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Research Article

On New Sequences of p-Binomial and Catalan Transforms of the k-Mersenne Numbers and Associated Generating Functions

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Abstract

In this work, we investigate the binomial transforms and Catalan transform of the k-Mersenne and k-Mersenne-Lucas numbers and examine the new integer sequences. We apply the p-binomial, rising p-binomial, and falling p-binomial transforms to the k-Mersenne sequences and present the associated generating and exponential generating functions for these transforms. Lastly, we provide the corresponding Binet-type formulas and recurrence relations for binomial transforms of the k-Mersenne (k-Mersenne-Lucas) numbers. These results are supported by numerical illustrations.

1. Introduction and Preliminaries

Number sequences represent ordered sets of numbers that reveal underlying mathematical patterns with significant implications for problem-solving. Various types of number sequences exist, including Fibonacci, arithmetic, and geometric progressions, each exhibiting unique characteristics. Notably, the Fibonacci sequence $(0,1,1,2,3,5,8,13...)$, where each term is the sum of the two preceding terms, appears in nature, art, and architecture, showcasing the inherent beauty of mathematical patterns in the world around us.

Fibonacci numbers have attracted considerable attention among number theorists due to their fascinating properties, leading to extensive research on their characteristics, extensions, generalizations, and applications. For properties and application of Fibonacci like numbers one can see [1–3], the journals 'Fibonacci Quarterly', 'Journal of Integer Sequences', 'INTEGERS', etc. Among others, one of these generalizations is Mersenne numbers which are of the kind 2^n-1, $n\in\mathbb{N}$ and one of the generalizations of Mersenne numbers is termed as k-Mersenne numbers. By this study, we examine various new integer sequences and their generating functions through binomial, p-binomial, and Catalan transforms with the k-Mersenne numbers.

For $n\geq 0$, the sequence $a_{n+1}=3a_n-2a_{n-1}$ generates Mersenne numbers $\{M_n\}$ [4] when $a_0=0,\ a_1=1$, and Mersenne-Lucas numbers $\{m_n\}$ [5] when $a_0=2,\ a_1=3$. For recent developments, generalizations and applications of the Mersenne numbers, one can see [5–12]. Let us restate the definitions and some useful results of the Mersenne numbers.

Definition 1.1. *[5, 11] Let $k\in\mathbb{R}^+$ and $n\in\mathbb{N}$, then the k-Mersenne numbers $\{M_{k,n}\}$ and k-Mersenne-Lucas numbers $\{m_{k,n}\}$ are given as*

$$M_{k,n+1}=3kM_{k,n}-2M_{k,n-1},\quad \text{with } M_{k,0}=0,\ M_{k,1}=1 \tag{1.1}$$
$$\text{and}\quad m_{k,n+1}=3km_{k,n}-2m_{k,n-1},\quad \text{with } m_{k,0}=2,\ m_{k,1}=3k.$$

For (1.1) the characteristic equation is $x^2-3kx+2=0$ and its roots are $r_1=(3k+\sqrt{9k^2-8})/2$ and $r_2=(3k-\sqrt{9k^2-8})/2$ which satisfy the following relations:

$$r_1+r_2=3k,\quad r_1-r_2=\sqrt{9k^2-8},\quad r_1r_2=2. \tag{1.2}$$

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The Binet's formula for these numbers are

$$M_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \quad \text{and} \quad m_{k,n} = r_1^n + r_2^n.$$

The first few terms of these numbers are as follows:

n	$M_{k,n}$	$m_{k,n}$
0	0	2
1	1	3k
2	$3k$	$9k^2-4$
3	$9k^2-2$	$27k^3-18k$
4	$27k^3-12k$	$81k^4-72k^2+8$
5	$81k^4-54k^2+4$	$243k^5-270k^3+60k$
6	$243k^5-216k^3+36k$	$729k^6-972k^4+324k^2-16$
7	$729k^6-810k^4+216k^2-8$	$2187k^7-3402k^5+1512k^3-168k$
8	$2187k^7-2916k^5+1080k^3-96k$	$6561k^8-11664k^6+6480k^4-1152k^2+32$
$\vdots$	$\vdots$	$\vdots$

At instance, for $k=1$, the k-Mersenne sequence gives the classic Mersenne sequence $\{1,3,7,15,31,63,127,255,...\}$: $[A001595]$, for $k=2$, $\{1,6,34,192,1084,6120,34552,195072,...\}$: $[A154244]$ and for $k=3$, $\{1,9,79,693,6079,53325,467767,4103253,...\}$, etc. where sequence [A154244] is the binomial transform of [A126473].
Similarly in the k-Mersenne-Lucas sequence, $k=1$ gives the classic Mersenne-Lucas numbers i.e. $\{2,3,5,9,17,33,65,129,...\}$: $[A000051]$, for $k=2$, $\{2,6,32,180,1016,5736,32384,182832,...\}$ and for $k=3$, $\{2,9,77,675,5921,51939,455609,3996603,...\}$, etc.

