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Soft Intersection Bi-quasi Ideals of Semigroup

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Abstract: Mathematicians find it valuable to extend the concept of ideals within algebraic structures. The bi-quasi (ƁQ) ideal was introduced as a broader version of quasi-ideal, bi-ideal, and left (right) ideals in semigroups. This paper applies this concept to soft set theory and semigroups, introducing the "Soft intersection (S-int) ƁQ ideal." The goal is to explore the relationships between S-int ƁQ ideals and other types of S-int ideals in semigroups. It is shown that every S-int bi-ideal, S-int ideal, S-int quasi-ideal, and S-int interior ideal of an idempotent soft set are S-int ƁQ ideals. Counterexamples demonstrate that the reverse is not always true unless the semigroup is simple* or regular. For soft simple* semigroups, the S-int ƁQ ideal coincides with the S-int bi-ideal, S-int left (right) ideal, and S-int quasi-ideal. The main theorem shows that if a subsemigroup of a semigroup is a ƁQ ideal, its soft characteristic function is an S-int ƁQ ideal, and vice versa. This connects semigroup theory with soft set theory. The paper also discusses how this concept integrates into classical semigroup structures, providing characterizations and analysis using soft set operations, soft image, and soft inverse image, supported by examples.

Yarıgrupların Esnek Kesişimsel Bi-quasi İdealleri

Anahtar Kelimeler
Esnek kümeler, Yarıgruplar, Bi-quasi ideals, Esnek kesişimsel bi-quasi ideals, Basit* yarıgrup

Öz: Matematikçiler, cebirsel yapılardaki ideal kavramını genişletmeyi değerli bulmaktadır. Bi-quasi (ƁQ) ideal, yarıgruplarda quasi-ideal, bi-ideal ve sol (sağ) idealin daha geniş bir versiyonu olarak tanıtılmıştır. Bu makale, bu kavramı esnek küme teorisi ve yarıgruplara uygulayarak "Esnek kesişimsel (EK) ƁQ ideali" tanıtmaktadır. Amaç, EK ƁQ idealleri ile diğer EK ideal türleri arasındaki ilişkileri incelemektir. Bir idempotent esnek küme için her EK-bi-ideal, EK-ideal, EK-quasi-ideal ve EK-iç idealin aynı zamanda bir EK*ƁQ ideal olduğu gösterilmiştir. Ancak, tersinin her zaman geçerli olmadığı, yalnızca yarıgrubun basit* veya regüler olduğunda sağlandığı aksine örneklerle gösterilmiştir. Esnek basit* yarıgruplarda, EK-ƁQ idealin EK-bi-ideal, EK-sol (sağ) ideal ve EK-wuazi-ideal ile çakıştığı kanıtlanmıştır.Ana teorem, bir yarıgrubun alt yarıgrubu bir ƁQ ideal ise, onun esnek karakteristik fonksiyonunun bir EK-ƁQ ideal olduğunu ve bunun tersinin de geçerli olduğunu göstermektedir. Bu sonuç, yarıgrup teorisi ile esnek küme teorisi arasındaki bağlantıyı kurmaktadır. Ayrıca, bu kavramın klasik yarıgrup yapılarıyla nasıl bütünleştiği tartışılmakta ve esnek küme işlemleri, esnek görüntü ve esnek ters görüntü kullanılarak çeşitli karakterizasyonlar ve analizler yapılmıştır. Bulgular örneklerle desteklenmiştir.

1. INTRODUCTION

Semigroups are crucial in various areas of mathematics as they provide the abstract algebraic foundation for "memoryless" systems, which reset after every iteration. Initially studied in the early 1900s, semigroups serve as key models for linear time-invariant systems in applied mathematics. Their connection to finite automata makes the study of finite semigroups particularly important in theoretical computer science. The concept of ideals is vital for understanding the structure and applications of mathematical systems, and thus, many mathematicians have focused on extending the theory of ideals in algebraic structures. By utilizing the concept and properties of generalized ideals, mathematicians have made significant contributions by characterizing algebraic of algebraic structures. Dedekind introduced ideals in the context of algebraic number theory and Noether expanded this concept to include associative rings.

In 1952, Good and Hughes [1] introduced the concept of bi-ideals for semigroups. Steinfeld [2] was the first to present the idea of quasi-ideals for semigroups, later extending it to rings. Quasi-ideals generalize right and left ideals, while bi-ideals are a further generalization of quasi-ideals. The concept of interior ideals was initially introduced by Lajos [3] and later explored by Szasz [4,5]. Interior ideals represent a generalization of the traditional ideal concept. Rao [6-9] developed several novel types of semigroup ideals that generalize existing ones, such as bi-interior ideals, bi-quasi ideals, quasi-ideal, interior ideals, weak-interior ideals, and bi-quasi-interior ideals. Baupradist et al. [10] proposed the concept of essential ideals in semigroups. As a more generalized form of various types of ideals, the notion of "almost" ideals was introduced, with a thorough examination of their characteristics and the relationships between them. The idea of almost ideals for semigroups was first introduced in [11], and a subsequent paper [12] expanded the concept to include almost bi-ideals. The concept of almost quasi-ideals was first presented in [13], and the study of almost interior ideals and weakly almost interior ideals followed in [14]. The authors proposed various types of soft intersection (S-int) almost ideals of semigroups in [15–18]. Additionally, in [13, 15–20], several fuzzy almost ideal types for semigroups were explored.

In 1999, Molodtsov [21] introduced "Soft Set Theory" to address problems involving uncertainty and to develop effective solutions for them. Since its inception, significant research has been conducted on various aspects of soft sets, particularly in relation to soft set operations. Maji et al. [22] provided definitions for soft sets and introduced several operations on them. Pei and Mia [23], along with Ali et al. [24], expanded on the operations of soft sets. For a more comprehensive overview of the growing body of research on soft set operations, we refer to [25-37].

The concept and operations of soft sets were further refined by Çağman and Enginoğlu [38]. Building on this work, Çağman et al. [39] introduced the concept of S-int groups, which spurred the investigation of various soft algebraic systems. In the context of semigroup theory, Sezer et al. [40,41] applied soft sets to define and explore soft intersection (S-int) semigroups, as well as left, right, and two-sided ideals, interior ideals, quasi-ideals, and generalized bi-ideals of semigroups, thoroughly analyzing their key properties. Sezgin and Orbay [42] further studied the soft intersection (S-int) substructures of semigroups, classifying various types, including semisimple semigroups, duo semigroups, and different categories of zero and simple semigroups, along with the semi-lattices of left and right simple semigroups, left and right groups, and groups. S-int almost ideals were introduced and examined as a generalization of various types of S-int ideals in [43-54]. Additionally, the soft versions of different algebraic structures were explored in [55-67].

As a result of the reviews conducted in the literature, some important studies on bi-quasi ideals are identified. The first of these is the study by Rao [69] on the bi-quasi ideals of Γ-semigroups and the fuzzy bi-quasi ideals of these semigroups. Rao [70,71] provided an extensive study on the bi-quasi ideals of semirings. Additionally, the bi-quasi ideals of Γ-semirings were examined by Rao, Venkateswarlu and Rafi [72]. Similarly, Rao [8] made significant contributions to the study of bi-quasi ideals of semigroups. In this paper, we extend the concept to soft set theory and semigroups by introducing "Soft intersection (S-int) bi-quasi (ƁQ) ideals of semigroups." We explore the relationships between S-int ƁQ ideals and other types of S-int ideals within a semigroup. Under certain necessary conditions, it is demonstrated that an S-int ideal (bi-ideal, quasi-ideal, or interior ideal) is indeed an S-int ƁQ ideal of a semigroup. Counterexamples are provided to show that the reverse of these statements does not always hold. It is also proven that for the converse to be true, the semigroup must be a soft simple* (see Definition 2.19) or regular semigroup. Our key theorem reveals that if a nonempty subset of a semigroup is a ƁQ ideal, its soft characteristic function is an S-int ƁQ ideal, and vice versa. This result facilitates the integration of semigroup theory with soft set theory. We illustrate how this concept connects to established algebraic structures in classical semigroup theory by utilizing this theorem. Moreover, we offer conceptual characterizations and analyses of the new idea in the context of soft set operations, soft image, and soft inverse image, supporting our findings with detailed and insightful examples.

The paper is organized into four sections. Section 1 presents an introduction to the subject, whereas Section 2 delves into the basic concepts of semigroups and soft set ideals, detailing their essential definitions and significance. In Section 3, we define S-int ƁQ ideals, examine their properties, and discuss their relationships with other forms of S-int ideals, supported by practical examples. Finally, Section 4 offers a summary of our findings and suggests potential avenues for future research.

2. MATERIAL AND METHOD

In this study, S is used to represent a semigroup. A nonempty subset Ƙ of S is called a subsemigroup of S if ƘƘ ⊆ Ƙ, is called a bi-ideal of S if ƘƘ ⊆ Ƙ and ƘSƘ ⊆ Ƙ, is called an interior ideal of S if SƘS ⊆ Ƙ, and is called a quasi-ideal of S if ƘS ∩ SƘ ⊆ Ƙ.

A subsemigroup Ƙ of S is called a left (ʟ-) ƁQ ideal of S if SƘ ∩ ƘSƘ ⊆ Ƙ, is called a right (ʀ-) ƁQ ideal of S if ƘS ∩ ƘSƘ ⊆ Ƙ, and is called a ƁQ ideal of S if it is both ʟ-ƁQ ideal of S and ʀ-ƁQ ideal [8].

Definition 2.1. [21] Let E be the parameter set, U be the universal set, $P(U)$ be the power set of U, and Ɗ ⊆ E. The soft set (SS) $\varrho_{Ɗ}$ over U is a function such that $\varrho_{Ɗ}: E \rightarrow P(U)$, where for all ɤ ∉ Ɗ, $\varrho_{Ɗ}$(ɤ) $= \emptyset$. That is,

$$\varrho_{D}=\{(\varphi,\varrho_{D}(\varphi)):\varphi\in E,\varrho_{D}(\varphi)\in P(U)\}$$

The set of all SSs over U is designated by $S_E(U)$ throughout this paper.

Definition 2.2. [38] Let $\varrho_D \in S_E(U)$. If $\varrho_D(t)=\emptyset$ for all $t\in E$, then ϱ_D is called a null SS and indicated by $\emptyset_E$.

Definition 2.3. [38] Let $\varrho_{\mathcal{M}},\varrho_N \in S_E(U)$. If $\varrho_{\mathcal{M}}(\omega)\subseteq \varrho_N(\omega)$, for all $\omega\in E$, then $\varrho_{\mathcal{M}}$ is a soft subset of ϱ_N and indicated by $\varrho_{\mathcal{M}}\tilde{\subseteq}\varrho_N$. If $\varrho_{\mathcal{M}}(\varsigma)=\varrho_N(\varsigma)$, for all $\varsigma\in E$, then $\varrho_{\mathcal{M}}$ is called soft equal to ϱ_N and denoted by $\varrho_{\mathcal{M}}=\varrho_N$.

Definition 2.4. [38] Let $\mathrm{q}_{\mathcal{M}},\mathrm{q}_{\mathcal{N}}\in S_E(U)$. The union (intersection) of $\mathrm{q}_{\mathcal{M}}$ and $\mathrm{q}_{\mathcal{N}}$ is the SS $\mathrm{q}_{\mathcal{M}}\tilde{\cup}\mathrm{q}_{\mathcal{N}}$ ($\mathrm{q}_{\mathcal{M}}\tilde{\cap}\mathrm{q}_{\mathcal{N}}$), where $(\mathrm{q}_{\mathcal{M}}\tilde{\cup}\mathrm{q}_{\mathcal{N}})(\upsilon)=\mathrm{q}_{\mathcal{M}}(\upsilon)\cup\mathrm{q}_{\mathcal{N}}(\upsilon)$ $\left((\mathrm{q}_{\mathcal{M}}\tilde{\cap}\mathrm{q}_{\mathcal{N}})(\upsilon)=\mathrm{q}_{\mathcal{M}}(\upsilon)\cap\mathrm{q}_{\mathcal{N}}(\upsilon)\right)$, for all $\upsilon\in E$, respectively.

Definition 2.5. [39] Let $f_{Ƙ},f_{H}\in S_E(U)$, and ϕ be a function from $Ƙ$ to H. Then, the soft image of $f_{Ƙ}$ under ϕ, and the soft pre-image (or soft inverse image) of f_{H} under ϕ are the SSs $\phi(f_{Ƙ})$ and $\phi^{-1}(f_{H})$ such that

$$\left(\phi(f_{Ƙ})\right)(r)=\begin{cases}\bigcup\{f_{Ƙ}(t)\mid t\in Ƙ \text{ and } \phi(t)=r\}, & \text{if } \phi^{-1}(r)\neq\emptyset\\ \emptyset, & \text{otherwise}\end{cases}$$

for all $r\in H$ and $\left(\phi^{-1}(f_{H})\right)(t)=f_{H}(\phi(t))$ for all $t\in\mathcal{H}$.

Definition 2.6. [39] Let $f_{Ƙ}\in S_E(U)$ and $\alpha\subseteq U$. Then, upper α-inclusion of $f_{Ƙ}$, denoted by $\mathcal{U}(f_{Ƙ};\alpha)$, is defined as $\mathcal{U}(f_{Ƙ};\alpha)=\{x\in Ƙ \mid f_{Ƙ}(x)\supseteq\alpha\}$.

Definition 2.7. [40] Let $\jmath_S, g_S\in S_S(U)$. S-int product $\jmath_S\circ g_S$ is defined by

$$(\jmath_S\circ g_S)(\varphi)=\begin{cases}\bigcup_{\varphi=yz}\{\jmath_S(y)\cap g_S(z)\}, & \text{if } \exists y,z\in S \text{ such that } \varphi=yz\\ \emptyset, & \text{otherwise}\end{cases}$$

Theorem 2.8. [40] Let $\mathfrak{h}_S,\mathfrak{p}_S,\mathfrak{n}_S\in S_S(U)$. Then,

i. $(\mathfrak{h}_S\circ\mathfrak{p}_S)\circ\mathfrak{n}_S=\mathfrak{h}_S\circ(\mathfrak{p}_S\circ\mathfrak{n}_S)$
ii. $\mathfrak{h}_S\circ\mathfrak{p}_S\neq\mathfrak{p}_S\circ\mathfrak{h}_S$
iii. $\mathfrak{h}_S\circ(\mathfrak{p}_S\tilde{\cup}\mathfrak{n}_S)=(\mathfrak{h}_S\circ\mathfrak{p}_S)\tilde{\cup}(\mathfrak{h}_S\circ\mathfrak{n}_S)$
$(\mathfrak{h}_S\tilde{\cup}\mathfrak{p}_S)\circ\mathfrak{n}_S=(\mathfrak{h}_S\circ\mathfrak{n}_S)\tilde{\cup}(\mathfrak{p}_S\circ\mathfrak{n}_S)$
iv. $\mathfrak{h}_S\circ(\mathfrak{p}_S\tilde{\cap}\mathfrak{n}_S)=(\mathfrak{h}_S\circ\mathfrak{p}_S)\tilde{\cap}(\mathfrak{h}_S\circ\mathfrak{n}_S)$
$(\mathfrak{h}_S\tilde{\cap}\mathfrak{p}_S)\circ\mathfrak{n}_S=(\mathfrak{h}_S\circ\mathfrak{n}_S)\tilde{\cap}(\mathfrak{p}_S\circ\mathfrak{n}_S)$
v. If $\mathfrak{h}_S\tilde{\subseteq}\mathfrak{p}_S$, then $\mathfrak{h}_S\circ\mathfrak{n}_S\tilde{\subseteq}\mathfrak{p}_S\circ\mathfrak{n}_S$ and $\mathfrak{n}_S\circ\mathfrak{h}_S\tilde{\subseteq}\mathfrak{n}_S\circ\mathfrak{p}_S$
vi. If $\mathfrak{j}_S,\mathfrak{u}_S\in S_S(U)$ such that $\mathfrak{j}_S\tilde{\subseteq}\mathfrak{h}_S$ and $\mathfrak{u}_S\tilde{\subseteq}\mathfrak{p}_S$, then $\mathfrak{j}_S\circ\mathfrak{u}_S\tilde{\subseteq}\mathfrak{h}_S\circ\mathfrak{p}_S$.

