}{\alpha}(e^{\alpha} - e^{-\alpha}) - 1.$$

This concludes the proof. □

In the proposition below, more sophisticated moment measures are considered, which implicitly use the cumulative distribution and probability density functions of the RE distribution as the main transformations of X.

Proposition 4.4. *Let $\alpha \in \mathbb{R}\backslash\{0\}$, $\beta \in \mathbb{R}\backslash\{0\}$ and X be a random variable with the RE distribution of parameter α.*

1. *We have*
$$\mathbb{E}\left\{\left[\log\left(\frac{1+e^{\alpha}}{1+e^{-\alpha X}}\right)\right]^{\beta}\right\} = \frac{\alpha^{\beta}}{\beta+1}.$$

2. *We have*

$$\mathbb{E}\left[\frac{e^{\alpha X}}{(1+e^{\alpha X})^{\beta}}\right] = \frac{e^{\alpha\beta}-1}{\alpha\beta(1+e^{\alpha})^{\beta}}.$$

Proof.

1. Identifying the expression of the cumulative distribution function $F(x;\alpha)$, we get

$$\mathbb{E}\left\{\left[\log\left(\frac{1+e^{\alpha}}{1+e^{-\alpha X}}\right)\right]^{\beta}\right\} = \alpha^{\beta}\mathbb{E}\left\{\left[\frac{1}{\alpha}\log\left(\frac{1+e^{\alpha}}{1+e^{-\alpha X}}\right)\right]^{\beta}\right\}$$
$$= \alpha^{\beta}\mathbb{E}\left\{[F(X;\alpha)]^{\beta}\right\}.$$

It is well known that $U = F(X;\alpha)$ has the uniform distribution over $[0,1]$, as for any continuous random variable X with a certain cumulative distribution function denoted $F(x;\alpha)$. So we have

$$\mathbb{E}\left\{[F(X;\alpha)]^{\beta}\right\} = \mathbb{E}(U^{\beta}) = \int_0^1 u^{\beta}du = \left[\frac{1}{\beta+1}u^{\beta+1}\right]_{u=0}^{u=1} = \frac{1}{\beta+1}.$$

We deduce that

$$\mathbb{E}\left\{\left[\log\left(\frac{1+e^{\alpha}}{1+e^{-\alpha X}}\right)\right]^{\beta}\right\} = \frac{\alpha^{\beta}}{\beta+1}.$$

2. Using the law of the unconscious statistician and standard primitives, we obtain

$$\mathbb{E}\left[\frac{e^{\alpha X}}{(1+e^{\alpha X})^{\beta}}\right] = \int_{-\infty}^{+\infty}\frac{e^{\alpha x}}{(1+e^{\alpha x})^{\beta}}f(x;\alpha)dx = \int_{-1}^{1}\frac{e^{\alpha x}}{(1+e^{\alpha x})^{\beta}}\times\frac{1}{1+e^{\alpha x}}dx$$
$$= \int_{-1}^{1}\frac{e^{\alpha x}}{(1+e^{\alpha x})^{\beta+1}}dx = \left[-\frac{1}{\alpha\beta(1+e^{\alpha x})^{\beta}}\right]_{x=-1}^{x=1}$$
$$= \frac{1}{\alpha\beta}\left[\frac{1}{(1+e^{-\alpha})^{\beta}} - \frac{1}{(1+e^{\alpha})^{\beta}}\right] = \frac{1}{\alpha\beta}\left[\frac{e^{\alpha\beta}}{(1+e^{\alpha})^{\beta}} - \frac{1}{(1+e^{\alpha})^{\beta}}\right]$$
$$= \frac{e^{\alpha\beta}-1}{\alpha\beta(1+e^{\alpha})^{\beta}}.$$

This concludes the proof. □

Moment measures associated with technical one-parameter even transformations of X are given in the proposition below.

Proposition 4.5. *Let $\alpha \in \mathbb{R}\backslash\{0\}$, $\beta \in \mathbb{R}\backslash\{0\}$ and X be a random variable with the RE distribution of parameter α.*

1. *We have*

$$\mathbb{E}(|X|^{\beta}) = \frac{1}{\beta+1},$$

2. *We have*

$$\mathbb{E}\left(\frac{1}{1+\beta X^2}\right) = \frac{1}{\sqrt{\beta}}\arctan[\sqrt{\beta}],$$

3. *We have*

$$\mathbb{E}[\cos(\beta X)] = \frac{\sin(\beta)}{\beta},$$

4. *We have*

$$\mathbb{E}[\cosh(\beta X)] = \frac{\sinh(\beta)}{\beta},$$

Proof. The proof is centered on Proposition 4.1.