1.1. Binomial and Catalan transforms

In literature, there are various transforms that performs on number sequences. For instance, the Binomial Transform (BT) [13, 14], Catalan Transform (CT) [15, 16], Hankel Transform (HT), Discrete Fourier Transform (DFT), etc. The overall estimation of BT and CT is based on integers, not based on the floating numbers, make them faster and more reliable method of transform unlikely DFT, DCT (Discrete Cosine Transform), etc.
The Catalan numbers $\{C_n\}$ [A000108] (due to Eugene Charles Catalan) are defined by

$$C_n = \frac{1}{1+n}\binom{2n}{n} = \frac{2n!}{(1+n)!(n)!}.$$

Thus, $\{1,1,2,5,14,42,132,429,...\}$ are first few terms of Catalan sequence. The Catalan numbers C_n also satisfy the following relation:

$$\frac{C_{n+1}}{C_n} = \frac{2(2n+1)}{n+2}.$$

Barry [15] give the Catalan transform $\{CA_n\}$ corresponding to a number sequence $\{A_n\}$ as follows:

$$CA_n = \sum_{r=0}^{n} \frac{r}{2n-r}\binom{2n-r}{n-r}A_r,\ n \geq 1, \ \text{ where } CA_0 \text{ is proviced.} \tag{1.3}$$

We should note that for the Catalan numbers, the generating function $c(x)$ is given as

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}. \tag{1.4}$$

Also, the binomial transform $\{b_n\}$ for an integer sequence $\{a_n\}$ is given as

$$b_n = \sum_{r=0}^{n}\binom{n}{r}a_r,$$

which is an invertible transformation and inverse transformation is given as

$$a_n = \sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}b_r.$$

Recently, Falcon and Plaza [13] obtained the binomial transforms for k-Fibonacci numbers and then Falcon [16] obtained the Catalan transform of this sequence. Tastan and Ozkan [17] and Prasad et al. [18] obtained the Catalan transform of the k-Pell and extended k-Horadam sequences, respectively. Yilmaz and Taskara [19] studied the recurrent binomial transforms for a sequence of matrices associated with the Padovan and Perrin numbers. Özkan et al. [20] studied the Catalan transform for the Jacobsthal numbers and polynomials. The binomial transform, Catalan transform and the Catalan triangle are versatile mathematical tools that have applications in various fields, particularly in combinatorics, number theory, Encoding/Decoding, various counting problems, applied fields like signal and image processing, etc.
Some recent developments on binomial and Catalan transforms with a number sequence can be seen in [13, 15, 16, 21–25].

2. Binomial and Catalan Transforms on k-Mersenne Sequences

Here, we start by applying the binomial and Catalan transform to the k-Mersenne numbers and investigate the newly generated integer sequences.
Let us define the binomial transform $B_k = \{BM_{k,n}\}$ of k-Mersenne sequence and $C_k = \{Bm_{k,n}\}$ of k-Mersenne-Lucas sequence, where

$$BM_{k,n} = \sum_{r=0}^{n} \binom{n}{r} M_{k,r} \quad \text{and} \quad Bm_{k,n} = \sum_{r=0}^{n} \binom{n}{r} m_{k,r}. \tag{2.1}$$

On running n over $\mathbb{N} \cup \{0\}$, the binomial transform for both the sequences are, respectively:

$$B_k = \{0, 1,\ 3k+2,\ 9k^2+9k+1,\ 27k^3+36k^2+6k-4, \ldots\}, \tag{2.2}$$
$$C_k = \{2, 2+3k, 9k^2+6k-2,\ 27k^3+27k^2-9k-10,\ 81k^4+108k^3-18k^2-60k-14, \ldots\}. \tag{2.3}$$

Thus, setting $k = 1,2,3,4,5$ in (2.2) and (2.3), the binomial transforms are:

$B_1 = \{0,1,5,19,65,211,665,\ldots\} : A001047$
$B_2 = \{0,1,8,55,368,2449,16280,\ldots\}$
$B_3 = \{0,1,11,109,1067,10429,101915,\ldots\}$
$B_4 = \{0,1,14,181,2324,29821,382634,\ldots\}$
$B_5 = \{0,1,17,271,4301,68239,1082645,\ldots\}$