Definition 2.9. [40] Let $\emptyset\neq\mathcal{T}\subseteq S$. The soft characteristic function (SCF) of $\mathcal{T}$, denoted by $S_{\mathcal{T}}$, is defined as

$$S_{\mathcal{T}}(v)=\begin{cases}U, & \text{if } v\in\mathcal{T}\\ \emptyset, & \text{if } v\in S\backslash\mathcal{T}\end{cases}$$

Theorem 2.10. [40, 49] Let $\mathfrak{F},\mathfrak{T}\subseteq S$. Then,

i. $\mathfrak{F}\subseteq\mathfrak{T}$ if and only if (iff) $S_{\mathfrak{F}}\tilde{\subseteq}S_{\mathfrak{T}}$
ii. $S_{\mathfrak{F}}\tilde{\cap}S_{\mathfrak{T}}=S_{\mathfrak{F}\cap\mathfrak{T}}$ and $S_{\mathfrak{F}}\tilde{\cup}S_{\mathfrak{H}}=S_{\mathfrak{F}\cup\mathfrak{T}}$
iii. $S_{\mathfrak{F}}\circ S_{\mathfrak{T}}=S_{\mathfrak{F}\mathfrak{T}}$

Definition 2.11. [40] An SS $\mathfrak{h}_S$ over U is called an S-int subsemigroup of S if $\mathfrak{h}_S(\varsigma v)\supseteq\mathfrak{h}_S(\varsigma)\cap\mathfrak{h}_S(v)$ for all $\varsigma,v\in S$.

Note that in [40], the definition of "S-int subsemigroup of S" is given as "S-int semigroup of S"; however in this paper, without loss of generality, we prefer to use "S-int subsemigroup of S".

Definition 2.12. [40] An SS $\mathfrak{h}_S$ over U is called an S-int L-(R-) ideal of S if $\mathfrak{h}_S(zv)\supseteq\mathfrak{h}_S(v)$ $\left(\mathfrak{h}_S(zv)\supseteq\mathfrak{h}_S(z)\right)$ for all $z,y\in S$, and is called an S-int two-sided ideal (S-int ideal) of S if it is both S-int L-ideal of S over U and S-int R-ideal of S over U. An S-int subsemigroup $\mathfrak{h}_S$ is called an S-int bi-ideal of S if $\mathfrak{h}_S(ryv)\supseteq\mathfrak{h}_S(r)\cap\mathfrak{h}_S(v)$ for all $r,y,v\in S$. An SS $\mathfrak{h}_S$ over U is called an S-int interior ideal of S if $\mathfrak{h}_S(ryv)\supseteq\mathfrak{h}_S(y)$ for all $r,y,v\in S$.

It is easy to see that if $\mathfrak{h}_S(v)=U$ for all $v\in S$, then $\mathfrak{h}_S$ is an S-int subsemigroup (L-ideal, R-ideal, ideal, bi-ideal, interior ideal). We denote such a kind of S-int subsemigroup (L-ideal, R-ideal, ideal, bi-ideal, interior ideal) by $\tilde{\mathbb{S}}$. It is obvious that $\tilde{\mathbb{S}}=S_S$, that is, $\tilde{\mathbb{S}}(v)=U$ for all $v\in S$ [40].

Definition 2.13. [41] An SS $\mathfrak{h}_S$ over U is called an S-int quasi-ideal of S over U if $\left(\tilde{\mathbb{S}}\circ\mathfrak{h}_S\right)\tilde{\cap}\left(\mathfrak{h}_S\circ\tilde{\mathbb{S}}\right)\tilde{\subseteq}\mathfrak{h}_S$.

Theorem 2.14. [40] Let $\mathfrak{h}_S\in S_S(U)$. Then,

i. $\tilde{\mathbb{S}}\circ\tilde{\mathbb{S}}\tilde{\subseteq}\tilde{\mathbb{S}}$
ii. $\tilde{\mathbb{S}}\circ\mathfrak{h}_S\tilde{\subseteq}\tilde{\mathbb{S}}$ and $\mathfrak{h}_S\circ\tilde{\mathbb{S}}\tilde{\subseteq}\tilde{\mathbb{S}}$
iii. $\mathfrak{h}_S\tilde{\cup}\tilde{\mathbb{S}}=\tilde{\mathbb{S}}$ and $\mathfrak{h}_S\tilde{\cap}\tilde{\mathbb{S}}=\mathfrak{h}_S$

Theorem 2.15. [40, 41] Let $\mathfrak{D}$ be a nonempty subset of a semigroup S. Then, $\mathfrak{D}$ is a subsemigroup (L-ideal, R-ideal, two-sided ideal, bi-ideal, interior ideal, quasi-ideal) of S iff $S_{\mathfrak{D}}$ is an S-int subsemigroup (L-ideal, R-ideal, two-sided ideal, bi-ideal, interior ideal, quasi-ideal).

Theorem 2.16. [40, 41] Let $\mathfrak{h}_S\in S_S(U)$. Then,

i. $\mathfrak{h}_S$ is an S-int subsemigroup $\Leftrightarrow(\mathfrak{h}_S\circ\mathfrak{h}_S)\tilde{\subseteq}\mathfrak{h}_S$,
ii. $\mathfrak{h}_S$ is an S-int L-(R-) ideal $\Leftrightarrow\left(\tilde{\mathbb{S}}\circ\mathfrak{h}_S\right)\tilde{\subseteq}\mathfrak{h}_S$ and $\left(\mathfrak{h}_S\circ\tilde{\mathbb{S}}\right)\tilde{\subseteq}\mathfrak{h}_S$,

iii. $ꜧ_S$ is an S-int bi-ideal $\Leftrightarrow (ꜧ_S \circ ꜧ_S) \tilde{\subseteq} ꜧ_S$ and $\left(ꜧ_S \circ \widetilde{S} \circ ꜧ_S\right) \tilde{\subseteq} ꜧ_S$,

iv. $ꜧ_S$ is an S-int interior ideal $\Leftrightarrow \left(\widetilde{S} \circ ꜧ_S \circ \widetilde{S}\right) \tilde{\subseteq} ꜧ_S$.

Theorem 2.17. [40, 41] The following assertions hold:

i. Every S-int L-(R-/two-sided) ideal is an S-int subsemigroup (S-int bi-ideal/S-int quasi-ideal),
ii. Every S-int ideal is an S-int bi-ideal.

Proposition 2.18. [40] Let $ꜧ_S \in S_S(U)$, α be a subset of U, $Im(ꜧ_S)$ be the image of $ꜧ_S$ such that $\alpha \in Im(ꜧ_S)$. If $ꜧ_S$ is an S-int subsemigroup of S, then $\mathcal{U}(ꜧ_S; \alpha)$ is a subsemigroup of S.

Definition 2.19. [68] Let $ꜧ_S \in S_S(U)$. Then, S is called a soft left simple* semigroup (with respect to $ꜧ_S$) if $\widetilde{S} = \widetilde{S} \circ ꜧ_S$, is called a soft right simple* semigroup (with respect to $ꜧ_S$) if $\widetilde{S} = ꜧ_S \circ \widetilde{S}$, is called a soft simple* semigroup (with respect to $ꜧ_S$) if $\widetilde{S} = \widetilde{S} \circ ꜧ_S = ꜧ_S \circ \widetilde{S}$. If S is a soft (left/right) simple* semigroup with respect to all soft sets over U, then it is called a soft (left/right) simple* semigroup .

For the sake of brevity, soft (left/right) simple* semigroup is abbreviated by soft (L-/ R-) simple*.

Corollary 2.20. [40] For a semigroup S, the following conditions are equivalent:

i. S is regular.
ii. $ꜧ_S \circ ꝑ_S = ꜧ_S \tilde{\cap} ꝑ_S$ for every S-int ideals $ꜧ_S$ and $ꝑ_S$ of S over U.

3. RESULTS

Definition 3.1. A soft set η_S over U is called a soft intersection left (right) (L-(R-) bi-quasi ideal of S over U if

$$\left(\widetilde{S} \circ \eta_S\right) \tilde{\cap} \left(\eta_S \circ \widetilde{S} \circ \eta_S\right) \tilde{\subseteq} \eta_S$$

$$\left(\left(\eta_S \circ \widetilde{S}\right) \tilde{\cap} \left(\eta_S \circ \widetilde{S} \circ \eta_S\right) \tilde{\subseteq} \eta_S\right)$$

A soft set over U is called a soft intersection bi-quasi ideal of S if it is both a soft intersection left bi-quasi ideal and a soft intersection right bi-quasi ideal of S over U. For the sake of brevity, soft intersection bi-quasi ideal of S over U is abbreviated by S-int ɃQ ideal.

Example 3.2. Consider the semigroup $S = \{ꝼ, h, ɍ\}$ defined by the following table:

Table 1. Cayley table of '♦' binary operation.

♦	ꝼ	h	ɍ
ꝼ	ꝼ	ɍ	ɍ
h	ɍ	h	ɍ
ɍ	ɍ	ɍ	ɍ

Let η_S and $ꞵ_S$ be SSs over $U = D_3 = \{< x, y >: x^3 = y^2 = e,\ xy = yx^2\} = \{e, x, x^2, y, yx, yx^2\}$ as follows:

$$\eta_S = \{(ꝼ, \{e, x, x^2\}), (h, \{e, x\}), (ɍ, \{e, x, x^2, y\})\}$$

$$ꞵ_S = \{(ꝼ, \{e, x, y\}), (h, \{e, x\}), (ɍ, \{e, x^2, y, yx^2\})\}$$

It can be readily proven that η_S is an S-int ɃQ ideal of S. Here, we find it appropriate to give a few concrete examples of elements for ease of illustration in order to be more understandable. In fact,

$$\left[\left(\widetilde{S} \circ \eta_S\right) \tilde{\cap} \left(f_S \circ \widetilde{S} \circ \eta_S\right)\right](ꝼ) = \eta_S(ꝼ) \subseteq \eta_S(ꝼ)$$

$$\left[\left(\widetilde{S} \circ \eta_S\right) \tilde{\cap} \left(\eta_S \circ \widetilde{S} \circ \eta_S\right)\right](h) = \eta_S(h) \subseteq \eta_S(h)$$

$$\left[\left(\widetilde{S} \circ \eta_S\right) \tilde{\cap} \left(\eta_S \circ \widetilde{S} \circ \eta_S\right)\right](ɍ) = \eta_S(h) \cup \eta_S(ɍ) \cup \eta_S(ꝼ) \subseteq \eta_S(ɍ)$$

It can be easily shown that the SS η_S satisfies the S-int L-ɃQ ideal condition for all other element combinations of the set S. Similarly,

$$\left[\left(\eta_S \circ \widetilde{S}\right) \tilde{\cap} \left(\eta_S \circ \widetilde{S} \circ \eta_S\right)\right](ꝼ) \subseteq \eta_S(ꝼ)$$

$$\left[\left(\eta_S \circ \widetilde{S}\right) \tilde{\cap} \left(\eta_S \circ \widetilde{S} \circ \eta_S\right)\right](h) \subseteq \eta_S(h)$$

$$\left[\left(\eta_S \circ \widetilde{S}\right) \tilde{\cap} \left(\eta_S \circ \widetilde{S} \circ \eta_S\right)\right](ɍ) \subseteq \eta_S(ɍ)$$

It can be easily shown that the SS η_S satisfies the S-int R-ɃQ ideal condition for all other element combinations of the set S, thus η_S is an S-int ɃQ ideal. However, since

$$\left[\left(\widetilde{S} \circ ꞵ_S\right) \tilde{\cap} \left(ꞵ_S \circ \widetilde{S} \circ ꞵ_S\right)\right](ɍ) = [ꞵ_S(h) \cup ꞵ_S(ɍ) \cup ꞵ_S(ꝼ)] \nsubseteq ꞵ_S(ɍ)$$

$ꞵ_S$ is not an S-int ɃQ ideal.

Corollary 3.3. $\widetilde{S}$ and $\emptyset_S$ are S-int ɃQ ideals.

Theorem 3.4. Let Ꜧ be a subsemigroup of S. Then, Ꜧ is a ɃQ ideal of S iff $S_{Ꜧ}$, the SCF of Ꜧ, is an S-int ɃQ ideal.
Proof: Let Ꜧ be a ɃQ ideal of S. Then, $SꜦ \cap ꜦSꜦ \tilde{\subseteq} Ꜧ$ and $ꜦS \cap ꜦSꜦ \tilde{\subseteq} Ꜧ$. By Theorem 2.10,

$$\left(\widetilde{S} \circ S_{Ꜧ}\right) \tilde{\cap} \left(S_{Ꜧ} \circ \widetilde{S} \circ S_{Ꜧ}\right) = (S_S \circ S_{Ꜧ}) \tilde{\cap} \left(S_{Ꜧ} \circ S_S \circ S_{Ꜧ}\right) = S_{SꜦ} \tilde{\cap} S_{ꜦSꜦ} = S_{SꜦ \cap ꜦSꜦ} \tilde{\subseteq} S_{Ꜧ}$$

and

$$\left(S_{Ƕ}\circ\widetilde{\mathbb{S}}\right)\tilde{\cap}\left(S_{Ƕ}\circ\widetilde{\mathbb{S}}\circ S_{Ƕ}\right)=\left(S_{Ƕ}\circ S_S\right)\tilde{\cap}\left(S_{Ƕ}\circ S_S\circ S_{Ƕ}\right)$$
$$=S_{ǶS}\,\tilde{\cap}\,S_{ǶSǶ}=S_{ǶS\cap ǶSǶ}\,\tilde{\subseteq}\,S_{Ƕ}$$

Hence, $S_{Ƕ}$ is an S-int ƁQ ideal.