1. Applying Proposition 4.1. with the even function $g(x) = |x|^\beta$, we have

$$\mathbb{E}(|X|^\beta) = \mathbb{E}[g(X)] = \int_0^1 g(x)dx = \int_0^1 x^\beta dx = \frac{1}{\beta+1}\left[x^{\beta+1}\right]_{x=0}^{x=1} = \frac{1}{\beta+1}.$$

2. Applying Proposition 4.1. with the even function $g(x) = 1/(1+\beta x^2)$, we have

$$\mathbb{E}\left(\frac{1}{1+\beta X^2}\right) = \mathbb{E}[g(X)] = \int_0^1 g(x)dx = \int_0^1 \frac{1}{1+\beta x^2}dx = \int_0^1 \frac{1}{1+[\sqrt{\beta}x]^2}dx$$
$$= \frac{1}{\sqrt{\beta}}\left[\arctan[\sqrt{\beta}x]\right]_{x=0}^{x=1} = \frac{1}{\sqrt{\beta}}\arctan[\sqrt{\beta}].$$

3. Applying Proposition 4.1. with the even function $g(x) = \cos(\beta x)$, we have

$$\mathbb{E}[\cos(\beta X)] = \mathbb{E}[g(X)] = \int_0^1 g(x)dx = \int_0^1 \cos(\beta x)dx = \frac{1}{\beta}\left[\sin(\beta x)\right]_{x=0}^{x=1} = \frac{\sin(\beta)}{\beta}.$$

4. Applying Proposition 4.1. with the even function $g(x) = \cosh(\beta x)$, we have

$$\mathbb{E}[\cosh(\beta X)] = \mathbb{E}[g(X)] = \int_0^1 g(x)dx = \int_0^1 \cosh(\beta x)dx = \frac{1}{\beta}\left[\sinh(\beta x)\right]_{x=0}^{x=1} = \frac{\sinh(\beta)}{\beta}.$$

This ends the proof. □

Using standard integral calculus techniques, these results show how the RE distribution can be used for further moment analysis.

5. New Distributions

The previous work has highlighted the novelty and characteristics of the RE distribution. We can also think of using it to create new distributions of interest, which is the aim of this section.

5.1 Some new distributions with support [0, 1]

As outlined in the introductive section, creating new and simple distributions with support [0, 1] is of interest because this interval is common in modeling proportions, probabilities, and scores in various fields. We refer again to Altun et al. (2024); Chesneau (2023); Korkmaz (2020); Korkmaz and Korkmaz (2021); Mazucheli, Menezes, and Chakraborty (2019); Mazucheli, Menezes, and Dey (2019); Sarhan and Sobh (2025).

In this subsection, we show how the RE distribution can be adapted for this purpose.

Proposition 5.1. *Let $\alpha \in \mathbb{R}$ and X be a random variable with the RE distribution of parameter α.*

1. *Let us set $Y = (X+1)/2$. Then the distribution of Y is with support $[0, 1]$, and is defined by the following probability density function:*

$$f_\star(x;\alpha) = \frac{2}{1+e^{\alpha(2x-1)}}, \quad x \in [0,1],$$

completed by $f_\star(x;\alpha) = 0$ for any $x \notin [0,1]$.

2. *Let us set $Z = |X|$. Then the distribution of Z is with support $[0,1]$, and is defined by the following probability density function:*

$$f_\circ(x;\alpha) = \frac{2 + e^{\alpha x} + e^{-\alpha x}}{(1 + e^{\alpha x})(1 + e^{-\alpha x})}, \quad x \in [0,1],$$

completed by $f_\circ(x;\alpha) = 0$ for any $x \notin [0,1]$.

3. *Let us set $W = \log(2 + X)/\log(3)$. Then the distribution of W is with support $[0,1]$, and is defined by the following probability density function:*

$$f_\dagger(x;\alpha) = \log(3)\frac{3^x}{1 + e^{\alpha(3^x - 2)}}, \quad x \in [0,1],$$

completed by $f_\dagger(x;\alpha) = 0$ for any $x \notin [0,1]$.

Proof.

1. Since the support of X is $[-1,1]$, the support of $Y = (X+1)/2$ is clearly $[0,1]$. This implies that $f_\star(x;\alpha) = 0$ for any $x \notin [0,1]$. For any $x \in [0,1]$, introducing the cumulative distribution function of Y, we have

$$F_\star(x;\alpha) = \mathbb{P}(Y \leq x) = \mathbb{P}\left(\frac{X+1}{2} \leq x\right) = \mathbb{P}(X \leq 2x - 1) = F(2x - 1;\alpha),$$

so that

$$f_\star(x;\alpha) = [F_\star(x;\alpha)]' = [F(2x-1;\alpha)]' = 2F'(2x-1;\alpha) = 2f(2x-1;\alpha)$$
$$= \frac{2}{1 + e^{\alpha(2x-1)}}.$$