$C_1 = \{2,5,13,35,97,275,793,\ldots\} : A007689$
$C_2 = \{2,8,46,296,1954,12968,86158,\ldots\}$
$C_3 = \{2,11,97,935,9121,89111,870769\ldots\}$
$C_4 = \{2,14,166,2114,27106,347774,4462246\ldots\}$
$C_5 = \{2,17,253,3995,63361,1005227,15948361,\ldots\}.$

2.1. The p-binomial, Rising p-binomial, and Falling p-binomial transforms

Analogous to binomial transform, Spivey and Steil [14] introduced three kinds of the binomial transform with two inputs: an integer sequence A_n and a fixed quantity (scalar) p and referred them as the p-binomial, rising p-binomial and falling p-binomial transform. Falcon and Plaza [13] studied these binomial transforms for k-Fibonacci sequences. Building upon the work of Spivey [14], this section examines transformations for k-Mersenne and k-Mersenne-Lucas sequences. Closed-form definitions for these transformations are presented, along with several specific examples.
The p-binomial transforms $\mathscr{B}M_p$ and $\mathscr{B}m_p$ for sequences $M_{k,n}$ and $m_{k,n}$, respectively, are the sequences $\mathscr{B}M_p = \{x_{p,n}\}_{n\geq 0}$ and $\mathscr{B}m_p = \{y_{p,n}\}_{n\geq 0}$, where $x_{p,n}$ and $y_{p,n}$ are given as

$$x_{p,n} = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} (p)^n M_{k,i} & \text{for } p \neq 0,\ n \neq 0, \\ M_{k,0} & \text{for } p = 0 \text{ or } n = 0 \end{cases} \tag{2.4}$$

and

$$y_{p,n} = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} (p)^n m_{k,i} & \text{for } p \neq 0, n \neq 0, \\ m_{k,0} & \text{for } p = 0 \text{ or } n = 0. \end{cases} \tag{2.5}$$

Let $n \geq 0$ then for the k-Mersenne sequence $\{M_{k,n}\}$, the rising p-binomial transform $\mathscr{R}M_p$ and falling p-binomial transform $\mathscr{F}M_p$ are the sequence $\mathscr{R}M_p = \{r_{p,n}\}$ and $\mathscr{F}M_p = \{f_{p,n}\}$, respectively, where $r_{p,n}$ and $f_{p,n}$ are defined as

$$r_{p,n} = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} (p)^i M_{k,i} & \text{for } p \neq 0, \\ M_{k,0} & \text{for } p = 0, \end{cases} \quad \text{and} \quad f_{p,n} = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} (p)^{n-i} M_{k,i} & \text{for } p \neq 0, \\ M_{k,0} & \text{for } p = 0. \end{cases} \tag{2.6}$$

Similarly for k-Mersenne-Lucas sequence, $\mathscr{R}m_p = \{s_{p,n}\}_{n\geq 0}$ and $\mathscr{F}m_p = \{t_{p,n}\}_{n\geq 0}$, where $s_{p,n}$ and $t_{p,n}$ are given as

$$s_{p,n} = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} (p)^i m_{k,i} & \text{for } p \neq 0, \\ m_{k,0} & \text{for } p = 0, \end{cases} \quad \text{and} \quad t_{p,n} = \begin{cases} k\sum_{i=0}^{n} \binom{n}{i} (p)^{n-i} m_{k,i} & \text{for } p \neq 0, \\ m_{k,0} & \text{for } p = 0. \end{cases} \tag{2.7}$$

We should note that for $p = 1$, the above three p-binomial transforms overlap with the binomial transform B_k and C_k.
From (2.4) and (2.5), after performing the necessary calculations, we have

$$\mathscr{B}M_p = \{0, p, p^2(3k+2), p^3(9k^2+9k+1), p^4(27k^3+36k^2+6k-4), \ldots\},$$
$$\mathscr{B}m_p = \{2, p(2+3k), p^2(9k^2+6k-2), p^3(27k^3+27k^2-9k-10), p^4(81k^4+108k^3-18k^2-60k-14), \ldots\}.$$

Setting $p = 1,2,3,4$ in the above p-binomial transforms $\mathscr{B}M_p$ and $\mathscr{B}m_p$, we get

$$\mathscr{B}M_1 = \{0, 1, (3k+2),\ (9k^2+9k+1),\ (27k^3+36k^2+6k-4), \ldots\},$$
$$\mathscr{B}M_2 = \{0, 2, 4(3k+2), 8(9k^2+9k+1), 16(27k^3+36k^2+6k-4), \ldots\},$$
$$\mathscr{B}M_3 = \{0, 3, 9(3k+2), 27(9k^2+9k+1), 81(27k^3+36k^2+6k-4), \ldots\},$$
$$\mathscr{B}M_4 = \{0, 4, 16(3k+2), 64(9k^2+9k+1), 256(27k^3+36k^2+6k-4), \ldots\},$$