Conversely, let $S_{Ƕ}$ be an S-int ƁQ ideal and Ƕ be a subsemigroup of S. Then,

$$\left(\widetilde{\mathbb{S}}\circ S_{Ƕ}\right)\tilde{\cap}\left(S_{Ƕ}\circ\widetilde{\mathbb{S}}\circ S_{Ƕ}\right)\tilde{\subseteq}S_{Ƕ}$$

and

$$\left(S_{Ƕ}\circ\widetilde{\mathbb{S}}\right)\tilde{\cap}\left(S_{Ƕ}\circ\widetilde{\mathbb{S}}\circ S_{Ƕ}\right)\tilde{\subseteq}S_{Ƕ}.$$

Let $ɍ\in SǶ\cap ǶSǶ$. Then,

$$S_{Ƕ}(ɍ)\supseteq\left(\widetilde{\mathbb{S}}\circ S_{Ƕ}\right)(ɍ)\cap\left(S_{Ƕ}\circ\widetilde{\mathbb{S}}\circ S_{Ƕ}\right)(ɍ)$$
$$\supseteq S_{SǶ}(ɍ)\cap S_{ǶSǶ}(ɍ)\supseteq S_{SǶ\cap ǶSǶ}(ɍ)$$
$$=U$$

Thus, $S_{Ƕ}(ɍ)=U$ and so $ɍ\in Ƕ$, implying that $SǶ\cap ǶSǶ\subseteq Ƕ$. Hence, Ƕ is an ʟ-ƁQ ideal of S. Similarly, let $ƶ\in ǶS\cap ǶSǶ$. Then,

$$S_{Ƕ}(ƶ)\supseteq\left(S_{Ƕ}\circ\widetilde{\mathbb{S}}\right)(ƶ)\cap\left(S_{Ƕ}\circ\widetilde{\mathbb{S}}\circ S_{Ƕ}\right)(ƶ)\supseteq S_{ǶS}(ƶ)\cap S_{ǶSǶ}(ƶ)\supseteq S_{ǶS\cap ǶSǶ}(ƶ)=U$$

Thus, $S_{Ƕ}(ƶ)=U$, and so $ƶ\in Ƕ$, implying that $ǶS\cap ǶSǶ\subseteq Ƕ$. Hence, Ƕ is an ʀ-ƁQ ideal of S. Therefore, Ƕ is a ƁQ ideal of S.

Example 3.5. We consider the semigroup in Example 3.2. One can show that $B=\{ƒ,ɍ\}$ is a ƁQ ideal of S. By the definition of SCF, $S_B=\{(ƒ,U),(h,\emptyset),(ɍ,U)\}$. One can easily show that S_B is an S-int ƁQ ideal. Conversely, by choosing the S-int ƁQ ideal as $\eta_S=\{(ƒ,\emptyset),(h,U),(ɍ,U)\}$, which is the SCF of $K=\{h,ɍ\}$, one can show that K is a ƁQ ideal of S.

Now, we continue with the relationships between S-int ƁQ ideals and other types of S-int ideals of S.

Proposition 3.6. Every S-int bi-ideal is an S-int ʀ-ƁQ ideal.

Proof: Let $ƕ_S$ be an S-int bi-ideal of S. Then, $ƕ_S\circ\widetilde{\mathbb{S}}\circ ƕ_S\,\tilde{\subseteq}\,ƕ_S$. Thus,

$$\left(ƕ_S\circ\widetilde{\mathbb{S}}\right)\tilde{\cap}\left(ƕ_S\circ\widetilde{\mathbb{S}}\circ ƕ_S\right)\tilde{\subseteq}ƕ_S\circ\widetilde{\mathbb{S}}\circ ƕ_S\,\tilde{\subseteq}\,ƕ_S$$

Hence, $ƕ_S$ is an S-int ʀ-ƁQ ideal of S.

We show with a counterexample that the converse of Proposition 3.6 is not true:

Example 3.7. Consider the semigroup $S=\{ɚ,ƴ,ɼ,ꞩ\}$ defined by the following table:

Table 1. Cayley table of '⸙' binary operation.

⸙	ɚ	ƴ	ɼ	ꞩ
ɚ	ɚ	ɚ	ɚ	ɚ
ƴ	ɚ	ɚ	ɚ	ɚ
ɼ	ɚ	ɚ	ɚ	Y
ꞩ	ɚ	ɚ	ƴ	ɼ

Let $ƕ_S$ be an SS over $U=N$ as follows:

$$ƕ_S=\{(ɚ,\{1,2,3,4\}),(ƴ,\{1,2,3\}),(ɼ,\{4\}),(ꞩ,\{1,2\})\}$$

Here, $ƕ_S$ is an S-int ʀ-ƁQ ideal. In fact,

$$\left[\left(ƕ_S\circ\widetilde{\mathbb{S}}\right)\tilde{\cap}\left(ƕ_S\circ\widetilde{\mathbb{S}}\circ ƕ_S\right)\right](ɚ)$$
$$=ƕ_S(ɚ)\cup ƕ_S(ƴ)\cup ƕ_S(ɼ)\cup ƕ_S(ꞩ)$$
$$\subseteq ƕ_S(ɚ)$$

$$\left[\left(ƕ_S\circ\widetilde{\mathbb{S}}\right)\tilde{\cap}\left(ƕ_S\circ\widetilde{\mathbb{S}}\circ ƕ_S\right)\right](ƴ)=ƕ_S(ꞩ)\subseteq ƕ_S(ƴ)$$

$$\left[\left(ƕ_S\circ\widetilde{\mathbb{S}}\right)\tilde{\cap}\left(ƕ_S\circ\widetilde{\mathbb{S}}\circ ƕ_S\right)\right](ɼ)=\emptyset\subseteq ƕ_S(ɼ)$$

$$\left[\left(ƕ_S\circ\widetilde{\mathbb{S}}\right)\tilde{\cap}\left(ƕ_S\circ\widetilde{\mathbb{S}}\circ ƕ_S\right)\right](ꞩ)=\emptyset\subseteq ƕ_S(ꞩ)$$

Thus, $ƕ_S$ is an S-int ʀ-ƁQ ideal of S. However, since $(ƕ_S\circ ƕ_S)(ɼ)=ƕ_S(ꞩ)\cap ƕ_S(ꞩ)\nsubseteq ƕ_S(ɼ)$, $ƕ_S$ is not an S-int bi-ideal.

Proposition 3.8 shows that the converse of Proposition 3.6 holds for soft ʟ-simple* semigroups.

Proposition 3.8. Let $ƕ_S\in S_S(U)$ and S be a soft ʟ-simple* semigroup. Then, the following conditions are equivalent:

(1) $ƕ_S$ is an S-int bi-ideal.
(2) $ƕ_S$ is an S-int ʀ-ƁQ ideal.

Proof: (1) implies (2) is obvious by Proposition 3.6. Assume that $ƕ_S$ is an S-int ʀ-ƁQ ideal. By assumption, $\widetilde{\mathbb{S}}=\widetilde{\mathbb{S}}\circ ƕ_S$. Thus,

$$ƕ_S\circ ƕ_S=(ƕ_S\circ ƕ_S)\tilde{\cap}(ƕ_S\circ ƕ_S)\tilde{\subseteq}\left(ƕ_S\circ\widetilde{\mathbb{S}}\right)\tilde{\cap}\left(ƕ_S\circ\widetilde{\mathbb{S}}\right)$$
$$=\left(ƕ_S\circ\widetilde{\mathbb{S}}\right)\tilde{\cap}\left(ƕ_S\circ\widetilde{\mathbb{S}}\circ ƕ_S\right)\tilde{\subseteq}ƕ_S$$

Hence, $ƕ_S$ is an S-int subsemigroup. Moreover,

$$ƕ_S\circ\widetilde{\mathbb{S}}\circ ƕ_S=\left(ƕ_S\circ\widetilde{\mathbb{S}}\circ ƕ_S\right)\tilde{\cap}\left(ƕ_S\circ\widetilde{\mathbb{S}}\circ ƕ_S\right)$$
$$=\left(ƕ_S\circ\widetilde{\mathbb{S}}\right)\tilde{\cap}\left(ƕ_S\circ\widetilde{\mathbb{S}}\circ ƕ_S\right)\tilde{\subseteq}ƕ_S$$

Thus, $ƕ_S$ is an S-int bi-ideal.

Proposition 3.9. Every S-int bi-ideal is an S-int ʟ-ƁQ ideal.

Proof: Let $ƕ_S$ be an S-int bi-ideal of S. Then, $ƕ_S\circ\widetilde{\mathbb{S}}\circ ƕ_S\,\tilde{\subseteq}\,ƕ_S$. Thus,

$$\left(\widetilde{\mathbb{S}} \circ ҕ_S\right) \widetilde{\cap} \left(ҕ_S \circ \widetilde{\mathbb{S}} \circ ҕ_S\right) \mathrel{\widetilde{\subseteq}} ҕ_S \circ \widetilde{\mathbb{S}} \circ ҕ_S \mathrel{\widetilde{\subseteq}} ҕ_S$$

Hence, $ҕ_S$ is an S-int ʟ-ƁQ ideal of S.

We show with a counterexample that the converse of Proposition 3.9 is not true:

Example 3.10. Consider the SS $ҕ_S$ in Example 3.7. The SS $ҕ_S$ is an S-int ʟ-ƁQ ideal. Since,

$$\begin{aligned}\left[\left(\widetilde{\mathbb{S}} \circ ҕ_S\right) \widetilde{\cap} \left(ҕ_S \circ \widetilde{\mathbb{S}} \circ ҕ_S\right)\right](ɚ) \\ &= ҕ_S(ɚ) \cup ҕ_S(ƴ) \cup ҕ_S(\mathfrak{r}) \cup ҕ_S(\mathfrak{s}) \\ &\subseteq ҕ_S(ɚ)\end{aligned}$$

$$\left[\left(\widetilde{\mathbb{S}} \circ ҕ_S\right) \widetilde{\cap} \left(ҕ_S \circ \widetilde{\mathbb{S}} \circ ҕ_S\right)\right](ƴ) = ҕ_S(\mathfrak{s}) \subseteq ҕ_S(ƴ)$$

$$\left[\left(\widetilde{\mathbb{S}} \circ ҕ_S\right) \widetilde{\cap} \left(ҕ_S \circ \widetilde{\mathbb{S}} \circ ҕ_S\right)\right](\mathfrak{r}) = \emptyset \subseteq ҕ_S(\mathfrak{r})$$

$$\left[\left(\widetilde{\mathbb{S}} \circ ҕ_S\right) \widetilde{\cap} \left(ҕ_S \circ \widetilde{\mathbb{S}} \circ ҕ_S\right)\right](\mathfrak{s}) = \emptyset \subseteq ҕ_S(\mathfrak{s})$$

Hence, $ҕ_S$ is an S-int ʟ-ƁQ ideal. However, since

$$(ҕ_S \circ ҕ_S)(\mathfrak{r}) = ҕ_S(\mathfrak{s}) \cap ҕ_S(\mathfrak{s}) \nsubseteq ҕ_S(\mathfrak{r})$$

$ҕ_S$ is not an S-int bi-ideal.

Proposition 3.11 shows that the converse of Proposition 3.9 holds for soft ʀ-simple* semigroups.

Proposition 3.11. Let $ҕ_S \in S_S(U)$ and S be a soft ʀ-simple* semigroup. Then, the following conditions are equivalent:

(1) $ҕ_S$ is an S-int bi-ideal.
(2) $ҕ_S$ is an S-int ʟ-ƁQ ideal.

Proof: (1) implies (2) is obvious by Theorem 3.9. Assume that $ҕ_S$ is an S-int ʟ-ƁQ ideal. By assumption, $\widetilde{\mathbb{S}} = ҕ_S \circ \widetilde{\mathbb{S}}$. Thus,

$$\begin{aligned}ҕ_S \circ ҕ_S = (ҕ_S \circ ҕ_S) \widetilde{\cap} (ҕ_S \circ ҕ_S) &\mathrel{\widetilde{\subseteq}} \left(ҕ_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left(ҕ_S \circ \widetilde{\mathbb{S}}\right) \\ &= \left(ҕ_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left(ҕ_S \circ \widetilde{\mathbb{S}} \circ ҕ_S\right) \mathrel{\widetilde{\subseteq}} ҕ_S\end{aligned}$$

Hence, $ҕ_S$ is an S-int subsemigroup. Moreover,

$$\begin{aligned}ҕ_S \circ \widetilde{\mathbb{S}} \circ ҕ_S &= \left(ҕ_S \circ \widetilde{\mathbb{S}} \circ ҕ_S\right) \widetilde{\cap} \left(ҕ_S \circ \widetilde{\mathbb{S}} \circ ҕ_S\right) \\ &= (\widetilde{\mathbb{S}} \circ ҕ_S) \widetilde{\cap} \left(ҕ_S \circ \widetilde{\mathbb{S}} \circ ҕ_S\right) \mathrel{\widetilde{\subseteq}} ҕ_S\end{aligned}$$

Thus, $ҕ_S$ is an S-int bi-ideal.

Theorem 3.12. Every S-int bi-ideal is an S-int ƁQ ideal.

Proof: It is followed by Proposition 3.6 and Proposition 3.9.

Theorem 3.13 shows that the converse of Theorem 3.12 holds for soft simple* semigroup.

Theorem 3.13. Let $ҕ_S \in S_S(U)$ and S be a soft simple* semigroup. Then, the following conditions are equivalent:

(1) $ҕ_S$ is an S-int bi-ideal.
(2) $ҕ_S$ is an S-int ƁQ ideal.

Proof: (1) implies (2) is obvious by Theorem 3.12. Assume that $ҕ_S$ is an S-int ƁQ ideal. Then, by Definition 2.19, S is both a soft ʟ-simple* and a soft ʀ-simple* semigroup. The rest of the proof follows from Proposition 3.8 and Proposition 3.11.

Proposition 3.14. Every S-int ʀ-ideal is an S-int ʀ-ƁQ ideal.

Proof: Let η_S be an S-int ʀ-ideal of S. Then, $\eta_S \circ \widetilde{\mathbb{S}} \mathrel{\widetilde{\subseteq}} \eta_S$. Thus, $\left(\eta_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right) \mathrel{\widetilde{\subseteq}} \eta_S \circ \widetilde{\mathbb{S}} \mathrel{\widetilde{\subseteq}} \eta_S$. Hence, η_S is an S-int ʀ-ƁQ ideal of S.

Additionally, since η_S is an S-int ʀ-ideal, by Theorem 2.17, it is an S-int bi-ideal. Therefore, by Proposition 3.6, η_S is an S-int ʀ-ƁQ ideal.

We show with a counterexample that the converse of Proposition 3.14 is not true:

Example 3.15. Consider the semigroup $S = \{ɣ, ʓ\}$ defined by the following table:

Table 3: Cayley table of '☼' binary operation.

☼	ɣ	ʓ
ɣ	ɣ	ʓ
ʓ	ɣ	ʓ

Let η_S be an SS over $U = \mathbb{Z}$ as follows:

$$\eta_S = \{(ɣ, \{1,3\}), (ʓ, \{1,2\})\}$$

Here, η_S is an S-int ʀ-ƁQ ideal. In fact,

$$\begin{aligned}\left[\left(\eta_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right)\right](ɣ) \\ &= (\eta_S \circ \widetilde{\mathbb{S}})(ɣ) \cap \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right)(ɣ) \\ &= \eta_S(ɣ) \subseteq \eta_S(ɣ)\end{aligned}$$

$$\begin{aligned}\left[\left(\eta_S \circ \widetilde{\mathbb{S}}\right) \widetilde{\cap} \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right)\right](ʓ) \\ &= \left(\eta_S \circ \widetilde{\mathbb{S}}\right)(ʓ) \cap \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right)(ʓ) \\ &= \eta_S(ʓ) \subseteq \eta_S(ʓ)\end{aligned}$$

Thus, η_S is an S-int ʀ-ƁQ ideal of S. However, since

$$\begin{aligned}\left(\eta_S \circ \widetilde{\mathbb{S}}\right)(ɣ) &= \left[\eta_S(ɣ) \cap \widetilde{\mathbb{S}}(ɣ)\right] \cup \left[\eta_S(ʓ) \cap \widetilde{\mathbb{S}}(ɣ)\right] \\ &= \eta_S(ɣ) \cup \eta_S(ʓ) \nsubseteq \eta_S(ɣ)\end{aligned}$$

$$\left(\eta_S \circ \widetilde{\mathbb{S}}\right)(z) = \left[\eta_S(y) \cap \widetilde{\mathbb{S}}(z)\right] \cup \left[\eta_S(z) \cap \widetilde{\mathbb{S}}(z)\right] = \eta_S(y) \cup \eta_S(z) \nsubseteq \eta_S(z)$$

η_S is not an S-int R-ideal.