2. Since the support of X is $[-1,1]$, the support of $Z = |X|$ is clearly $[0,1]$. This implies that $f_\circ(x;\alpha) = 0$ for any $x \notin [0,1]$. For any $x \in [0,1]$, introducing the cumulative distribution function of Z, we get

$$F_\circ(x;\alpha) = \mathbb{P}(Z \leq x) = \mathbb{P}(|X| \leq x) = \mathbb{P}(-x \leq X \leq x) = \mathbb{P}(X \leq x) - \mathbb{P}(X \leq -x)$$
$$= F(x;\alpha) - F(-x;\alpha),$$

so that

$$f_\circ(x;\alpha) = [F_\circ(x;\alpha)]' = [F(x;\alpha) - F(-x;\alpha)]' = F'(x;\alpha) + F'(-x;\alpha)$$
$$= f(x;\alpha) + f(-x;\alpha) = \frac{1}{1 + e^{\alpha x}} + \frac{1}{1 + e^{-\alpha x}} = \frac{2 + e^{\alpha x} + e^{-\alpha x}}{(1 + e^{\alpha x})(1 + e^{-\alpha x})}.$$

3. Since the support of X is $[-1,1]$, the support of $W = \log(2+X)/\log(3)$ is clearly $[0,1]$. This implies that $f_\dagger(x;\alpha) = 0$ for any $x \notin [0,1]$. For any $x \in [0,1]$, introducing the cumulative distribution function of W, we find that

$$F_\dagger(x;\alpha) = \mathbb{P}(W \leq x) = \mathbb{P}\left[\frac{1}{\log(3)}\log(2+X) \leq x\right] = \mathbb{P}\left[\log(2+X) \leq x\log(3)\right]$$
$$= \mathbb{P}(2 + X \leq e^{x\log(3)}) = \mathbb{P}(X \leq 3^x - 2) = F(3^x - 2;\alpha),$$

so that

$$f_\dagger(x;\alpha) = \left[F_\dagger(x;\alpha)\right]' = \{F(3^x - 2;\alpha)\}' = \log(3)3^x F'(3^x - 2;\alpha)$$
$$= \log(3)3^x f(3^x - 2;\alpha) = \log(3)\frac{3^x}{1 + e^{\alpha(3^x - 2)}}.$$

The desired expressions are obtained, ending the proof. □

Each of the introduced distributions in this proposition can be the subject of an independent study. In particular, it may be interesting to see how they apply to the analysis of proportional data. We leave this idea for the future.

5.2 Negation distribution

In this subsection, we apply the notion of negation distribution to the context of the RE distribution. Let us first recall some generalities. We consider a distribution with compact support, say $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$, with a probability density function f continuous on $[a, b]$. Then the associated negation distribution is defined by the following probability density function:

$$f_{-}(x) = \frac{1}{(b-a)\left[\sup_{t\in[a,b]} f(t)\right] - 1}\left\{\left[\sup_{t\in[a,b]} f(t)\right] - f(x)\right\}, \quad x \in [a, b],$$

and $f_{-}(x) = 0$ for any $x \notin [a, b]$. The main interest of this probability density function is to have forms that are almost horizontally symmetric with those of f. Full information and technical details are given in Chesneau (2024c).

The negation distribution associated with the RE distribution is studied in the proposition below.

Proposition 5.2. *Let $\alpha \in \mathbb{R}\backslash\{0\}$. The negation distribution associated with the RE distribution of parameter α is defined by the following probability distribution function:*

$$f_{-}(x; \alpha) = \frac{1}{1 - e^{-|\alpha|}}\left[\frac{e^{\alpha x} - e^{-|\alpha|}}{1 + e^{\alpha x}}\right], \quad x \in [-1, 1],$$

completed by $f_{-}(x; \alpha) = 0$ for any $x \notin [-1, 1]$.

Proof. By adapting the definition of a negation distribution to the context of the RE distribution, the new probability density function is defined as follows:

$$f_{-}(x; \alpha) = \frac{1}{2\left[\sup_{t\in[-1,1]} f(t; \alpha)\right] - 1}\left\{\left[\sup_{t\in[-1,1]} f(t; \alpha)\right] - f(x; \alpha)\right\}, \quad x \in [-1, 1],$$

completed by $f_{-}(x; \alpha) = 0$ for any $x \notin [-1, 1]$. For $\alpha \in \mathbb{R}\backslash\{0\}$ and any $x \in [-1, 1]$, we have

$$f'(x; \alpha) = -\alpha \frac{e^{\alpha x}}{(1 + e^{\alpha x})^2},$$

which is of the contrary sign to α. Therefore, $f(x; \alpha)$ is monotonic. This implies that

$$\sup_{t\in[-1,1]} f(t; \alpha) = \max[f(-1; \alpha), f(1; \alpha)] = \max\left[\frac{1}{1 + e^{-\alpha}}, \frac{1}{1 + e^{\alpha}}\right] = \frac{1}{1 + e^{-|\alpha|}}.$$

For any $x \in [-1, 1]$, we therefore have

$$\begin{aligned} f_{-}(x; \alpha) &= \frac{1}{2/(1 + e^{-|\alpha|}) - 1}\left[\frac{1}{1 + e^{-|\alpha|}} - \frac{1}{1 + e^{\alpha x}}\right] \\ &= \frac{1}{1 - e^{-|\alpha|}}\left[1 - \frac{1 + e^{-|\alpha|}}{1 + e^{\alpha x}}\right] = \frac{1}{1 - e^{-|\alpha|}}\left[\frac{e^{\alpha x} - e^{-|\alpha|}}{1 + e^{\alpha x}}\right]. \end{aligned}$$

This concludes the proof. □

To the best of our knowledge, the negation distribution associated with the RE distribution is also a new distribution with support $[-1, 1]$. By construction, the forms of the corresponding probability density function are almost horizontally symmetric to those of the probability density function associated with the RE distribution.