and

$$\mathscr{B}m_1 = \{2,\ (2+3k),\ (9k^2+6k-2),\ (27k^3+27k^2-9k-10),\ (81k^4+108k^3-18k^2-60k-14),...\},$$
$$\mathscr{B}m_2 = \{2, 2(2+3k), 4(9k^2+6k-2), 8(27k^3+27k^2-9k-10), 16(81k^4+108k^3-18k^2-60k-14),...\},$$
$$\mathscr{B}m_3 = \{2, 3(2+3k), 9(9k^2+6k-2), 27(27k^3+27k^2-9k-10), 81(81k^4+108k^3-18k^2-60k-14),...\},$$
$$\mathscr{B}m_4 = \{2, 4(2+3k), 16(9k^2+6k-2), 64(27k^3+27k^2-9k-10), 256(81k^4+108k^3-18k^2-60k-14),...\}.$$

Similarly, on performing the necessary calculations with the rising p-binomial transforms given in (2.6) and (2.7), we have

$$\mathscr{R}M_p = \{0, p, 2p+3kp^2, 3p+9kp^2+(9k^2-2)p^3, 4p+18kp^2+(36k^2-8)p^3+(27k^3-12k)p^4,...\},$$
$$\mathscr{R}m_p = \{2,\ 2+3kp,\ 2+6kp+9k^2p^2-4p^2,\ 2+9kp+(27k^2-12)p^2+(27k^3-18k)p^3,...\}.$$

Thus, on setting $p=1,2,3,4$ in the above rising p-binomial transforms $\mathscr{R}M_p$ and $\mathscr{R}m_p$, we get

$$\mathscr{R}M_1 = \{0, 1, 2+3k, 1+9k+9k^2, -4+6k+36k^2+27k^3,...\},$$
$$\mathscr{R}M_2 = \{0, 2, 4+12k, -10+36k+72k^2, -56-120k+288k^2+432k^3,...\},$$
$$\mathscr{R}M_3 = \{0, 3, 6+27k, -45+81k+243k^2, -204-810k+972k^2+2187k^3,...\},$$
$$\mathscr{R}M_4 = \{0, 4, 8+48k, -116+144k+576k^2, -496-2784k+2304k^2+6912k^3,...\},$$

and

$$\mathscr{R}m_1 = \{2, 2+3k, -2+6k+9k^2, -10-9k+27k^2+27k^3,...\},$$
$$\mathscr{R}m_2 = \{2, 2+6k, -14+12k+36k^2, -46-126k+108k^2+216k^3,...\},$$
$$\mathscr{R}m_3 = \{2, 2+9k, -34+18k+81k^2, -106-459k+243k^2+729k^3,...\},$$
$$\mathscr{R}m_4 = \{2, 2+12k, -62+24k+144k^2, -190-1116k+432k^2+1728k^3,...\}.$$

Finally, performing the necessary calculations with the falling p-binomial transform for $\{M_{k,n}\}$ and $\{m_{k,n}\}$ gives the following sequences in p:

$$\mathscr{F}M_p = \{0, 1, 2p+3k, 3p^2+9kp+(9k^2-2), 4p^3+18kp^2+(36k^2-8)p+(27k^3-12k),...\},$$
$$\mathscr{F}m_p = \{2, 2p+3k, 2p^2+6kp+9k^2-4, 2p^3+9kp^2+(27k^2-12)p+(27k^3-18k),...\}.$$

Thus, first few falling p-binomial transforms are:

$$\mathscr{F}M_1 = \{0, 1, 2+3k, 1+9k+9k^2, -4+6k+36k^2+27k^3,...\},$$
$$\mathscr{F}M_2 = \{0, 1, 4+3k, 10+18k+9k^2, 16+60k+72k^2+27k^3,...\},$$
$$\mathscr{F}M_3 = \{0, 1, 6+3k, 25+27k+27k^2, 84+150k+108k^2+27k^3,...\},$$
$$\mathscr{F}M_4 = \{0, 1, 8+3k, 46+36k+9k^2, 224+276k+144k^2+27k^3,...\},$$

and

$$\mathscr{F}m_1 = \{2, 2+3k, -2+6k+9k^2, -10-9k+27k^2+27k^3,...\},$$
$$\mathscr{F}m_2 = \{2, 4+3k, 4+12k+9k^2, -8+18k+54k^2+27k^3,...\},$$
$$\mathscr{F}m_3 = \{2, 6+3k, 14+18k+9k^2, 18+63k+81k^2+27k^3,...\},$$
$$\mathscr{F}m_4 = \{2, 8+3k, 28+24k+9k^2, 80+126k+108k^2+27k^3,...\}.$$

2.2. Catalan transform and Catalan triangle

The Catalan transform and the Catalan triangle are mathematical tools that have applications in various fields, particularly in combinatorics, number theory, Encoding/Decoding, various counting problems, etc. More recently, Özkan, et al. [26] studied the k-Mersenne sequences where they examined the Catalan transform and other properties. Some recent developments on Catalan transforms with a number sequence can be seen in [15, 16, 20, 23].