Proposition 3.16 shows that the converse of Proposition 3.14 holds for soft L-simple* semigroups.

Proposition 3.16. Let $\eta_S \in S_S(U)$ and S be a soft L-simple* semigroup. Then, the following conditions are equivalent:

(1) η_S is an S-int R-ideal.
(2) η_S is an S-int R-BQ ideal.

Proof: (1) implies (2) is obvious by Proposition 3.14. Assume that η_S is an S-int R-BQ ideal. By assumption, $\widetilde{\mathbb{S}} = \widetilde{\mathbb{S}} \circ \eta_S$. Thus,

$$\left(\eta_S \circ \widetilde{\mathbb{S}}\right) = \left(\eta_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\eta_S \circ \widetilde{\mathbb{S}}\right) = \left(\eta_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right) \tilde{\subseteq} \eta_S$$

Hence, η_S is an S-int R-ideal.

Proposition 3.17. Every S-int R-ideal is an S-int L-BQ ideal.

Proof: Let η_S be an S-int R-ideal of S. Then, $\eta_S \circ \widetilde{\mathbb{S}} \tilde{\subseteq} \eta_S$ and $\eta_S \circ \eta_S \tilde{\subseteq} \eta_S$. Thus,

$$\left(\widetilde{\mathbb{S}} \circ \eta_S\right) \tilde{\cap} \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right) \tilde{\subseteq} \eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S \tilde{\subseteq} \eta_S \circ \eta_S \tilde{\subseteq} \eta_S$$

Hence, η_S is an S-int L-BQ ideal of S.

Additionally, since η_S is an S-int R-ideal, by Theorem 2.17, it is an S-int bi-ideal. Therefore, by Proposition 3.9, η_S is an S-int L-BQ ideal.

We show with a counterexample that the converse of Proposition 3.17 is not true:

Example 3.18. Consider the SS η_S in Example 3.15. The SS η_S is an S-int L-BQ ideal. Since,

$$\left[\left(\widetilde{\mathbb{S}} \circ \eta_S\right) \tilde{\cap} \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right)\right](y) = \left(\widetilde{\mathbb{S}} \circ \eta_S\right)(y) \cap \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right)(y) = \eta_S(y) \subseteq \eta_S(y)$$

$$\left[\left(\widetilde{\mathbb{S}} \circ \eta_S\right) \tilde{\cap} \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right)\right](z) = \left(\widetilde{\mathbb{S}} \circ \eta_S\right)(z) \cap \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right)(z) = \eta_S(z) \subseteq \eta_S(z)$$

Hence, η_S is an S-int L-BQ ideal. However, since

$$\left(\eta_S \circ \widetilde{\mathbb{S}}\right)(y) = \left[\eta_S(y) \cap \widetilde{\mathbb{S}}(y)\right] \cup \left[\eta_S(z) \cap \widetilde{\mathbb{S}}(y)\right] = \eta_S(y) \cup \eta_S(z) \nsubseteq \eta_S(y)$$

$$\left(\eta_S \circ \widetilde{\mathbb{S}}\right)(z) = \left[\eta_S(y) \cap \widetilde{\mathbb{S}}(z)\right] \cup \left[\eta_S(z) \cap \widetilde{\mathbb{S}}(z)\right] = \eta_S(y) \cup \eta_S(z) \nsubseteq \eta_S(z)$$

η_S is not an S-int R-ideal.

Proposition 3.19 shows that the converse of Proposition 3.17 holds for soft simple* semigroups.

Proposition 3.19. Let $\eta_S \in S_S(U)$ and S be a soft simple* semigroup. Then, the following conditions are equivalent:

(1) η_S is an S-int R-ideal.
(2) η_S is an S-int L-BQ ideal.

Proof: (1) implies (2) is obvious by Theorem 3.17. Assume that η_S is an S-int L-BQ ideal. By assumption, $\widetilde{\mathbb{S}} = \eta_S \circ \widetilde{\mathbb{S}} = \widetilde{\mathbb{S}} \circ \eta_S$. Thus,

$$\left(\eta_S \circ \widetilde{\mathbb{S}}\right) = \left(\eta_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\eta_S \circ \widetilde{\mathbb{S}}\right) = \left(\widetilde{\mathbb{S}} \circ \eta_S\right) \tilde{\cap} \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right) \tilde{\subseteq} \eta_S$$

η_S is an S-int R-ideal.

Theorem 3.20. Every S-int R-ideal is an S-int BQ ideal.

Proof: It is followed by Proposition 3.14 and Proposition 3.17.

Here note that the converse of Theorem 3.20 is not true follows from Example 3.15 and Example 3.18. Theorem 3.21 shows that the converse of Theorem 3.20 holds for soft simple* semigroup.

Theorem 3.21. Let $f_S \in S_S(U)$ and S be a soft simple* semigroup. Then, the following conditions are equivalent:

(1) η_S is an S-int R-ideal.
(2) η_S is an S-int BQ ideal.

Proof: (1) implies (2) is obvious by Theorem 3.20. (2) implies (1) is obvious by Proposition 3.16 and Proposition 3.19.

Proposition 3.22. Every S-int L-ideal is an S-int R-BQ ideal.

Proof: Let f_S be an S-int L-ideal of S. Then, $\widetilde{\mathbb{S}} \circ f_S \tilde{\subseteq} f_S$ and $f_S \circ f_S \tilde{\subseteq} f_S$. Thus,

$$\left(f_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(f_S \circ \widetilde{\mathbb{S}} \circ f_S\right) \tilde{\subseteq} f_S \circ \widetilde{\mathbb{S}} \circ f_S \tilde{\subseteq} f_S \circ f_S \tilde{\subseteq} f_S$$

Hence, f_S is an S-int R-BQ ideal of S.

Additionally, since f_S is an S-int L-ideal, by Theorem 2.17, it is an S-int bi-ideal. Therefore, by Proposition 3.6, f_S is an S-int R-BQ ideal.

We show with a counterexample that the converse of Proposition 3.22 is not true:

Example 3.23. Consider the semigroup $S = \{\varrho, \mathfrak{Q}\}$ defined by the following table:

Table 4: Cayley table of '$\circledast$' binary operation.

$\circledast$	ϱ	$\mathfrak{Q}$
ϱ	ϱ	ϱ
$\mathfrak{Q}$	$\mathfrak{Q}$	$\mathfrak{Q}$

Let q_S be an SS over $U = \mathbb{Z}$ as follows:

$$\mathrm{q}_S = \{(\varrho, \{3,6\}), (\mathfrak{Q}, \{3,9\})\}$$

Here, q_S is an S-int R-ƁQ ideal. In fact,

$$\left[\left(\mathrm{q}_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\mathrm{q}_S \circ \widetilde{\mathbb{S}} \circ \mathrm{q}_S\right)\right](\varrho) = (\mathrm{q}_S \circ \widetilde{\mathbb{S}})(\varrho) \cap \left(\mathrm{q}_S \circ \widetilde{\mathbb{S}} \circ \mathrm{q}_S\right)(\varrho) = \mathrm{q}_S(\varrho) \subseteq \mathrm{q}_S(\varrho)$$

$$\left[\left(\mathrm{q}_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\mathrm{q}_S \circ \widetilde{\mathbb{S}} \circ \mathrm{q}_S\right)\right](\mathfrak{Q}) = (\mathrm{q}_S \circ \widetilde{\mathbb{S}})(\mathfrak{Q}) \cap \left(\mathrm{q}_S \circ \widetilde{\mathbb{S}} \circ \mathrm{q}_S\right)(\mathfrak{Q}) = \mathrm{q}_S(\mathfrak{Q}) \subseteq \mathrm{q}_S(\mathfrak{Q})$$

Thus, q_S is an S-int R-ƁQ ideal of S. However, since

$$\left(\widetilde{\mathbb{S}} \circ \mathrm{q}_S\right)(\varrho) = \left[\widetilde{\mathbb{S}}(\varrho) \cap \mathrm{q}_S(\varrho)\right] \cup \left[\widetilde{\mathbb{S}}(\varrho) \cap \mathrm{q}_S(\mathfrak{Q})\right] = \mathrm{q}_S(\varrho) \cup \mathrm{q}_S(\mathfrak{Q}) \nsubseteq \mathrm{q}_S(\varrho)$$

$$\left(\widetilde{\mathbb{S}} \circ \mathrm{q}_S\right)(\mathfrak{Q}) = \left[\widetilde{\mathbb{S}}(\mathfrak{Q}) \cap \mathrm{q}_S(\varrho)\right] \cup \left[\widetilde{\mathbb{S}}(\mathfrak{Q}) \cap \mathrm{q}_S(\mathfrak{Q})\right] = \mathrm{q}_S(\varrho) \cup \mathrm{q}_S(\mathfrak{Q}) \nsubseteq \mathrm{q}_S(\mathfrak{Q})$$

q_S is not an S-int L-ideal.

Proposition 3.24 shows that the converse of Proposition 3.22 holds for soft simple* semigroups.

Proposition 3.24. Let $\mathrm{q}_S \in S_S(U)$ and S be a soft simple* semigroup. Then, the following conditions are equivalent:

(1) q_S is an S-int L-ideal.
(2) q_S is an S-int R-ƁQ ideal.

Proof: (1) implies (2) is obvious by Proposition 3.22. Assume that q_S is an S-int R-ƁQ ideal. By assumption, $\widetilde{\mathbb{S}} = \mathrm{q}_S \circ \widetilde{\mathbb{S}} = \widetilde{\mathbb{S}} \circ \mathrm{q}_S$. Thus,

$$\widetilde{\mathbb{S}} \circ \mathrm{q}_S = \left(\widetilde{\mathbb{S}} \circ \mathrm{q}_S\right) \tilde{\cap} \left(\widetilde{\mathbb{S}} \circ \mathrm{q}_S\right) = (\mathrm{q}_S \circ \widetilde{\mathbb{S}}) \tilde{\cap} \left(\mathrm{q}_S \circ \widetilde{\mathbb{S}} \circ \mathrm{q}_S\right) \tilde{\subseteq} \mathrm{q}_S$$

q_S is an S-int L-ideal.

Proposition 3.25. Every S-int L-ideal is an S-int L-ƁQ ideal.

Proof: Let q_S be an S-int L ideal of S. Then, $\widetilde{\mathbb{S}} \circ \mathrm{q}_S \tilde{\subseteq} \mathrm{q}_S$. Thus, $\left(\widetilde{\mathbb{S}} \circ \mathrm{q}_S\right) \tilde{\cap} \left(\mathrm{q}_S \circ \widetilde{\mathbb{S}} \circ \mathrm{q}_S\right) \tilde{\subseteq} \widetilde{\mathbb{S}} \circ \mathrm{q}_S \tilde{\subseteq} \mathrm{q}_S$. Hence, q_S is an S-int L-ƁQ ideal of S.

Additionally, since q_S is an S-int L-ideal, by Theorem 2.17, it is an S-int bi-ideal. Therefore, by Proposition 3.9, q_S is an S-int L-ƁQ ideal.

We show with a counterexample that the converse of Proposition 3.25 is not true:

Example 3.26. Consider the SS q_S in Example 3.23. The SS q_S is an S-int L-ƁQ ideal. Since,

$$\left[\left(\widetilde{\mathbb{S}} \circ \mathrm{q}_S\right) \tilde{\cap} \left(\mathrm{q}_S \circ \widetilde{\mathbb{S}} \circ \mathrm{q}_S\right)\right](\varrho) = \left(\widetilde{\mathbb{S}} \circ \mathrm{q}_S\right)(\varrho) \cap \left(\mathrm{q}_S \circ \widetilde{\mathbb{S}} \circ \mathrm{q}_S\right)(\varrho) = \mathrm{q}_S(\varrho) \subseteq \mathrm{q}_S(\varrho)$$

$$\left[\left(\widetilde{\mathbb{S}} \circ \mathrm{q}_S\right) \tilde{\cap} \left(\mathrm{q}_S \circ \widetilde{\mathbb{S}} \circ \mathrm{q}_S\right)\right](\mathfrak{Q}) = (\widetilde{\mathbb{S}} \circ \mathrm{q}_S)(\mathfrak{Q}) \cap \left(\mathrm{q}_S \circ \widetilde{\mathbb{S}} \circ \mathrm{q}_S\right)(\mathfrak{Q}) = \mathrm{q}_S(\mathfrak{Q}) \subseteq \mathrm{q}_S(\mathfrak{Q})$$

Hence, q_S is an S-int L-ƁQ ideal. However, since

$$\left(\widetilde{\mathbb{S}} \circ \mathrm{q}_S\right)(\varrho) = \left[\widetilde{\mathbb{S}}(\varrho) \cap \mathrm{q}_S(\varrho)\right] \cup \left[\widetilde{\mathbb{S}}(\varrho) \cap \mathrm{q}_S(\mathfrak{Q})\right] = \mathrm{q}_S(\varrho) \cup \mathrm{q}_S(\mathfrak{Q}) \nsubseteq \mathrm{q}_S(\varrho)$$

$$\left(\widetilde{\mathbb{S}} \circ \mathrm{q}_S\right)(\mathfrak{Q}) = \left[\widetilde{\mathbb{S}}(\mathfrak{Q}) \cap \mathrm{q}_S(\varrho)\right] \cup \left[\widetilde{\mathbb{S}}(\mathfrak{Q}) \cap \mathrm{q}_S(\mathfrak{Q})\right] = \mathrm{q}_S(\varrho) \cup \mathrm{q}_S(\mathfrak{Q}) \nsubseteq \mathrm{q}_S(\mathfrak{Q})$$

q_S is not an S-int L-ideal.

Proposition 3.27 shows that the converse of Proposition 3.25 holds for soft R-simple* semigroups.

Proposition 3.27. Let $\mathrm{q}_S \in S_S(U)$ and S be a soft R-simple* semigroup. Then, the following conditions are equivalent:

(1) q_S is an S-int L-ideal.
(2) q_S is an S-int L-ƁQ ideal.

Proof: (1) implies (2) is obvious by Theorem 3.25. Assume that q_S is an S-int L-ƁQ ideal. By assumption, $\widetilde{\mathbb{S}} = \mathrm{q}_S \circ \widetilde{\mathbb{S}}$. Thus,

$$\widetilde{\mathbb{S}} \circ \mathrm{q}_S = \left(\widetilde{\mathbb{S}} \circ \mathrm{q}_S\right) \tilde{\cap} \left(\widetilde{\mathbb{S}} \circ \mathrm{q}_S\right) = (\widetilde{\mathbb{S}} \circ \mathrm{q}_S) \tilde{\cap} \left(\mathrm{q}_S \circ \widetilde{\mathbb{S}} \circ \mathrm{q}_S\right) \tilde{\subseteq} \mathrm{q}_S$$

q_S is an S-int L-ideal.