5.3 Notion of opposite distribution

The proposition below provides a general way to construct new distributions with support $[-1, 1]$ based on another. It follows the general idea of negation of distributions as described in Chesneau (2024c), but with more flexibility since a bound is considered instead of the exact supremum of the initial probability density function.

Proposition 5.3. *Let f be a probability density function of a distribution with support $[-1, 1]$. We assume that, for any $x \in [-1, 1]$, we have $f(x) \leq 1$. Then the following function defines a valid probability density function:*

$$f_{\ddagger}(x) = 1 - f(x), \quad x \in [-1, 1], \tag{5.1}$$

completed by $f_{\ddagger}(x) = 0$ for any $x \notin [-1, 1]$.

Proof. We need to check that $f_{\ddagger}$ satisfies the necessary assumptions to be a probability density function. Since $f(x) \leq 1$ for any $x \in [-1, 1]$, it is clear that $f_{\ddagger}(x) = 1 - f(x) \geq 0$ for any $x \in \mathbb{R}$. Let us now verify the integral unitary property, i.e., $\int_{-\infty}^{+\infty} f_{\ddagger}(x)dx = 1$. Since f is a probability density function with support $[-1, 1]$, we have $\int_{-1}^{1} f(x)dx = 1$, which implies that

$$\int_{-\infty}^{+\infty} f_{\ddagger}(x)dx = \int_{-1}^{1} [1 - f(x)]dx = \int_{-1}^{1} dx - \int_{-1}^{1} f(x)dx = [t]_{x=-1}^{x=1} - 1 = 2 - 1 = 1.$$

As a result, $f_{\ddagger}$ is a valid probability density function. This concludes the proof. □

Obviously, the distribution defined by the probability density function given in Equation (5.1) is with support $[-1, 1]$. Let us call it the opposite distribution of the distribution defined by the probability density function f, provided that f satisfies the necessary bounded assumption, i.e., $f(x) \leq 1$ for any $x \in [-1, 1]$. To the best of our knowledge, this is the first time that the notion of opposite distribution is introduced.

Based on this new notion, the proposition below investigates the opposite distribution of the RE distribution.

Proposition 5.4. *Let $\alpha \in \mathbb{R}\backslash\{0\}$. The opposite distribution of the RE distribution of parameter α is the RE distribution with parameter $-\alpha$.*

Proof. First of all, we note that the opposite distribution of the RE distribution of parameter α is well defined since the RE distribution is with support $[-1, 1]$ and, for any $x \in \mathbb{R}$, we have $f(x) \leq 1$ since $e^{\alpha x} \geq 0$. Therefore, based on Proposition 5.3, the desired opposite distribution is defined by the probability density function $f_{\ddagger}(x; \alpha)$ such that $f_{\ddagger}(x; \alpha) = 0$ for any $x \notin [-1, 1]$, and, for any $x \in [-1, 1]$,

$$f_{\ddagger}(x; \alpha) = 1 - f(x; \alpha) = 1 - \frac{1}{1 + e^{\alpha x}} = \frac{e^{\alpha x}}{1 + e^{\alpha x}} = \frac{1}{1 + e^{-\alpha x}} = f(x; -\alpha).$$

We recognize the probability density function associated with the RE distribution with parameter $-\alpha$, ending the proof. □

This proposition thus highlights an original feature of the RE distribution. We also believe that more work can be done with the notion of opposite distribution beyond the framework of the RE distribution. We can think of exploring it with the distributions presented in Table 1, among others.

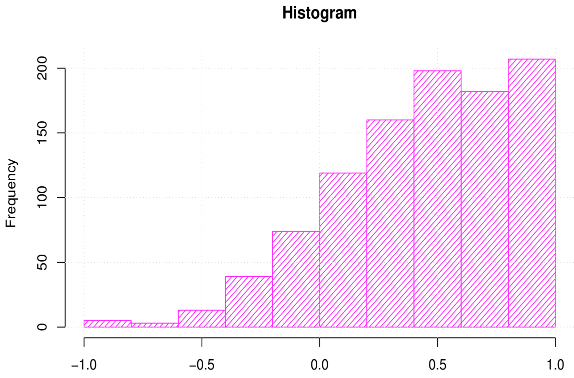
6. Numerical Studies

This section is devoted to numerical studies related to the RE distribution, mainly, the data generation and the estimation of the parameter α by a well-established method.