For $n \geq 0$, let us define the Catalan transform $CW_k = \{CW_{k,n}\}_{n\geq 0}$ of the sequence $\{W_{k,n}\}$ following (1.3), where $W_{k,n} = M_{k,n}$ or $m_{k,n}$, as follows:

$$CW_{k,n} = \sum_{r=0}^{n} \frac{r}{2n-r}\binom{2n-r}{n-r} W_{k,r} \quad \text{with } CW_{k,0} = 0. \tag{2.8}$$

Thus, we have

$$CW_{k,1} = \sum_{r=0}^{1} \frac{r}{2-r}\binom{2-r}{1-r} W_{k,r} = 0W_{k,0} + 1W_{k,1} = W_{k,1},$$
$$CW_{k,2} = \sum_{r=0}^{2} \frac{r}{4-r}\binom{4-r}{2-r} W_{k,r} = \frac{1}{3}\binom{3}{1}W_{k,1} + \frac{2}{2}\binom{4}{0}W_{k,2} = W_{k,1} + W_{k,2},$$
$$CW_{k,3} = \sum_{r=0}^{3} \frac{r}{6-r}\binom{6-r}{3-r} W_{k,r} = \frac{1}{5}\binom{5}{2}W_{k,1} + \frac{2}{4}\binom{4}{1}W_{k,2} + \frac{3}{3}\binom{3}{0}W_{k,3} = 2W_{k,1} + 2W_{k,2} + W_{k,3},$$

Similarly,

$$CW_{k,4}=\sum_{r=0}^{4}\frac{r}{8-r}\binom{8-r}{4-r}W_{k,r}=5W_{k,1}+5W_{k,2}+3W_{k,3}+W_{k,4},$$

$$CW_{k,5}=\sum_{r=0}^{5}\frac{r}{10-r}\binom{10-r}{5-r}W_{k,r}=14W_{k,1}+14W_{k,2}+9W_{k,3}+4W_{k,4}+W_{k,5},$$

$$CW_{k,6}=\sum_{r=0}^{6}\frac{r}{12-r}\binom{12-r}{6-r}W_{k,r}=42W_{k,1}+42W_{k,2}+28W_{k,3}+14W_{k,4}+5W_{k,5}+W_{k,6},$$

$$CW_{k,7}=132W_{k,1}+132W_{k,2}+90W_{k,3}+48W_{k,4}+20W_{k,5}+6W_{k,6}+W_{k,7}.$$

The above transforms can be represented in matrix form as $\mathscr{G}W_k^T=LX^T$ with $X=[W_{k,1},W_{k,2},W_{k,3},...]$ and $\mathscr{G}W_k=[CW_{k,1},CW_{k,2},CW_{k,3},...]$, where $L=[a_{ij}]_{i,j\geq 1}$ is given as

$$L=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 5 & 5 & 3 & 1 & 0 & 0 & 0 & \dots \\ 14 & 14 & 9 & 4 & 1 & 0 & 0 & \dots \\ 42 & 42 & 28 & 14 & 5 & 1 & 0 & \dots \\ 132 & 132 & 90 & 48 & 20 & 6 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We should note that from matrix L, we have
(a). entries of the first column is the Catalan numbers and for $i\geq 2$ the second column is equal to first column.
(b). for $i\geq j>1$, $a_{i,j}=\sum_{r=j-1}^{i-1}a_{i-1,r}$.
The lower triangular matrix $L_{n,n-i}$ gives the Catalan triangle whose entries are defined by relation

$$a_{n+1,n-i}=\frac{(i+1)(2n-i)!}{(n+1)!(n-i)!},\quad 0\leq i\leq n$$

and under the assumption $n-i=k$, it becomes $a_{n+1,k}=\binom{n+k}{k}-\binom{n+k}{k-1}=\frac{n+1-k}{n+1}\binom{n+k}{k}$, where $k=n,n-1,...,0$.
Thus the Catalan triangles associated with the Catalan transform for sequences $M_{k,n}$ and $m_{k,n}$ are shown in the following tables:

CM_0							0						
CM_1						0		1					
CM_2					0		3		1				
CM_3				0		9		6		0			
CM_4			0		27		27		3		-1		
CM_5		0		81		108		27		-6		0	
CM_6	0		243		405		162		-18		-6		6