Theorem 3.28. Every S-int L-ideal is an S-int ƁQ ideal.
Proof: It is followed by Proposition 3.22 and Proposition 3.25.

Here note that the converse of Theorem 3.28 is not true follows from Example 3.23 and Example 3.26.

Theorem 3.29 shows that the converse of Theorem 3.28 holds for soft simple* semigroup.

Theorem 3.29. Let $\mathfrak{q}_S \in S_S(U)$ and S be a soft simple* semigroup. Then, the following conditions are equivalent:

(1) $\mathfrak{q}_S$ is an S-int L-ideal.
(2) $\mathfrak{q}_S$ is an S-int BQ ideal.

Proof: (1) implies (2) is obvious by Theorem 3.28. (2) implies (1) is obvious by Proposition 3.24 and Proposition 3.27.

Theorem 3.30. Every S-int ideal is an S-int BQ ideal.

Proof: It is followed by Theorem 3.20 and Theorem 3.28. Theorem 3.31 shows that the converse of Theorem 3.30 holds for soft simple* semigroup.

Theorem 3.31. Let $\mathfrak{q}_S \in S_S(U)$ and S be a soft simple* semigroup. Then, the following conditions are equivalent:

(1) $\mathfrak{q}_S$ is an S-int ideal.
(2) $\mathfrak{q}_S$ is an S-int BQ ideal.

1.

Proof: (1) implies (2) is obvious by Theorem 3.30. (2) implies (1) is obvious by Proposition 3.21 and Proposition 3.28.

Proposition 3.32. Every S-int quasi-ideal is an S-int R-BQ ideal.

Proof: Let f_S be an S-int quasi-ideal of S. Then, $(\mathfrak{h}_S \circ \tilde{\mathbb{S}}) \tilde{\cap} (\tilde{\mathbb{S}} \circ \mathfrak{h}_S) \tilde{\subseteq} \mathfrak{h}_S$.

Thus,

$$\left(\mathfrak{h}_S \circ \tilde{\mathbb{S}}\right) \tilde{\cap} \left(\mathfrak{h}_S \circ \tilde{\mathbb{S}} \circ \mathfrak{h}_S\right) \tilde{\subseteq} \left(\mathfrak{h}_S \circ \tilde{\mathbb{S}}\right) \tilde{\cap} \left(\tilde{\mathbb{S}} \circ \tilde{\mathbb{S}} \circ \mathfrak{h}_S\right) \tilde{\subseteq} (\mathfrak{h}_S \circ \tilde{\mathbb{S}}) \tilde{\cap} (\tilde{\mathbb{S}} \circ \mathfrak{h}_S) \tilde{\subseteq} \mathfrak{h}_S$$

Hence, $\mathfrak{h}_S$ is an S-int R-BQ ideal of S.

We show with a counterexample that the converse of Proposition 3.32 is not true:

Example 3.33. Consider the SS $\mathfrak{h}_S$ in Example 3.7. The SS $\mathfrak{h}_S$ is an S-int R-BQ ideal. Since,

$$\left[\left(\mathfrak{h}_S \circ \tilde{\mathbb{S}}\right) \tilde{\cap} \left(\tilde{\mathbb{S}} \circ \mathfrak{h}_S\right)\right](y) = \mathfrak{h}_S(\mathfrak{r}) \cup \mathfrak{h}_S(\mathfrak{s}) \not\subseteq \mathfrak{h}_S(y)$$

Hence, $\mathfrak{h}_S$ is not an S-int quasi ideal.

Proposition 3.34 shows that the converse of Proposition 3.32 holds for soft R-simple* semigroups.

Proposition 3.34. Let $\mathfrak{h}_S \in S_S(U)$ and S be a soft R-simple* semigroup. Then, the following conditions are equivalent:

(1) $\mathfrak{h}_S$ is an S-int quasi-ideal.
(2) $\mathfrak{h}_S$ is an S-int R-BQ ideal.

Proof: (1) implies (2) is obvious by Theorem 3.32. Assume that $\mathfrak{h}_S$ is an S-int R-BQ ideal. By assumption, $\tilde{\mathbb{S}} = \mathfrak{h}_S \circ \tilde{\mathbb{S}}$. Thus,

$$\left(\mathfrak{h}_S \circ \tilde{\mathbb{S}}\right) \tilde{\cap} \left(\tilde{\mathbb{S}} \circ \mathfrak{h}_S\right) = \left(\mathfrak{h}_S \circ \tilde{\mathbb{S}}\right) \tilde{\cap} \left(\mathfrak{h}_S \circ \tilde{\mathbb{S}} \circ \mathfrak{h}_S\right) \tilde{\subseteq} \mathfrak{h}_S$$

$\mathfrak{h}_S$ is an S-int quasi-ideal.

Proposition 3.35. Every S-int quasi-ideal is an S-int L-BQ ideal.

Proof: Let g_S be an S-int quasi-ideal of S. Then, $(g_S \circ \tilde{\mathbb{S}}) \tilde{\cap} (\tilde{\mathbb{S}} \circ g_S) \tilde{\subseteq} g_S$. Thus,

$$\left(\tilde{\mathbb{S}} \circ g_S\right) \tilde{\cap} \left(g_S \circ \tilde{\mathbb{S}} \circ g_S\right) \tilde{\subseteq} \left(\tilde{\mathbb{S}} \circ g_S\right) \tilde{\cap} \left(g_S \circ \tilde{\mathbb{S}} \circ \tilde{\mathbb{S}}\right) \tilde{\subseteq} (\tilde{\mathbb{S}} \circ g_S) \tilde{\cap} (g_S \circ \tilde{\mathbb{S}}) \tilde{\subseteq} g_S$$

Hence, g_S is an S-int L-BQ ideal of S.

We show with a counterexample that the converse of Proposition 3.35 is not true:

Example 3.36. Consider the SS $\mathfrak{h}_S$ in Example 3.7. The SS $\mathfrak{h}_S$ is an S-int L-BQ ideal. Since,

$$\left[\left(\mathfrak{h}_S \circ \tilde{\mathbb{S}}\right) \tilde{\cap} \left(\tilde{\mathbb{S}} \circ \mathfrak{h}_S\right)\right](y) = \mathfrak{h}_S(\mathfrak{r}) \cup \mathfrak{h}_S(\mathfrak{s}) \not\subseteq \mathfrak{h}_S(y)$$

$\mathfrak{h}_S$ is not an S-int quasi-ideal.

Proposition 3.37 shows that the converse of Proposition 3.35 holds for soft simple* semigroups.

Proposition 3.37. Let $g_S \in S_S(U)$ and S be a soft simple* semigroup. Then, the following conditions are equivalent:

(1) g_S is an S-int quasi-ideal.
(2) g_S is an S-int L-BQ ideal.

Proof: (1) implies (2) is obvious by Theorem 3.35. Assume that g_S is an S-int L-BQ ideal. By assumption, $\tilde{\mathbb{S}} = g_S \circ \tilde{\mathbb{S}} = \tilde{\mathbb{S}} \circ g_S$. Thus,

$$\left(g_S \circ \tilde{\mathbb{S}}\right) \tilde{\cap} \left(\tilde{\mathbb{S}} \circ g_S\right) = \left(\tilde{\mathbb{S}} \circ g_S\right) \tilde{\cap} \left(g_S \circ \tilde{\mathbb{S}} \circ g_S\right) \tilde{\subseteq} g_S$$

g_S is an S-int quasi-ideal.

Theorem 3.38. Every S-int quasi-ideal is an S-int BQ ideal.

Proof: It is followed by Theorem 3.32 and Theorem 3.35. Here note that the converse of Theorem 3.38 is not true follows from Example 3.33 and Example 3.36.

Theorem 3.39 shows that the converse of Theorem 3.38 holds for soft simple* semigroup.

Theorem 3.39. Let $g_S \in S_S(U)$ and S be a soft simple* semigroup. Then, the following conditions are equivalent:

(1) $\mathscr{g}_S$ is an S-int quasi-ideal.
(2) $\mathscr{g}_S$ is an S-int ƁQ ideal.

Proof: (1) implies (2) is obvious by Theorem 3.38. (2) implies (1) is obvious by Proposition 3.34 and Proposition 3.37.

Proposition 3.40. Let ϑ_S be an idempotent SS over U. If ϑ_S is an S-int interior ideal, then ϑ_S is an S-int ʟ-ƁQ ideal.
Proof: Let ϑ_S be an idempotent S-int interior ideal of S. Then, $\vartheta_S \circ \vartheta_S = \vartheta_S$ and $\widetilde{\mathbb{S}} \circ \vartheta_S \circ \widetilde{\mathbb{S}} \tilde{\subseteq} \vartheta_S$. Thus,

$$\begin{aligned}(\widetilde{\mathbb{S}} \circ \vartheta_S) \tilde{\cap} (\vartheta_S \circ \widetilde{\mathbb{S}} \circ \vartheta_S) &\tilde{\subseteq} \widetilde{\mathbb{S}} \circ \vartheta_S \\ &= \widetilde{\mathbb{S}} \circ \vartheta_S \circ \vartheta_S \tilde{\subseteq} \widetilde{\mathbb{S}} \circ \vartheta_S \circ \widetilde{\mathbb{S}} \tilde{\subseteq} \vartheta_S\end{aligned}$$

Hence, ϑ_S is an S-int ʟ-ƁQ ideal of S.

Proposition 3.41. Let ϑ_S be an idempotent SS over U. If ϑ_S is an S-int interior ideal, then ϑ_S is an S-int ʀ-ƁQ ideal.

Proof: Let ϑ_S be an idempotent S-int interior ideal of S. Then, $\vartheta_S \circ \vartheta_S = \vartheta_S$ and $\widetilde{\mathbb{S}} \circ \vartheta_S \circ \widetilde{\mathbb{S}} \tilde{\subseteq} \vartheta_S$. Thus,

$$\begin{aligned}(\vartheta_S \circ \widetilde{\mathbb{S}}) \tilde{\cap} (\vartheta_S \circ \widetilde{\mathbb{S}} \circ \vartheta_S) &\tilde{\subseteq} \vartheta_S \circ \widetilde{\mathbb{S}} \\ &= \vartheta_S \circ \vartheta_S \circ \widetilde{\mathbb{S}} \tilde{\subseteq} \widetilde{\mathbb{S}} \circ \vartheta_S \circ \widetilde{\mathbb{S}} \tilde{\subseteq} \vartheta_S\end{aligned}$$

Hence, ϑ_S is an S-int ʀ-ƁQ ideal of S.

Theorem 3.42. Let ϑ_S be an idempotent SS over U. If ϑ_S is an S-int interior ideal, then ϑ_S is an S-int ƁQ ideal.

Proof: It is followed by Theorem 3.40 and Theorem 3.41.

Proposition 3.43. Let $\vartheta_S \in S_S(U)$ and S be a soft simple* semigroup. Then, the following conditions are equivalent:

(1) ϑ_S is an S-int interior ideal.
(2) ϑ_S is an S-int ʟ-ƁQ ideal.

1.

Proof: First assume that (1) holds. Where ϑ_S is an S-int interior ideal of S. Then, $\widetilde{\mathbb{S}} \circ \vartheta_S \circ \widetilde{\mathbb{S}} \tilde{\subseteq} \vartheta_S$. By assumption, $\widetilde{\mathbb{S}} = \vartheta_S \circ \widetilde{\mathbb{S}} = \widetilde{\mathbb{S}} \circ \vartheta_S$. Thus,

$$\begin{aligned}(\widetilde{\mathbb{S}} \circ \vartheta_S) \tilde{\cap} (\vartheta_S \circ \widetilde{\mathbb{S}} \circ \vartheta_S) &\tilde{\subseteq} \vartheta_S \circ \widetilde{\mathbb{S}} \circ \vartheta_S \\ &= \widetilde{\mathbb{S}} \circ \vartheta_S \circ \vartheta_S \tilde{\subseteq} \widetilde{\mathbb{S}} \circ \vartheta_S \circ \widetilde{\mathbb{S}} \tilde{\subseteq} \vartheta_S\end{aligned}$$

ϑ_S is an S-int ʟ-ƁQ ideal.

Conversely, assume that (2) holds. Where ϑ_S is an S-int ʟ-ƁQ ideal of S. Then, $(\widetilde{\mathbb{S}} \circ \vartheta_S) \tilde{\cap} (\vartheta_S \circ \widetilde{\mathbb{S}} \circ \vartheta_S) \tilde{\subseteq} \vartheta_S$. In order to show that ϑ_S S-int interior ideal, we need to show that $\widetilde{\mathbb{S}} \circ \vartheta_S \circ \widetilde{\mathbb{S}} \tilde{\subseteq} \vartheta_S$. By assumption, $\widetilde{\mathbb{S}} = \vartheta_S \circ \widetilde{\mathbb{S}} = \widetilde{\mathbb{S}} \circ \vartheta_S$. Thus,

$$\begin{aligned}&\widetilde{\mathbb{S}} \circ \vartheta_S \circ \widetilde{\mathbb{S}} = \left(\widetilde{\mathbb{S}} \circ \vartheta_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\widetilde{\mathbb{S}} \circ \vartheta_S \circ \widetilde{\mathbb{S}}\right) \\ &= \left(\widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}} \circ \vartheta_S\right) \tilde{\cap} \left(\vartheta_S \circ \widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}}\right) \tilde{\subseteq} \left(\widetilde{\mathbb{S}} \circ \vartheta_S\right) \tilde{\cap} \left(\vartheta_S \circ \widetilde{\mathbb{S}}\right) \\ &= \left(\widetilde{\mathbb{S}} \circ \vartheta_S\right) \tilde{\cap} \left(\vartheta_S \circ \widetilde{\mathbb{S}} \circ \vartheta_S\right) \tilde{\subseteq} \vartheta_S\end{aligned}$$

Hence, ϑ_S is an S-int interior ideal.

Proposition 3.44. Let $\vartheta_S \in S_S(U)$ and S be a soft simple* semigroup. Then, the following conditions are equivalent:

(1) ϑ_S is an S-int interior ideal.
(2) ϑ_S is an S-int ʀ-ƁQ ideal.

Proof: First assume that (1) holds.Where ϑ_S is an S-int interior ideal of S. Then, $\widetilde{\mathbb{S}} \circ \vartheta_S \circ \widetilde{\mathbb{S}} \tilde{\subseteq} \vartheta_S$. By assumption, $\widetilde{\mathbb{S}} = \vartheta_S \circ \widetilde{\mathbb{S}} = \widetilde{\mathbb{S}} \circ \vartheta_S$. Thus,

$$\begin{aligned}(\vartheta_S \circ \widetilde{\mathbb{S}}) \tilde{\cap} (\vartheta_S \circ \widetilde{\mathbb{S}} \circ \vartheta_S) &\tilde{\subseteq} \vartheta_S \circ \widetilde{\mathbb{S}} \circ \vartheta_S \\ &= \widetilde{\mathbb{S}} \circ \vartheta_S \circ \vartheta_S \tilde{\subseteq} \widetilde{\mathbb{S}} \circ \vartheta_S \circ \widetilde{\mathbb{S}} \tilde{\subseteq} \vartheta_S\end{aligned}$$

Therefore, ϑ_S is an S-int ʀ-ƁQ ideal.