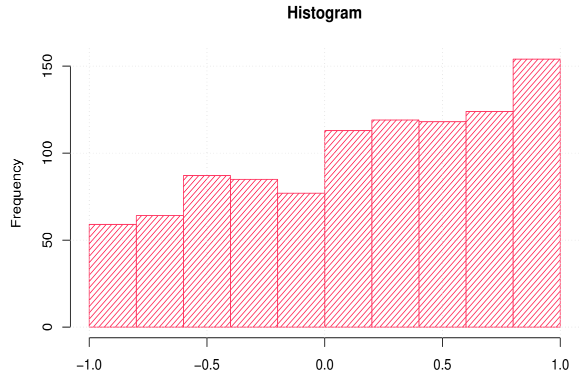
6.1 Data generation

It is possible to generate data from the RE distribution using the corresponding quantile function. More precisely, if we aim to generate n such data based on the RE distribution of parameter α, the numerical process is as follows:

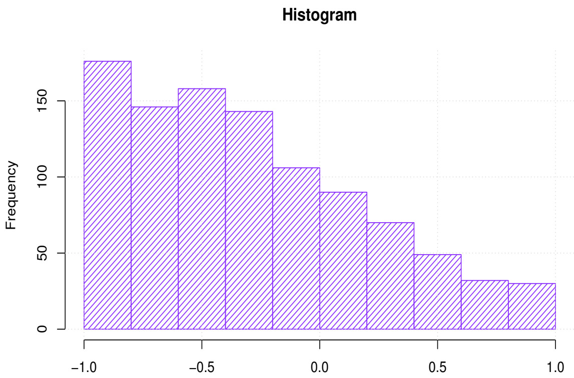
1. Generate n data from the uniform distribution over $[0, 1]$, say $x_1, \ldots, x_n$,
2. For any $i = 1, \ldots, n$, calculate $y_i = Q(x_i; \alpha)$,
3. Consider $y_1, \ldots, y_n$.



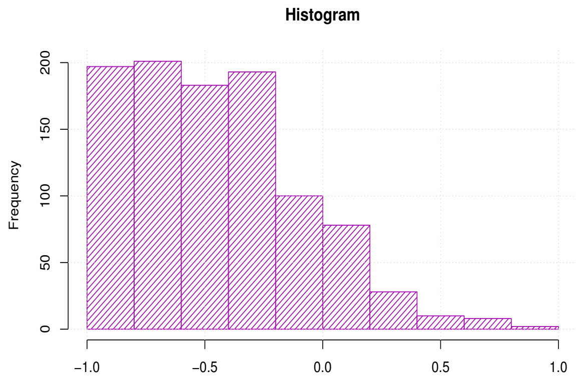
(a) $\alpha = -5$



(b) $\alpha = -1$



(c) $\alpha = 2$



(d) $\alpha = 6$

Figure 3. Frequency histograms of four sample of $n = 1000$ data generated from the RE distribution of parameter α for the following values: (a) $\alpha = -5$, (b) $\alpha = -1$, (c) $\alpha = 2$ and (d) $\alpha = 6$

To illustrate this process concretely, we generate four samples of $n = 1000$ data from the RE distribution by considering the parameters $\alpha = -5, -1, 2$ and 6, with each value for one sample, and we plot the corresponding frequency histograms in Figure 3. The aim is to get an idea of the patterns of the data in the light of our knowledge of the RE distribution.

The general forms of the frequency histograms correspond to the typical monotonic forms observed for the probability density function of the RE distribution, as those shown in Figure 1. More precisely, although the scales are different, we can draw a parallel between the form of the histogram in Figure 3 (a) and that of $f(x;\alpha)$ with $\alpha = -5$, the form of the histogram in Figure 3 (b) and that of $f(x;\alpha)$ with $\alpha = -1$, the form of the histogram in Figure 3 (c) and that of $f(x;\alpha)$ with $\alpha = 2$, and the form of the histogram in Figure 3 (d) and that of $f(x;\alpha)$ with $\alpha = 6$.

6.2 Parameter estimation

We can consider the RE distribution in a statistical scenario dealing with data in $[-1, 1]$, assuming that α is unknown. The first step is to estimate this parameter from the data using one of the existing parametric estimation methods, such as the maximum likelihood method. Full details of this method can be found in Casella and Berger (2002), among others. The basics in the context of the RE distribution are presented below. Let n be the number of data and $x_1, \ldots, x_n$ be data assumed to be associated with the RE distribution, i.e., with a frequency histogram that has a monotonic pattern of the S or Ƨ forms. Then the maximum likelihood estimate of α is given by

$$\hat{\alpha} = \operatorname{argmax}_{\alpha\in\mathbb{R}} \sum_{i=1}^{n} \log[f(x_i;\alpha)] = \operatorname{argmax}_{\alpha\in\mathbb{R}} \sum_{i=1}^{n} \log\left(\frac{1}{1+e^{\alpha x_i}}\right),$$

which corresponds to

$$\hat{\alpha} = \text{argmin}_{\alpha \in \mathbb{R}} \sum_{i=1}^{n} \log(1 + e^{\alpha x_i}).$$

There is no closed form expression for $\hat{\alpha}$, but it can be easily computed with the use of scientific software, like R with the basic function `nlminb`.