Table 2.1: Catalan triangle of the k-Mersenne sequence

Cm_0							0						
Cm_1						3		0					
Cm_2					9		3		-4				
Cm_3				27		18		-12		-8			
Cm_4			81		81		-27		-39		28		
Cm_5		243		324		-27		-162		-60		-24	
Cm_6	729		1215		162		-594		-306		-78		-72

Table 2.2: Catalan triangle of the k-Mersenne-Lucas sequence

Example 2.1. *Let $k=1,2,3,4,5,6$, then using (2.8) the Catalan transforms for sequences $M_{k,n}$ and $m_{k,n}$ are:*

$\mathscr{G}M_1=\{0,1,4,15,56,...\}$
$\mathscr{G}M_2=\{0,1,7,48,329,...\}$
$\mathscr{G}M_3=\{0,1,10,99,980,...\}$
$\mathscr{G}M_4=\{0,1,13,168,2171,...\}$
$\mathscr{G}M_5=\{0,1,16,255,4064,...\}$
$\mathscr{G}M_6=\{0,1,19,360,6821,...\}$

$\mathscr{G}m_1=\{0,3,8,25,124,...\}$
$\mathscr{G}m_2=\{0,6,38,256,1786,...\}$
$\mathscr{G}m_3=\{0,9,86,847,8416,...\}$
$\mathscr{G}m_4=\{0,12,152,1960,25360,...\}$
$\mathscr{G}m_5=\{0,15,236,3757,59908,...\}$
$\mathscr{G}m_6=\{0,18,338,6400,121294,...\}$.

3. Generating Functions of the Binomial and Catalan Transforms

From [11] and [5], we should note that the generating functions $M(t)$ and $m(t)$ for the k-Mersenne and k-Mersenne-Lucas sequences, respectively, are:

$$M(t)=\frac{t}{1-3kt+2t^2} \quad \text{and} \quad m(t)=\frac{2-3kt}{1-3kt+2t^2}. \tag{3.1}$$

By the virtue of [15], note that if A_n is any sequence and $A(t)$ is its generating function then the generating function $B(t)$ of the binomial transform of A_n is given by

$$B(t)=\frac{1}{1-t}A\left(\frac{t}{1-t}\right). \tag{3.2}$$

Also, if $c(t)$ is the ordinary generating function (see (1.4)) for Catalan numbers then the generating function for the corresponding Catalan transform of the sequence A_n is given by $A(tc(t))$.

Theorem 3.1. *For the binomial and Catalan transforms of the k-Mersenne sequence, the generating functions $M_B(t)$ and $M_C(t)$ are given by*

$$M_B(t)=\frac{t}{1-t(2+3k)+3t^2(1+k)}$$

and

$$M_C(t)=\frac{1-\sqrt{1-4t}}{4-3k-4t+(3k-2)\sqrt{1-4t}}.$$

Proof. Using generating function $M(t)$ (See (3.1)) of the k-Mersenne sequence, the generating function $M_B(t)$ of the corresponding binomial transform is given by (3.2) as

$$\begin{aligned} M_B(t)&=\frac{1}{1-t}M\left(\frac{t}{1-t}\right)\\ &=\frac{1}{1-t}\left(\frac{\frac{t}{1-t}}{1-3k(\frac{t}{1-t})+2(\frac{t}{1-t})^2}\right) \quad \text{(using (3.1))}\\ &=\frac{1}{1-t}\left(\frac{(1-t)t}{(1-t)^2-3kt(1-t)+2t^2}\right). \end{aligned}$$

After some necessary calculations, we get

$$M_B(t)=\frac{t}{1-t(2+3k)+3t^2(1+k)}.$$

In a similar fashion, the generating function $M_C(t)$ is obtained by simplifying $M_C(t)=M(tc(t))$. □

Theorem 3.2. *For the binomial and Catalan transforms of the k-Mersenne-Lucas sequence, the generating functions $m_B(t)$ and $m_C(t)$ are given as*

$$m_B(t)=\frac{2-t(2+3k)}{1-t(2+3k)+3t^2(1+k)}$$

and

$$m_C(t)=\frac{4-3k(1-\sqrt{1-4t})}{4-3k-4t+(3k-2)\sqrt{1-4t}}.$$

Proof. Combining (3.1)) and (3.2) for the k-Mersenne-Lucas sequence, we have $m_B(t)=\frac{1}{1-t}m\left(\frac{t}{1-t}\right)$, thus

$$\begin{aligned} m_B(t)&=\frac{1}{1-t}\left(\frac{2-3k\left(\frac{t}{1-t}\right)}{1-3k(\frac{t}{1-t})+2(\frac{t}{1-t})^2}\right) \quad \text{(using (3.1))}\\ &=\frac{1}{1-t}\left(\frac{2(1-t)^2-3k(1-t)t}{(1-t)^2-3kt(1-t)+2t^2}\right)\\ &=\frac{2(1-t)-3kt}{1-t(2+3k)+3t^2(1+k)}. \end{aligned}$$