Conversely, assume that (2) holds, where ϑ_S is an S-int ʀ-ƁQ ideal of S. Then, $(\vartheta_S \circ \widetilde{\mathbb{S}}) \tilde{\cap} (\vartheta_S \circ \widetilde{\mathbb{S}} \circ \vartheta_S) \tilde{\subseteq} \vartheta_S$. In order to show that ϑ_S S-int interior ideal, we need to show that $\widetilde{\mathbb{S}} \circ \vartheta_S \circ \widetilde{\mathbb{S}} \tilde{\subseteq} \vartheta_S$. By assumption, $\widetilde{\mathbb{S}} = \vartheta_S \circ \widetilde{\mathbb{S}} = \widetilde{\mathbb{S}} \circ \vartheta_S$. Thus,

$$\begin{aligned}&\widetilde{\mathbb{S}} \circ \vartheta_S \circ \widetilde{\mathbb{S}} = \left(\widetilde{\mathbb{S}} \circ \vartheta_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\widetilde{\mathbb{S}} \circ \vartheta_S \circ \widetilde{\mathbb{S}}\right) \\ &= \left(\vartheta_S \circ \widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\vartheta_S \circ \widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}}\right) \tilde{\subseteq} \left(\vartheta_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\vartheta_S \circ \widetilde{\mathbb{S}}\right) \\ &= \left(\vartheta_S \circ \widetilde{\mathbb{S}}\right) \tilde{\cap} \left(\vartheta_S \circ \widetilde{\mathbb{S}} \circ \vartheta_S\right) \tilde{\subseteq} \vartheta_S\end{aligned}$$

Therefore, ϑ_S is an S-int interior ideal.

Theorem 3.45. Let $\vartheta_S \in S_S(U)$ and S be a soft simple* semigroup. Then, the following conditions are equivalent:

(1) ϑ_S is an S-int interior ideal.
(2) ϑ_S is an S-int ƁQ ideal.

Proof: It is followed by Theorem 3.43 and Theorem 3.44.

Proposition 3.46. Let $\mathfrak{p}_S$ and $\mathfrak{s}_S$ be S-int ʟ-(ʀ-) ƁQ ideals. Then, $\mathfrak{p}_S \tilde{\cap} \mathfrak{s}_S$ is an S-int ʟ-(ʀ-) ƁQ ideal.

Proof: The proof is presented only for S-int ʟ-ƁQ ideal, as the proof for S-int ʀ-ƁQ ideal can be shown similarly. Let $\mathfrak{p}_S$ and $\mathfrak{s}_S$ be S-int ʟ-ƁQ ideals of S. Then,

$$\left(\widetilde{\mathbb{S}} \circ \mathfrak{p}_S\right) \tilde{\cap} \left(\mathfrak{p}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{p}_S\right) \tilde{\subseteq} \mathfrak{p}_S$$

$$\left(\widetilde{\mathbb{S}} \circ \mathfrak{s}_S\right) \tilde{\cap} \left(\mathfrak{s}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{s}_S\right) \tilde{\subseteq} \mathfrak{s}_S$$

Thus,

$$\begin{aligned}&\left[\widetilde{\mathbb{S}} \circ (\mathfrak{p}_S \tilde{\cap} \mathfrak{s}_S)\right] \tilde{\cap} \left[(\mathfrak{p}_S \tilde{\cap} \mathfrak{s}_S) \circ \widetilde{\mathbb{S}} \circ (\mathfrak{p}_S \tilde{\cap} \mathfrak{s}_S)\right] \tilde{\subseteq} \\ &\left(\widetilde{\mathbb{S}} \circ \mathfrak{p}_S\right) \tilde{\cap} \left(\mathfrak{p}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{p}_S\right) \tilde{\subseteq} \mathfrak{p}_S\end{aligned}$$

$$\begin{aligned}&\left[\widetilde{\mathbb{S}} \circ (\mathfrak{p}_S \tilde{\cap} \mathfrak{s}_S)\right] \tilde{\cap} \left[(\mathfrak{p}_S \tilde{\cap} \mathfrak{s}_S) \circ \widetilde{\mathbb{S}} \circ (\mathfrak{p}_S \tilde{\cap} \mathfrak{s}_S)\right] \tilde{\subseteq} \\ &\left(\widetilde{\mathbb{S}} \circ \mathfrak{s}_S\right) \tilde{\cap} \left(\mathfrak{s}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{s}_S\right) \tilde{\subseteq} \mathfrak{s}_S\end{aligned}$$

Hence,

$$\left[\tilde{S}\circ(p_S\,\tilde{\cap}\,s_S)\right]\tilde{\cap}\left[(p_S\,\tilde{\cap}\,s_S)\circ\tilde{S}\circ(p_S\,\tilde{\cap}\,s_S)\right]\tilde{\subseteq}\,p_S\,\tilde{\cap}\,s_S$$

Thus, $p_S\,\tilde{\cap}\,s_S$ is an S-int L-BQ ideals.

Theorem 3.47. Let p_S and s_S be S-int BQ ideals. Then, $p_S\,\tilde{\cap}\,s_S$ is an S-int BQ ideals.

Corollary 3.48. The finite intersection of S-int BQ ideals is an S-int BQ ideal.

Proposition 3.49. Let q_S and s_S be S-int L-(R-) ideals. Then, $q_S\,\tilde{\cap}\,s_S$ is an S-int L-(R-) BQ ideal.

Proof: The proof is presented only for S-int L-BQ ideal, as the proof for S-int R-BQ ideal can be shown similarly. Let q_S and s_S be S-int L-ideals of S. Then, $\tilde{S}\circ q_S\,\tilde{\subseteq}\,g_S$ and $\tilde{S}\circ s_S\,\tilde{\subseteq}\,s_S$. Thus,

$$\left[\tilde{S}\circ(q_S\,\tilde{\cap}\,s_S)\right]\tilde{\cap}\left[(q_S\,\tilde{\cap}\,s_S)\circ\tilde{S}\circ(q_S\,\tilde{\cap}\,s_S)\right]\tilde{\subseteq}$$
$$(\tilde{S}\circ q_S)\,\tilde{\cap}\,(q_S\circ\tilde{S}\circ q_S)\,\tilde{\subseteq}\,\tilde{S}\circ q_S\,\tilde{\subseteq}\,q_S$$

$$\left[\tilde{S}\circ(q_S\,\tilde{\cap}\,s_S)\right]\tilde{\cap}\left[(q_S\,\tilde{\cap}\,s_S)\circ\tilde{S}\circ(q_S\,\tilde{\cap}\,s_S)\right]\tilde{\subseteq}$$
$$(\tilde{S}\circ s_S)\,\tilde{\cap}\,(s_S\circ\tilde{S}\circ s_S)\,\tilde{\subseteq}\,\tilde{S}\circ s_S\,\tilde{\subseteq}\,s_S$$

Hence,

$$\left[\tilde{S}\circ(q_S\,\tilde{\cap}\,s_S)\right]\tilde{\cap}\left[(q_S\,\tilde{\cap}\,s_S)\circ\tilde{S}\circ(q_S\,\tilde{\cap}\,s_S)\right]\tilde{\subseteq}\,q_S\,\tilde{\cap}\,s_S$$

Thus, $q_S\,\tilde{\cap}\,s_S$ is an S-int L-BQ ideals.

Theorem 3.50. Let q_S and s_S be S-int ideals. Then, $q_S\,\tilde{\cap}\,s_S$ is an S-int BQ ideals.

Theorem 3.51. Let q_S be an S-int R-ideal and s_S be an S-int L-ideal. Then, $q_S\,\tilde{\cap}\,s_S$ is an S-int BQ ideal.

Proof: Let q_S be an S-int R-ideal and h_S be an S-int L-ideal. Then, $q_S\circ\tilde{S}\,\tilde{\subseteq}\,q_S$, $\tilde{S}\circ s_S\,\tilde{\subseteq}\,s_S$, and $q_S\circ q_S\,\tilde{\subseteq}\,q_S, s_S\circ s_S\,\tilde{\subseteq}\,s_S$.Thus,

$$\left[\tilde{S}\circ(q_S\,\tilde{\cap}\,s_S)\right]\tilde{\cap}\left[(q_S\,\tilde{\cap}\,s_S)\circ\tilde{S}\circ(q_S\,\tilde{\cap}\,s_S)\right]\tilde{\subseteq}$$
$$\left(\tilde{S}\circ s_S\right)\tilde{\cap}\left(q_S\circ\tilde{S}\circ q_S\right)\tilde{\subseteq}\,s_S\,\tilde{\cap}\,(q_S\circ q_S)\,\tilde{\subseteq}\,s_S\,\tilde{\cap}\,q_S$$

Hence, $q_S\,\tilde{\cap}\,s_S$ is an S-int L-BQ ideal. Similarly, since

$$\left[(q_S\,\tilde{\cap}\,s_S)\circ\tilde{S}\right]\tilde{\cap}\left[(q_S\,\tilde{\cap}\,s_S)\circ\tilde{S}\circ(q_S\,\tilde{\cap}\,s_S)\right]\tilde{\subseteq}$$
$$(q_S\circ\tilde{S})\,\tilde{\cap}\,(s_S\circ\tilde{S}\circ s_S)\,\tilde{\subseteq}\,q_S\,\tilde{\cap}\,(s_S\circ s_S)\,\tilde{\subseteq}\,q_S\,\tilde{\cap}\,s_S$$

$q_S\,\tilde{\cap}\,s_S$ is an S-int R-BQ ideal. Therefore, $q_S\,\tilde{\cap}\,s_S$ is an S-int BQ ideal.

Theorem 3.52. Let ϑ_S be an S-int L-BQ ideal and t_S be an S-int L-ideal. Then, $\vartheta_S\,\tilde{\cap}\,t_S$ is an S-int BQ ideal.

Proof: Let ϑ_S be an S-int L-BQ ideal and t_S be an S-int L-ideal. Then, $(\tilde{S}\circ\vartheta_S)\,\tilde{\cap}\left(\vartheta_S\circ\tilde{S}\circ\vartheta_S\right)\tilde{\subseteq}\,\vartheta_S$ and $\tilde{S}\circ t_S\,\tilde{\subseteq}\,t_S$. Thus,

$$\left[\tilde{S}\circ(\vartheta_S\,\tilde{\cap}\,t_S)\right]\tilde{\cap}\left[(\vartheta_S\,\tilde{\cap}\,t_S)\circ\tilde{S}\circ(\vartheta_S\,\tilde{\cap}\,t_S)\right]\tilde{\subseteq}$$
$$(\tilde{S}\circ\vartheta_S)\,\tilde{\cap}\left(\vartheta_S\circ\tilde{S}\circ\vartheta_S\right)\tilde{\subseteq}\,\vartheta_S$$

$$\left[\tilde{S}\circ(\vartheta_S\,\tilde{\cap}\,t_S)\right]\tilde{\cap}\left[(\vartheta_S\,\tilde{\cap}\,t_S)\circ\tilde{S}\circ(\vartheta_S\,\tilde{\cap}\,t_S)\right]\tilde{\subseteq}$$
$$(\tilde{S}\circ t_S)\,\tilde{\cap}\left(t_S\circ\tilde{S}\circ t_S\right)\tilde{\subseteq}\,\tilde{S}\circ t_S\,\tilde{\subseteq}\,t_S$$

Hence,

$$\left[\tilde{S}\circ(\vartheta_S\,\tilde{\cap}\,t_S)\right]\tilde{\cap}\left[(\vartheta_S\,\tilde{\cap}\,t_S)\circ\tilde{S}\circ(\vartheta_S\,\tilde{\cap}\,t_S)\right]\tilde{\subseteq}\,\vartheta_S\,\tilde{\cap}\,t_S$$

Thus, $\vartheta_S\,\tilde{\cap}\,t_S$ is an S-int L-BQ ideal.

Theorem 3.53. Let s_S be an S-int L-ideal and p_S be an SS over U. Then, $s_S\circ p_S$ is an S-int L-BQ ideal.

Proof: Let s_S be an S-int L-ideal. Then, $\tilde{S}\circ s_S\,\tilde{\subseteq}\,s_S$. Thus,

$$\left[\tilde{S}\circ(s_S\circ p_S)\right]\tilde{\cap}\left[(s_S\circ p_S)\circ\tilde{S}\circ(s_S\circ p_S)\right]\tilde{\subseteq}\,\tilde{S}\circ(s_S\circ p_S)$$
$$=\left(\tilde{S}\circ s_S\right)\circ p_S\,\tilde{\subseteq}\,s_S\circ p_S$$

Hence, $s_S\circ p_S$ is an S-int L-BQ ideal.

Theorem 3.54. Let s_S be an S-int R-ideal and p_S be an SS over U. Then, $p_S\circ s_S$ is an S-int R-BQ ideal.

Proof: Let s_S be an S-int R-ideal. Then, $s_S\circ\tilde{S}\,\tilde{\subseteq}\,s_S$. Thus,

$$\left[(p_S\circ s_S)\circ\tilde{S}\right]\tilde{\cap}\left[(p_S\circ s_S)\circ\tilde{S}\circ(p_S\circ s_S)\right]\tilde{\subseteq}\,(p_S\circ s_S)\circ\tilde{S}$$
$$=p_S\circ(s_S\circ\tilde{S})\,\tilde{\subseteq}\,p_S\circ s_S$$

Hence, $p_S\circ s_S$ is an S-int R-BQ ideal.

Theorem 3.55. Let h_S be a nonempty SS over U. Then, every soft subset of h_S containing $\left(\tilde{S}\circ h_S\right)\tilde{\cup}\left(h_S\circ\tilde{S}\right)$ is an S-int BQ ideal.

Proof: Let p_S be a soft subset of h_S containing $\left(\tilde{S}\circ h_S\right)\tilde{\cup}\left(h_S\circ\tilde{S}\right)$. Since,

$$\tilde{S}\circ p_S\,\tilde{\subseteq}\,\tilde{S}\circ h_S\,\tilde{\subseteq}\left(\tilde{S}\circ h_S\right)\tilde{\cup}\left(h_S\circ\tilde{S}\right)\tilde{\subseteq}\,p_S$$

$\left(\tilde{S}\circ p_S\right)\tilde{\subseteq}\,p_S$ is obtained. Hence, p_S is an S-int L-ideal.

$$p_S\circ\tilde{S}\,\tilde{\subseteq}\,h_S\circ\tilde{S}\,\tilde{\subseteq}\,(\tilde{S}\circ h_S)\,\tilde{\cup}\left(h_S\circ\tilde{S}\right)\tilde{\subseteq}\,p_S$$

Thus, $p_S\circ\tilde{S}\,\tilde{\subseteq}\,p_S$. Hence, p_S is an S-int R-ideal. Therefore, p_S is an S-int BQ ideal. Thus, by Theorem 3.30, p_S is an S-int BQ ideal. Hence, every soft subset of h_S containing $\left(\tilde{S}\circ h_S\right)\tilde{\cup}\left(h_S\circ\tilde{S}\right)$ is an S-int BQ ideal.