6.3 Efficiency checks

As a first approach, we propose to test the efficiency of $\hat{\alpha}$ in a simple way, using simulated data from the RE distribution at given values of α. We use the framework described in the previous subsection.

First, we generate $n = 1000$ data from the RE distribution of parameter $\alpha = -5$. Then we proceed as if we did not know the value of this parameter. We use the maximum likelihood method, and obtain the following value: $\hat{\alpha} = -5.117507$, which is very close to the target value of -5. In Figure 4, we visualize this precision by showing the density histogram of the data and the estimated probability density function of the RE distribution defined by

$$\hat{f}(x) = f(x; \hat{\alpha}) = \frac{1}{1 + e^{\hat{\alpha}x}} = \frac{1}{1 + e^{-5.117507x}}, \quad x \in [-1, 1].$$

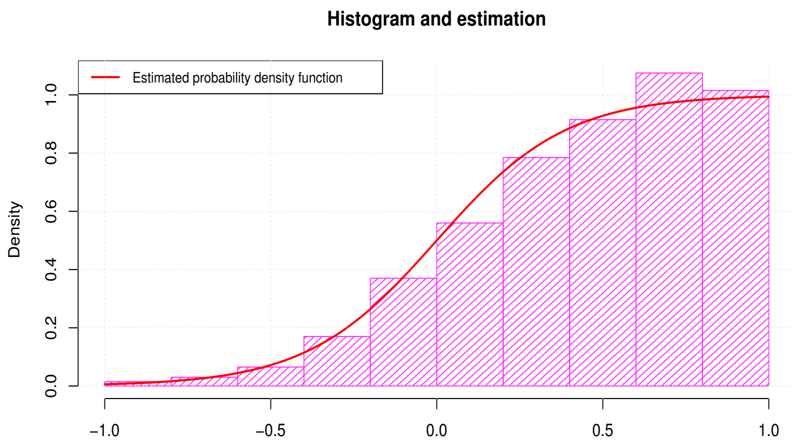


Figure 4. Density histogram of $n = 1000$ data generated from the RE distribution of parameter $\alpha = -5$ and estimated probability density function of the RE distribution (with $\hat{\alpha} = -5.117507$)

We see that the form of the estimated probability density function fits that of the density histogram quite well, illustrating the efficiency of our estimation method.

In the rest of this subsection, we apply this approach to other generated samples based on the parameter configurations adopted in Figure 3.

We thus perform the same study with $n = 1000$ data from the RE distribution of parameter $\alpha = -1$. The maximum likelihood method is then applied based on these data. We obtain the estimate $\hat{\alpha} = -0.9857152$. Figure 5 shows the density histogram of the data and the estimated probability density function of the RE distribution.

Similarly, we generate $n = 1000$ data from the RE distribution of parameter $\alpha = 2$. The maximum likelihood method gives the estimate $\hat{\alpha} = 2.069511$. Figure 6 shows the density histogram of the data and the estimated probability density function of the RE distribution.

We perform the same study with $n = 1000$ data from the RE distribution of parameter $\alpha = 6$. We obtain the estimate $\hat{\alpha} = 5.985817$. Figure 7 shows the density histogram of the data and the estimated probability density function of the RE distribution.

In all cases, the conclusion is the same: when used in the context of the RE distribution, the maximum likelihood method has a good efficiency. This is illustrated by the closeness of the estimated value to the true value of the parameter and the good fit of the estimated probability density function to the corresponding density histogram of the data.

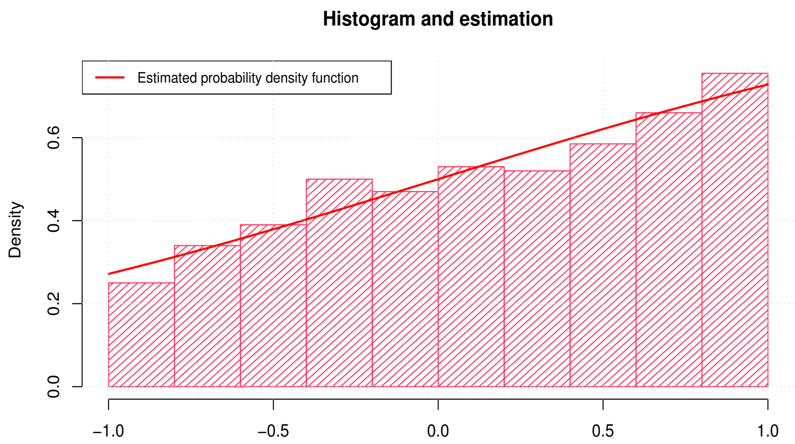


Figure 5. Density histogram of $n = 1000$ data generated from the RE distribution of parameter $\alpha = -1$ and estimated probability density function of the RE distribution (with $\hat{\alpha} = -0.9857152$)