In a similar fashion, the generating function $m_C(t)$ is obtained by simplifying $m_C(t)=m(tc(t))$. □

Theorem 3.3. *For the p-binomial transforms of the sequences $M_{k,n}$ and $m_{k,n}$, the generating functions $w_M(p,t)$ and $w_m(p,t)$ are given by*

$$w_M(p,t)=\frac{pt}{1-pt(2+3k)+3p^2t^2(1+k)}$$

and

$$w_m(p,t)=\frac{2-pt(2+3k)}{1-pt(2+3k)+3p^2t^2(1+k)}.$$

Proof. To prove the result, we use the fact from [14] that if $M(t)$ is the generating function for $\{M_{k,n}\}$ then for the associated p-binomial transform, the generating function is given by

$$\begin{aligned} w_M(p,t) &= \frac{1}{1-pt}M\Big(\frac{pt}{1-pt}\Big) \\ &= \frac{1}{1-pt}\left[\frac{\frac{pt}{1-pt}}{1-3k(\frac{pt}{1-pt})+2(\frac{pt}{1-pt})^2}\right] \\ &= \frac{1}{1-pt}\left[\frac{(1-pt)pt}{(1-pt)^2-3kpt(1-pt)+2p^2t^2}\right]. \end{aligned}$$

After some necessary calculations, we get

$$w_M(p,t) = \frac{pt}{1-pt(2+3k)+3p^2t^2(1+k)}.$$

Similarly the second identity holds. □

Theorem 3.4. *The exponential generating function $E_M(t)$ and $E_m(t)$ for sequences $M_{k,n}$ and $m_{k,n}$ are given by*

$$E_M(t) = \frac{e^{r_1 t} - e^{r_2 t}}{\sqrt{9k^2-8}} \qquad \text{and} \qquad E_m(t) = e^{r_1 t} + e^{r_2 t}.$$

Proof. It can be easily proved using Binet's formula of the respective sequences. □

Theorem 3.5. *For p-binomial transforms $\mathscr{B}M_p$ and $\mathscr{B}m_p$, rising p-binomial transforms $\mathscr{R}M_p$ and $\mathscr{R}m_p$ and falling p-binomial transforms $\mathscr{F}M_p$ and $\mathscr{F}m_p$, the exponential generating functions are given by*

1. $E_{\mathscr{B}M}(p,t) = \dfrac{e^{pt}(e^{r_1 pt} - e^{r_2 pt})}{\sqrt{9k^2-8}}$,
2. $E_{\mathscr{R}M}(p,t) = \dfrac{e^{t}(e^{r_1 pt} - e^{r_2 pt})}{\sqrt{9k^2-8}}$,
3. $E_{\mathscr{F}M}(p,t) = \dfrac{e^{pt}(e^{r_1 t} - e^{r_2 t})}{\sqrt{9k^2-8}}$,
4. $E_{\mathscr{B}m}(p,t) = e^{pt}(e^{r_1 pt} + e^{r_2 pt})$,
5. $E_{\mathscr{R}m}(p,t) = e^{t}(e^{r_1 pt} + e^{r_2 pt})$,
6. $E_{\mathscr{F}m}(p,t) = e^{pt}(e^{r_1 t} + e^{r_2 t})$.

Proof. In accordance with [[14], Theorem 5.1], we should note that "if $E(t)$ is the exponential generating function of a sequence A_n then the exponential generating functions for the p-binomial transforms, rising p-binomial transforms and falling p-binomial transforms of sequence A_n are given by, respectively, $e^{pt}E(pt)$, $e^{t}E(pt)$ and $e^{pt}E(t)$ respectively".
Thus, the results follows from the above fact. □

4. New Recurrences From the Binomial Transforms

Now we establish a recurrence relation for the above obtained binomial transforms. Then, we obtain their Binet type formula which help us to establish several identities and results.

Theorem 4.1. *For the binomial transforms $\{BM_{k,n}\}$ and $\{Bm_{k,n}\}$, we have*

1. $BM_{k,n+1} - BM_{k,n} = \displaystyle\sum_{a=0}^{n}\binom{n}{a}M_{k,a+1}$,
2. $Bm_{k,n+1} - Bm_{k,n} = \displaystyle\sum_{a=0}^{n}\binom{n}{a}m_{k,a+1}$.