Theorem 3.56. Let ϑ_S be a nonempty SS over U. Then, every soft subset of ϑ_S containing $\tilde{\mathbb{S}} \circ \vartheta_S$ is an S-int L-BQ ideal.

Proof: Let $\mathfrak{h}_S$ be a soft subset of ϑ_S containing $\tilde{\mathbb{S}} \circ \vartheta_S$. Since, $\tilde{\mathbb{S}} \circ \mathfrak{h}_S \tilde{\subseteq} \tilde{\mathbb{S}} \circ \vartheta_S \tilde{\subseteq} \mathfrak{h}_S$. Thus, $\tilde{\mathbb{S}} \circ \mathfrak{h}_S \tilde{\subseteq} \mathfrak{h}_S$. Hence, $\mathfrak{h}_S$ is an S-int L-ideal. Thus, by Theorem 3.25, $\mathfrak{h}_S$ is an S-int BQ ideal. Hence, every soft subset of ϑ_S containing $\tilde{\mathbb{S}} \circ \vartheta_S$ is an S-int L-BQ ideal.

Theorem 3.57. Let ϑ_S be a nonempty SS over U. Then, every soft subset of ϑ_S containing $(\tilde{\mathbb{S}} \circ \vartheta_S) \tilde{\cap} (\vartheta_S \circ \tilde{\mathbb{S}} \circ \vartheta_S)$ is an S-int L-BQ ideal.

Proof: Let $\mathfrak{h}_S$ be a soft subset of ϑ_S containing $(\tilde{\mathbb{S}} \circ \vartheta_S) \tilde{\cap} (\vartheta_S \circ \tilde{\mathbb{S}} \circ \vartheta_S)$. Then,

$$\tilde{\mathbb{S}} \circ \mathfrak{h}_S \tilde{\subseteq} \tilde{\mathbb{S}} \circ \vartheta_S \text{ and } \mathfrak{h}_S \circ \tilde{\mathbb{S}} \circ \mathfrak{h}_S \tilde{\subseteq} \vartheta_S \circ \tilde{\mathbb{S}} \circ \vartheta_S$$

Since,

$$\left(\tilde{\mathbb{S}} \circ \mathfrak{h}_S\right) \tilde{\cap} \left(\mathfrak{h}_S \circ \tilde{\mathbb{S}} \circ \mathfrak{h}_S\right) \tilde{\subseteq} \left(\tilde{\mathbb{S}} \circ \vartheta_S\right) \tilde{\cap} (\vartheta_S \circ \tilde{\mathbb{S}} \circ \vartheta_S) \tilde{\subseteq} \mathfrak{h}_S$$

Hence, $\mathfrak{h}_S$ is an S-int L-ideal.

Proposition 3.58. Let ρ_S be an S-int subsemigroup over U, σ be a subset of U, $Im(\rho_S)$ be the image of ρ_S such that $\sigma \in Im(\rho_S)$. If ρ_S is an S-int L −(R-) BQ ideal of S, then $\mathcal{U}(\rho_S;\sigma)$ is a L-(R-) BQ ideal of S.

Proof: The proof is presented only for S-int L-BQ ideal, as the proof for S-int R-BQ ideal can be shown similarly. Since, $\rho_S(\mathrm{x}) = \sigma$ for some $\mathrm{x} \in S$, $\emptyset \neq \mathcal{U}(\rho_S;\sigma) \subseteq S$. Let $\kappa \in \left(S.\mathcal{U}(\rho_S;\sigma)\right) \cap \left(\mathcal{U}(\rho_S;\sigma).S.\mathcal{U}(\rho_S;\sigma)\right)$. Then, there exist $\mathrm{x}, y, z \in \mathcal{U}(\rho_S;\sigma)$ and r, $\mathfrak{s} \in S$ such that $\kappa = \mathfrak{s}\mathrm{x} = yrz$. Thus, $\rho_S(x) \supseteq \sigma$, $f_S(y) \supseteq \sigma$ and $\rho_S(z) \supseteq \sigma$. Since ρ_S is an S-int L-BQ ideal,

$$\left(\tilde{\mathbb{S}} \circ \rho_S\right)(\kappa) = \bigcup_{\kappa=\mathfrak{m}n} \left\{\tilde{\mathbb{S}}(\mathfrak{m}) \cap \rho_S(n)\right\} \supseteq \tilde{\mathbb{S}}(\mathfrak{s}) \cap \rho_S(\mathrm{x}) = U \cap \rho_S(\mathrm{x}) = \rho_S(\mathrm{x}) \supseteq \sigma$$

$$\begin{aligned}\left(\rho_S \circ \tilde{\mathbb{S}} \circ \rho_S\right)(\kappa) &= \bigcup_{\kappa=\mathfrak{m}n} \left\{\rho_S(\mathfrak{m}) \cap \left(\tilde{\mathbb{S}} \circ \rho_S\right)(n)\right\} \\ &\supseteq \rho_S(\mathrm{x}) \cap \left(\tilde{\mathbb{S}} \circ \rho_S\right)(yz) \\ &= \rho_S(\mathrm{x}) \cap \bigcup_{yz=pq} \left\{\tilde{\mathbb{S}}(p) \cap \rho_S(q)\right\} \\ &\supseteq \rho_S(\mathrm{x}) \cap \rho_S(y) \cap \rho_S(z) \\ &\supseteq \sigma \cap \sigma \cap \sigma = \sigma\end{aligned}$$

Thus, $(\tilde{\mathbb{S}} \circ \rho_S)(\kappa) \cap (\rho_S \circ \tilde{\mathbb{S}} \circ \rho_S)(\kappa)) \supseteq \sigma$. Since ρ_S is an S-int L-BQ ideal,

$$\rho_S(\kappa) \supseteq (\tilde{\mathbb{S}} \circ \rho_S)(\kappa) \cap (\rho_S \circ \tilde{\mathbb{S}} \circ \rho_S)(\kappa) \supseteq \sigma$$

Thus, $\kappa \in \mathcal{U}(\rho_S;\sigma)$. Therefore,

$$[S.\mathcal{U}(\rho_S;\sigma)] \cap [\mathcal{U}(\rho_S;\sigma).S.\mathcal{U}(\rho_S;\sigma)]$$

Hence, $\mathcal{U}(\rho_S;\sigma)$ is a BQ ideal of S.

Theorem 3.59. Let ρ_S be an S-int subsemigroup over U, σ be a subset of U, $Im(\rho_S)$ be the image of ρ_S such that $\sigma \in Im(\rho_S)$. If ρ_S is an S-int BQ ideal of S, then $\mathcal{U}(\rho_S;\sigma)$ is a BQ ideal ideal of S.

We illustrate Theorem 3.59 with Example 3.60.

Example 3.60. Consider the SS η_S in Example 3.2. By considering the image set of η_S, that is,

$$Im(\eta_S) = \{\{e, x, x^2, y\}, \{e, x, x^2\}, \{e, x\}\}$$

we obtain the following:

$$\mathcal{U}(\eta_S;\sigma) = \begin{cases} \{\mathfrak{f}, h, \mathfrak{r}\}, & \sigma = \{e, x\} \\ \{\mathfrak{f}, \mathfrak{r}\}, & \sigma = \{e, x, x^2\} \\ \{\mathfrak{r}\}, & \sigma = \{e, x, x^2, y\} \end{cases}$$

Here, $\{f, h, \mathfrak{r}\}, \{f, \mathfrak{r}\}$ and $\{\mathfrak{r}\}$ are all BQ ideals of S. In fact, since

$$\{\mathfrak{r}\}.\{\mathfrak{r}\} \subseteq \{\mathfrak{r}\}, \{\mathfrak{f}, \mathfrak{r}\}.\{\mathfrak{f}, \mathfrak{r}\} \subseteq \{\mathfrak{f}, \mathfrak{r}\}, \{\mathfrak{f}, h, \mathfrak{r}\}.\{\mathfrak{f}, h, \mathfrak{r}\} \subseteq \{\mathfrak{f}, h, \mathfrak{r}\}$$

each $\mathcal{U}(\eta_S;\sigma)$ is a subsemigroup of S. Similarly, since

$$\begin{gathered}(S.\{\mathfrak{r}\}) \cap (\{\mathfrak{r}\}.S.\{\mathfrak{r}\}) \subseteq \{\mathfrak{r}\} \cap \{\mathfrak{r}\} \subseteq \{\mathfrak{r}\} \\ (S.\{\mathfrak{f}, \mathfrak{r}\}) \cap (\{\mathfrak{f}, \mathfrak{r}\}.S.\{\mathfrak{f}, \mathfrak{r}\}) \subseteq \{\mathfrak{f}, \mathfrak{r}\} \cap \{\mathfrak{f}, \mathfrak{r}\} \subseteq \{\mathfrak{f}, \mathfrak{r}\} \\ (S.\{\mathfrak{f}, h, \mathfrak{r}\}) \cap (\{\mathfrak{f}, h, \mathfrak{r}\}.S.\{\mathfrak{f}, h, \mathfrak{r}\}) \subseteq \{\mathfrak{f}, h, \mathfrak{r}\} \cap \{\mathfrak{f}, h, \mathfrak{r}\} \subseteq \{\mathfrak{f}, h, \mathfrak{r}\}\end{gathered}$$

each $\mathcal{U}(\eta_S;\sigma)$ is an L-BQ ideal of S. Similarly, since

$$(\{\mathfrak{r}\}.S) \cap (\{\mathfrak{r}\}.S.\{\mathfrak{r}\}) \subseteq \{\mathfrak{r}\} \cap \{\mathfrak{r}\} \subseteq \{\mathfrak{r}\}$$

$$(\{\mathfrak{f}, \mathfrak{r}\}.S) \cap (\{\mathfrak{f}, \mathfrak{r}\}.S.\{\mathfrak{f}, \mathfrak{r}\}) \subseteq \{\mathfrak{f}, \mathfrak{r}\} \cap \{\mathfrak{f}, \mathfrak{r}\} \subseteq \{\mathfrak{f}, \mathfrak{r}\}$$

$$(\{\mathfrak{f}, h, \mathfrak{r}\}.S) \cap (\{\mathfrak{f}, h, \mathfrak{r}\}.S.\{\mathfrak{f}, h, \mathfrak{r}\}) \subseteq \{\mathfrak{f}, h, \mathfrak{r}\} \cap \{\mathfrak{f}, h, \mathfrak{r}\} \subseteq \{\mathfrak{f}, h, \mathfrak{r}\}$$

each $\mathcal{U}(\eta_S;\sigma)$ is an R-BQ ideal of S, and thus each of $\mathcal{U}(\eta_S;\sigma)$ is a BQ ideal of S.

Now, consider the SS $\mathfrak{s}_S$ in Example 3.2. By taking into account

$$Im(\mathfrak{s}_S) = \{\{e, x^2, y, yx^2\}, \{e, x, y\}, \{e, x\}\}$$

we obtain the following:

$$\mathcal{U}(\mathfrak{s}_S;\sigma) = \begin{cases} \{\mathfrak{f}, h\}, & \sigma = \{e, x\} \\ \{\mathfrak{f}\}, & \sigma = \{e, x, y\} \\ \{\mathfrak{r}\}, & \sigma = \{e, x^2, y, yx^2\} \end{cases}$$

Here, $\{\mathfrak{f}, h\}$ is not a BQ ideal of S. In fact, since

$$(S.\{\mathfrak{f}, h\}) \cap (\{\mathfrak{f}, h\}.S.\{\mathfrak{f}, h\}) \subseteq \{\mathfrak{f}, h, \mathfrak{r}\} \cap \{\mathfrak{f}, h, \mathfrak{r}\} \nsubseteq \{\mathfrak{f}, h\}$$

one of the $\mathcal{U}(\mathcal{S}_S;\sigma)$ is not an L-ƁQ ideal of S, hence it is not a ƁQ ideal of S, It is seen that each of $\mathcal{U}(\mathcal{S}_S;\sigma)$ is not a ƁQ ideal of S. On the other hand, in Example 3.2 it was shown that $\mathcal{S}_S$ is not an S-int ƁQ ideal of S.

Proposition 3.61. For a semigroup S, the following conditions are equivalent:

(1) S is regular.

(2) $\eta_S = (\widetilde{\mathbb{S}} \circ \eta_S) \,\widetilde{\cap}\, \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right)$ for every S-int L-ƁQ ideal.

Proof: First assume that (1) holds. Let S be a regular semigroup, η_S be an S-int L-ƁQ ideal and $x \in S$. Then, $(\widetilde{\mathbb{S}} \circ \eta_S) \,\widetilde{\cap}\, \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right) \,\widetilde{\subseteq}\, \eta_S$ and there exist an element $y \in S$ such that $x = xyx$. Since

$$\left(\widetilde{\mathbb{S}} \circ \eta_S\right)(x) = \bigcup_{x=kn} \left\{\widetilde{\mathbb{S}}(k) \cap \eta_S(n)\right\} \supseteq \widetilde{\mathbb{S}}(xy) \cap \eta_S(x) = U \cap \eta_S(x) = \eta_S(x)$$

$$\begin{aligned}\left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right)(x) &= \bigcup_{x=kn} \left\{\eta_S(k) \cap (\widetilde{\mathbb{S}} \circ \eta_S)(n)\right\} \\ &\supseteq \eta_S(x) \cap \left(\widetilde{\mathbb{S}} \circ \eta_S\right)(yx) \\ &= \eta_S(x) \cap \bigcup_{yx=rs} \left\{\widetilde{\mathbb{S}}(r) \cap \eta_S(s)\right\} \\ &\supseteq \eta_S(x) \cap \widetilde{\mathbb{S}}(y) \cap \eta_S(x) \\ &= \eta_S(x) \cap U \cap \eta_S(x) = \eta_S(x)\end{aligned}$$

Thus,

$$\left(\widetilde{\mathbb{S}} \circ \eta_S\right)(x) \cap \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right)(x) \supseteq \eta_S(x) \cap \eta_S(x) \supseteq \eta_S(x)$$

implying that $\eta_S \,\widetilde{\subseteq}\, \left(\widetilde{\mathbb{S}} \circ \eta_S\right) \,\widetilde{\cap}\, \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right)$. Therefore, $\eta_S = (\widetilde{\mathbb{S}} \circ \eta_S) \,\widetilde{\cap}\, \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right)$.

Conversely, let $\eta_S = (\widetilde{\mathbb{S}} \circ \eta_S) \,\widetilde{\cap}\, \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right)$, where f_S is an S-int L-ƁQ ideal. In order to show that S is regular, we need to show that $\mathcal{P} = S\mathcal{P} \cap \mathcal{P}S\mathcal{P}$ for every L-ƁQ ideal of S. It is obvious that $S\mathcal{P} \cap \mathcal{P}S\mathcal{P} \subseteq \mathcal{P}$. Thus, it is enough to show that $\mathcal{P} \subseteq S\mathcal{P} \cap \mathcal{P}S\mathcal{P}$. Let $d \in \mathcal{P}$ and $\mathcal{P}$ be any L-ƁQ ideal of S. Thus, $S_{\mathcal{P}}$ is an S-int L-ƁQ ideal ideal. Hence,

$$\begin{aligned}S_{\mathcal{P}}(d) &= \left(\widetilde{\mathbb{S}} \circ S_{\mathcal{P}}\right)(d) \cap \left(S_{\mathcal{P}} \circ \widetilde{\mathbb{S}} \circ S_{\mathcal{P}}\right)(d) \\ &= (S_S \circ S_{\mathcal{P}})(d) \cap (S_{\mathcal{P}} \circ S_S \circ S_{\mathcal{P}})(d) \\ &= S_{S\mathcal{P} \cap \mathcal{P}S\mathcal{P}}(d) = U\end{aligned}$$

implying that $d \in S\mathcal{P} \cap \mathcal{P}S\mathcal{P}$. Hence, $\mathcal{P} = S\mathcal{P} \cap \mathcal{P}S\mathcal{P}$ so S is a regular semigroup.