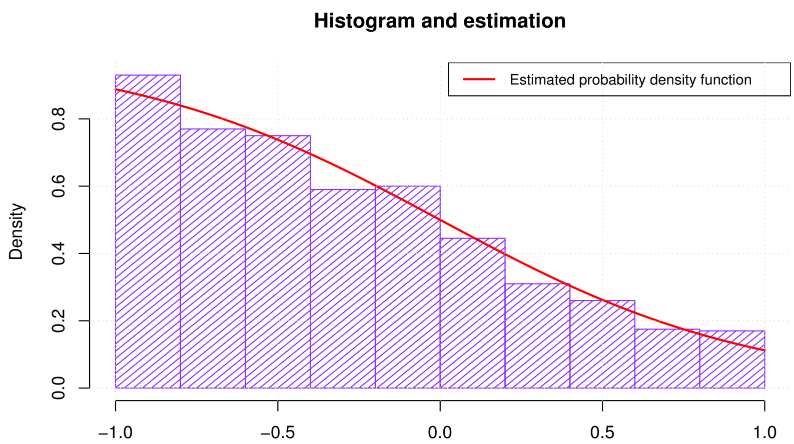


Figure 6. Density histogram of $n = 1000$ data generated from the RE distribution of parameter $\alpha = 2$ and estimated probability density function of the RE distribution (with $\hat{\alpha} = 2.069511$)

6.4 A score analysis scenario

Let us now present a potentially attractive scenario where the RE distribution can be applied directly. For this illustration, we use simulated (not real) data, as described below. We conduct a survey of 36 university students, asking them to rate their weekly engagement with video games using a specially designed scale. The scoring system is defined as follows:

- a score of -1 indicates no gaming activity at all,
- a score of 0 corresponds to a neutral attitude or infrequent gaming,
- a score of 1 represents intense and regular engagement in gaming activities.

The recorded data from the students are as follows:

-0.307, -0.302, -0.744, 0.326, -1.000, -0.163, -0.465, 0.581, 0.860, -0.535, 0.163, -0.047, 0.395, -0.721, -0.209, -0.302, 0.698, 0.395, 0.186, 0.298, 0.907, 0.805, 1.000, 0.977, 0.442, -0.116, 0.512, 0.605, 0.628, 0.163, 0.898, 0.516, 0.674, 0.121, 0.674, 0.293.

This data set therefore provides a range of values from -1 to 1. A simple histogram shows that the data have an increasing pattern, making the RE distribution suitable for analysis.

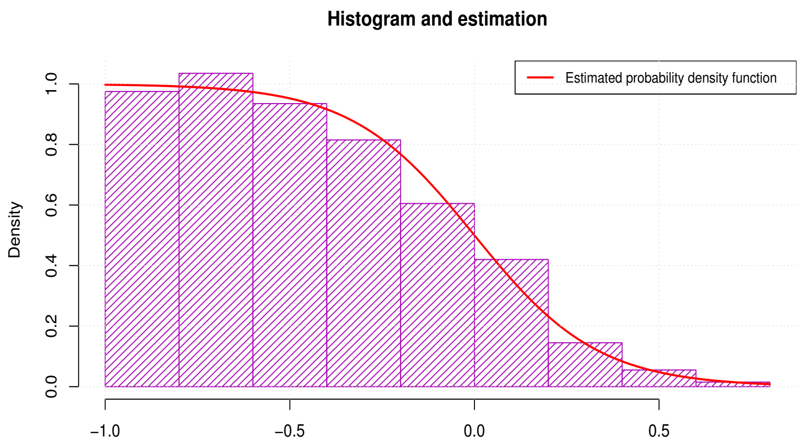


Figure 7. Density histogram of $n = 1000$ data generated from the RE distribution of parameter $\alpha = 6$ and estimated probability density function of the RE distribution (with $\hat{\alpha} = 5.985817$)

Since we do not know the true value of the parameter of this distribution, we estimate it using the maximum likelihood method. We obtain $\hat{\alpha} = -1.549555$. The estimated probability density function is thus given as

$$\hat{f}(x) = f(x; \hat{\alpha}) = \frac{1}{1 + e^{\hat{\alpha}x}} = \frac{1}{1 + e^{-1.549555x}}, \quad x \in [-1, 1].$$

We visualize the precision of the estimation by showing the density histogram of the data and the estimated probability density function in Figure 8.