Proposition 3.62. For a semigroup S, the following conditions are equivalent:

(1) S is regular.

(2) $ƀ_S = (ƀ_S \circ \widetilde{\mathbb{S}}) \,\widetilde{\cap}\, \left(ƀ_S \circ \widetilde{\mathbb{S}} \circ ƀ_S\right)$ for every S-int R-ƁQ ideal.

Proof: First assume that (1) holds. Let S be a regular semigroup, $ƀ_S$ be an S-int R-ƁQ ideal and $x \in S$. Then, $(ƀ_S \circ \widetilde{\mathbb{S}}) \,\widetilde{\cap}\, \left(ƀ_S \circ \widetilde{\mathbb{S}} \circ ƀ_S\right) \,\widetilde{\subseteq}\, ƀ_S$ and there exist an element $t \in S$ such that $x = xtx$. Since,

$$\left(ƀ_S \circ \widetilde{\mathbb{S}}\right)(x) = \bigcup_{x=kn} \left\{ƀ_S(k) \cap \widetilde{\mathbb{S}}(n)\right\} \supseteq ƀ_S(x) \cap \widetilde{\mathbb{S}}(tx) = ƀ_S(x) \cap U = ƀ_S(x)$$

$$\left(ƀ_S \circ \widetilde{\mathbb{S}} \circ ƀ_S\right)(x) = \cup_{x=kn}\left\{ƀ_S(k) \cap (\widetilde{\mathbb{S}} \circ ƀ_S)(n)\right\} \supseteq ƀ_S(x) \cap \left(\widetilde{\mathbb{S}} \circ ƀ_S\right)(tx) = ƀ_S(x) \cap \cup_{tx=qs}\left\{\widetilde{\mathbb{S}}(q) \cap ƀ_S(s)\right\} \supseteq ƀ_S(x) \cap \widetilde{\mathbb{S}}(y) \cap ƀ_S(x) = ƀ_S(x) \cap U \cap ƀ_S(x) = ƀ_S(x).$$

Thus,

$$\left(ƀ_S \circ \widetilde{\mathbb{S}}\right)(x) \cap \left(ƀ_S \circ \widetilde{\mathbb{S}} \circ ƀ_S\right)(x) \supseteq ƀ_S(x) \cap ƀ_S(x) \supseteq ƀ_S(x)$$

implying that $ƀ_S \,\widetilde{\subseteq}\, \left(ƀ_S \circ \widetilde{\mathbb{S}}\right) \,\widetilde{\cap}\, \left(ƀ_S \circ \widetilde{\mathbb{S}} \circ ƀ_S\right)$. Therefore, $ƀ_S = \left(ƀ_S \circ \widetilde{\mathbb{S}}\right) \,\widetilde{\cap}\, \left(ƀ_S \circ \widetilde{\mathbb{S}} \circ ƀ_S\right)$.

Conversely, let $ƀ_S = \left(ƀ_S \circ \widetilde{\mathbb{S}}\right) \,\widetilde{\cap}\, \left(ƀ_S \circ \widetilde{\mathbb{S}} \circ ƀ_S\right)$ where $ƀ_S$ is an S-int R-ƁQ ideal. In order to show that S is regular, we need to show that $\mathcal{M} = \mathcal{M}S \cap \mathcal{M}S\mathcal{M}$ for every R-ƁQ ideal of S. It is obvious that $\mathcal{M}S \cap \mathcal{M}S\mathcal{M} \subseteq \mathcal{M}$. Thus, it is enough to show that $\mathcal{M} \subseteq \mathcal{M}S \cap \mathcal{M}S\mathcal{M}$. Let $v \in \mathcal{M}$ and $\mathcal{M}$ be any R-ƁQ ideal of S. Thus, $S_{\mathcal{M}}$ is an S-int R-ƁQ ideal ideal. Hence,

$$\begin{aligned}S_{\mathcal{M}}(v) &= \left(S_{\mathcal{M}} \circ \widetilde{\mathbb{S}}\right)(v) \cap \left(S_{\mathcal{M}} \circ \widetilde{\mathbb{S}} \circ S_{\mathcal{M}}\right)(v) \\ &= \left(S_{\mathcal{M}} \circ S_S\right)(v) \cap \left(S_{\mathcal{M}} \circ S_S \circ S_{\mathcal{M}}\right)(v) \\ &= S_{\mathcal{M}S \cap \mathcal{M}S\mathcal{M}}(v) = U\end{aligned}$$

implying that $v \in \mathcal{M}S \cap \mathcal{M}S\mathcal{M}$. Hence, $\mathcal{M} = \mathcal{M}S \cap \mathcal{M}S\mathcal{M}$ so S is a regular semigroup.

Theorem 3.63. For a semigroup S, the following conditions are equivalent:

(1) S is regular.

(2) $\eta_S = \left(\widetilde{\mathbb{S}} \circ \eta_S\right) \,\widetilde{\cap}\, \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right) = \left(\eta_S \circ \widetilde{\mathbb{S}}\right) \,\widetilde{\cap}\, \left(\eta_S \circ \widetilde{\mathbb{S}} \circ \eta_S\right)$ for every S-int ƁQ ideal.

Proof: It is followed by Proposition 3.61 and Proposition 3.62.

Proposition 3.64. Let S be a regular semigroup. Then every S-int L-ƁQ ideal of a semigroup S is an S-int quasi ideal of semigroup.

Proof: Let f_S be an S-int L-ƁQ ideal of S. Then, $\left(\widetilde{S} \circ ɋ_S\right) \tilde{\cap} \left(ɋ_S \circ \widetilde{S} \circ ɋ_S\right) \tilde{\subseteq} ɋ_S$. We know that $ɋ_S \circ \widetilde{S}$ and $\widetilde{S} \circ ɋ_S$ are S-int R-and S-int L-ideals of the semigroup S respectively. By Corollary 2.20, we have $\left(ɋ_S \circ \widetilde{S}\right) \tilde{\cap} \left(\widetilde{S} \circ ɋ_S\right) = ɋ_S \circ \widetilde{S} \circ \widetilde{S} \circ ɋ_S$. Thus,

$$(ɋ_S \circ \widetilde{S}) \tilde{\cap} (\widetilde{S} \circ ɋ_S) \tilde{\subseteq} \widetilde{S} \circ ɋ_S \text{ and } \left(ɋ_S \circ \widetilde{S}\right) \tilde{\cap} \left(\widetilde{S} \circ ɋ_S\right) = ɋ_S \circ \widetilde{S} \circ \widetilde{S} \circ ɋ_S \tilde{\subseteq} ɋ_S \circ \widetilde{S} \circ ɋ_S$$

Hence,

$$\left(ɋ_S \circ \widetilde{S}\right) \tilde{\cap} \left(\widetilde{S} \circ ɋ_S\right) \tilde{\subseteq} \left(\widetilde{S} \circ ɋ_S\right) \tilde{\cap} (ɋ_S \circ \widetilde{S} \circ ɋ_S) \tilde{\subseteq} ɋ_S$$

Therefore, $ɋ_S$ is an S-int quasi ideal.

Proposition 3.65. Let S be a regular semigroup. Then every S-int R-ƁQ ideal of a semigroup S is an S-int quasi ideal of semigroup.

Proof: Let $ɋ_S$ be an S-int R-ƁQ ideal of S. Then, $\left(ɋ_S \circ \widetilde{S}\right) \tilde{\cap} \left(ɋ_S \circ \widetilde{S} \circ ɋ_S\right) \tilde{\subseteq} ɋ_S$. We know that $ɋ_S \circ \widetilde{S}$ and $\widetilde{S} \circ ɋ_S$ are S-int R-and S-int L-ideals of the semigroup S respectively. By Corollary 2.20, we have

$$\left(ɋ_S \circ \widetilde{S}\right) \tilde{\cap} \left(\widetilde{S} \circ ɋ_S\right) = ɋ_S \circ \widetilde{S} \circ \widetilde{S} \circ ɋ_S$$

Thus,

$$\left(ɋ_S \circ \widetilde{S}\right) \tilde{\cap} \left(\widetilde{S} \circ ɋ_S\right) \tilde{\subseteq} ɋ_S \circ \widetilde{S}$$

$$\left(ɋ_S \circ \widetilde{S}\right) \tilde{\cap} \left(\widetilde{S} \circ ɋ_S\right) = ɋ_S \circ \widetilde{S} \circ \widetilde{S} \circ ɋ_S \tilde{\subseteq} ɋ_S \circ \widetilde{S} \circ ɋ_S.$$

Hence,

$$\left(ɋ_S \circ \widetilde{S}\right) \tilde{\cap} \left(\widetilde{S} \circ ɋ_S\right) \tilde{\subseteq} \left(ɋ_S \circ \widetilde{S}\right) \tilde{\cap} (ɋ_S \circ \widetilde{S} \circ ɋ_S) \tilde{\subseteq} ɋ_S$$

Therefore, $ɋ_S$ is an S-int quasi ideal.

Theorem 3.66. Let S be a regular semigroup. Then every S-int ƁQ ideal of a semigroup S is an S-int quasi ideal of semigroup.

Proof: It is followed by Proposition 3.64 and Proposition 3.65.

4. DISCUSSION AND CONCLUSION

Rao [8] expanded the notions of quasi-ideal, bi-ideal, L-(R-) ideal, and ideal in semigroups by defining ƁQ ideals and examining their characteristics. In this study, we have applied the concept of "S-int ƁQ ideals of semigroups" to both SS theory and semigroup theory. It has been shown that every S-int bi-ideal, S-int ideal, S-int quasi-ideal, and S-int interior ideal of an idempotent SS is an S-int ƁQ ideal. Counterexamples show that the converse is not always true, and for the converse to hold, the semigroup must be simple* or regular. It has also been demonstrated that in a soft simple* semigroup, the S-int ƁQ ideal coincides with the S-int bi-ideal, S-int L-(R-) ideal, S-int quasi-ideal, and S-int interior ideal. To link SS theory and classical semigroup theory, it is shown that if a subsemigroup is an S-int ƁQ ideal, its upper α-inclusion set is also a ƁQ ideal. Furthermore, if a subsemigroup is a ƁQ ideal, its SCF is an S-int ƁQ ideal, and the reverse is also true. The finite soft intersections of S-int ƁQ ideals are shown to be S-int ƁQ ideals, as are the soft intersections of S-int ideals. Additionally, the relationship between regular semigroups and S-int ƁQ ideals is explored. In future studies, the characterization of S-int ƁQ ideals of semigroups can be conducted with respect to various types of semigroups, such as L-(R-) simple semigroups, L-(R-) zero semigroups, and intra-regular semigroups.

The relation between several S-int ideals and their generalized ideals is depicted in the following figure, where $\mathcal{A} \rightarrow \mathcal{B}$ denotes that $\mathcal{A}$ is $\mathcal{B}$ but $\mathcal{B}$ may not always be $\mathcal{A}$.

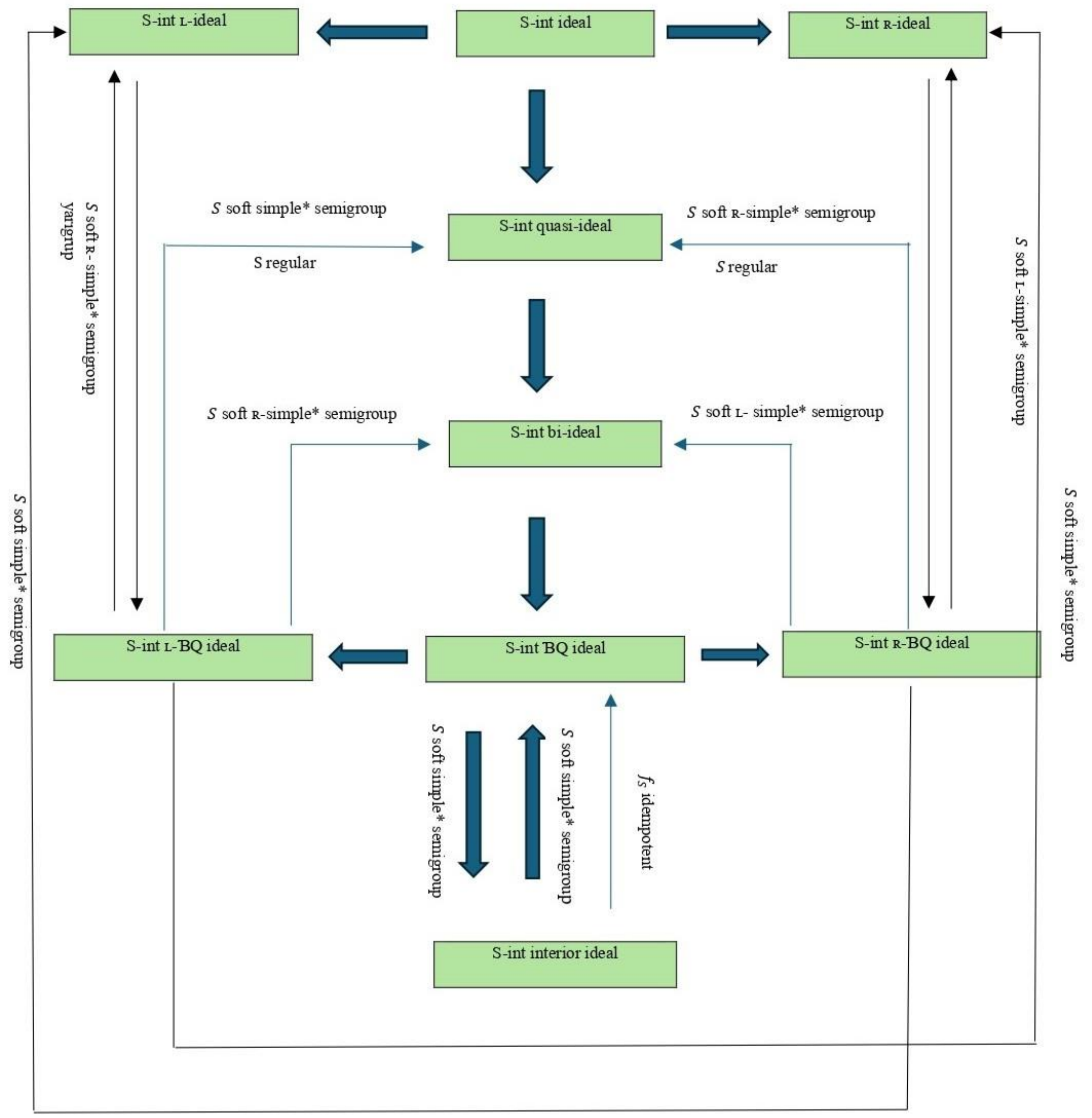

Figure 1. Diagram illustrating the relationships between some S-int ideals

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