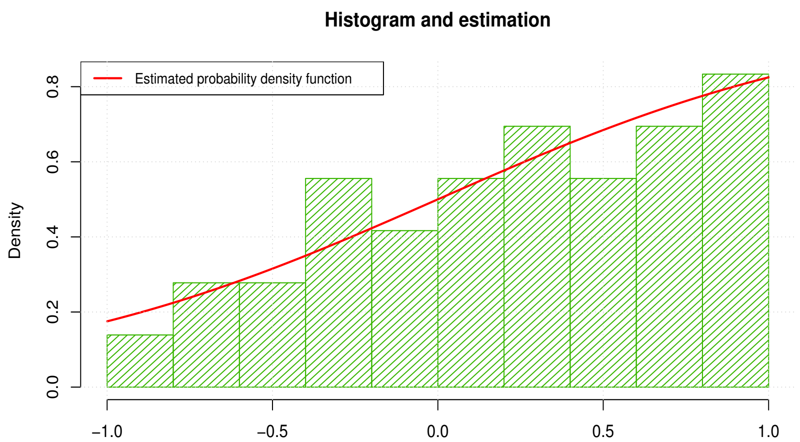


Figure 8. Density histogram of the data set and estimated probability density function of the RE distribution (with $\hat{\alpha} = -1.549555$)

We can see that the estimated probability density function fits well with the general form of the density histogram. This suggests that the RE distribution is an appropriate choice for analyzing such score-type data. To further validate this approach, we compute the Akaike information criterion (AIC) and Bayesian information criterion (BIC) as distribution selection metrics. We obtain

$$AIC = 45.898271, \quad BIC = 47.481790.$$

These values provide a basis for comparing the RE distribution with other distributions supported on $[-1, 1]$ for the given data. These criteria help to evaluate which distribution most effectively captures the underlying patterns. Since the RE distribution is unique among the distributions supported on $[-1, 1]$ due to its probability density function having different monotonic forms, it is challenging to compare it directly with classical distributions such as those listed in Table 1. This is a promising direction for future research. For the sake of transparency, the main R code used in this data analysis are provided in the Appendix.

7. Conclusions

In this article, we have introduced, motivated and studied a new distribution with support $[-1, 1]$ related to the logistic distribution and function. It is called the RE distribution. In the first part, we have emphasized its simplicity and original features, including monotonic forms for the corresponding probability density function, closed form expressions for the cumulative distribution and quantile functions. The moment measures have been studied from a theoretical and practical point of view. We have also shown how the RE distribution can be used to create new distributions with different support. These include the negation and opposite RE distributions, which provide new perspectives on distribution theory. Finally, numerical work has been carried out on the generation of data based on the RE distribution and the analysis of score-type data using the maximum likelihood method.

The perspective of this article is to use the RE distribution for further data analysis with data extracted from different and challenging application areas. The consideration of the presented distributions generated by the RE distribution may also lead to nice applications. In addition, it is conceptually interesting to further explore the notion of opposite distribution introduced in Proposition 5.3 with the distributions described in Table 1. We leave these perspectives for future work.

8. Appendix

The main R codes of the simulated example are presented and commented below.

```
# Simulated data for student video game engagement
datta <- c(-0.307, -0.302, -0.744,  0.326, -1.000, -0.163, -0.465,
             0.581,  0.860, -0.535,  0.163, -0.047,  0.395, -0.721,
            -0.209, -0.302,  0.698,  0.395,  0.186,  0.298,  0.907,
             0.805,  1.000,  0.977,  0.442, -0.116,  0.512,  0.605,
             0.628,  0.163,  0.898,  0.516,  0.674,  0.121,  0.674,
             0.293)

# Function to calculate the negative log-likelihood (RRc)
RRc1 <- function(theta, datta) {
  x <- datta  # Data input
  alpha <- theta[1]  # Extracting parameter alpha
  f <- 1 / (1 + exp(alpha * x))  # Probability density function of the RE
      distribution
  RRc <- -sum(log(f))  # Negative log-likelihood
  RRc  # Return the value
}

# Using nlminb to minimize the negative log-likelihood function
dd1 <- nlminb(start = c(1), RRc1, lower = c(-1000), upper = c(1000), datta =
    datta)
dd1$par  # Estimated parameter

# Probability density function of the RE distribution
f <- function(datta, theta) {
  x <- datta
  alpha <- theta[1]
  f <- 1 / (1 + exp(alpha * x))  # Probability density function of the RE
      distribution
}

# Plotting histogram of the data and estimated probability density function
hist(datta, main = "Histogram and Estimation", col="#36b612", xlab = "", density
     = 20,
```

```
     prob = TRUE, breaks = 12)  # Histogram with probability density
curve(f(x, c(dd1$par[1])), col = "red", lty = 1, lwd = 2, add = TRUE)  # Adding
    estimated curve
legend("topleft", legend = c("Estimated probability density function"),
       col = c("red"), lwd = 2, lty = 1:2, cex = 0.8)  # Adding legend
grid()  # Adding grid to the plot

# Computing the AIC and BIC
AIC <- 2 * dd1$objective + 1 * 2  # AIC formula
BIC <- 2 * dd1$objective + 1 * log(length(datta))  # BIC formula

# Output the estimated parameters and statistical criteria
c(dd1$par[1], AIC, BIC)  # Display the results
```

Article Information

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

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Affiliation

Christophe Chesneau
Address: Department of Mathematics, LMNO, University of Caen-Normandie, 14032 Caen, France.
E-mail: christophe.chesneau@gmail.com
Orcid: 0000-0002-1522-9292