Proof. Since, from the binomial theorem we have

$$\binom{n+1}{a} = \binom{n}{a} + \binom{n}{a-1}.$$

Thus, using (2.1), we have

$$\begin{aligned} BM_{k,n+1} &= \sum_{a=0}^{n+1}\binom{n+1}{a}M_{k,a} = \sum_{a=0}^{n+1}\Big[\binom{n}{a}+\binom{n}{a-1}\Big]M_{k,a} \\ &= \sum_{a=0}^{n+1}\binom{n}{a}M_{k,a} + \sum_{a=0}^{n+1}\binom{n}{a-1}M_{k,a} \\ &= \sum_{a=0}^{n}\binom{n}{a}M_{k,a} + \sum_{a=0}^{n}\binom{n}{a}M_{k,a+1} \qquad \Big(\text{Since, }\binom{n}{n+1}=0 \text{ and } \binom{n}{-1}=0\Big) \\ &= BM_{k,n} + \sum_{a=0}^{n}\binom{n}{a}M_{k,a+1} \qquad (\text{from } (2.1)). \end{aligned}$$

Thus, the required result. Similarly the second identity holds. □

From Theorem 4.1, we deduce that

$$BM_{k,n+1}=\sum_{a=0}^{n}\binom{n}{a}\left[M_{k,a}+M_{k,a+1}\right] \quad \text{and} \quad Bm_{k,n+1}=\sum_{a=0}^{n}\binom{n}{a}\left[m_{k,a}+m_{k,a+1}\right]. \tag{4.1}$$

Theorem 4.2. *For $n\geq 0$, the binomial transforms $\{BM_{k,n}\}$ and $\{Bm_{k,n}\}$ posses the following recurrence relations:*

$$\begin{aligned} BM_{k,n+2}&=(2+3k)BM_{k,n+1}-(3+3k)BM_{k,n} \quad \textit{with} \quad BM_{k,0}=0,\ BM_{k,1}=1,\\ Bm_{k,n+2}&=(2+3k)Bm_{k,n+1}-(3+3k)Bm_{k,n} \quad \textit{with} \quad Bm_{k,0}=2,\ Bm_{k,1}=2+3k. \end{aligned} \tag{4.2}$$

Proof. From (4.1), we can write

$$\begin{aligned} BM_{k,n+1}&=\sum_{a=1}^{n}\binom{n}{a}\left[M_{k,a}+M_{k,a+1}\right]+M_{k,0}+M_{k,1}\\ &=\sum_{a=1}^{n}\binom{n}{a}\left[M_{k,a}+3kM_{k,a}-2M_{k,a-1}\right]+1 \quad (\text{using } (1.1))\\ &=(1+3k)\sum_{a=1}^{n}\binom{n}{a}M_{k,a}-2\sum_{a=1}^{n}\binom{n}{a}M_{k,a-1}+1\\ &=(1+3k)BM_{k,n}-2\sum_{a=1}^{n}\binom{n}{a}M_{k,a-1}+1 \quad (\text{using } (2.1)). \end{aligned} \tag{4.3}$$

Now replacing n by $n+1$ in the above equation, we get

$$BM_{k,n+2}=(1+3k)BM_{k,n+1}-2\sum_{a=1}^{n+1}\binom{n+1}{a}M_{k,a-1}+1.$$

On some elementary calculations and simplification, we obtain

$$BM_{k,n+2}=(1+3k)BM_{k,n+1}-2\sum_{a=0}^{n}\binom{n}{a}M_{k,a}-2BM_{k,n}+1$$

or,

$$BM_{k,n+2}-(1+3k)BM_{k,n+1}+2BM_{k,n}=-2\sum_{a=0}^{n}\binom{n}{a}M_{k,a}+1.$$

Thus, on substituting from (4.3), we have

$$BM_{k,n+2}=(2+3k)BM_{k,n+1}-(3+3k)BM_{k,n}.$$

Similarly the second recurrence relation can be achieved. □

Some terms of the binomial transforms corresponding to the k-Mersenne and k-Mersenne-Lucas numbers are

$$\begin{aligned} &BM_{k,0}=0,\ BM_{k,1}=1,\ BM_{k,2}=3k+2,\ BM_{k,3}=9k^2+9k+1,\\ &BM_{k,4}=27k^3+36k^2+6k-4,\ldots, \end{aligned}$$

and

$$\begin{aligned} &Bm_{k,0}=2,\ Bm_{k,1}=2+3k,\ Bm_{k,2}=9k^2+6k-2,\\ &Bm_{k,3}=27k^3+27k^2-9k-10,\ Bm_{k,4}=81k^4+108k^3-18k^2-60k-14,\ldots. \end{aligned}$$

The characteristic equation corresponding to recurrence relation (4.2) is $x^2-(2+3k)x+(3+3k)=0$ which has the following two roots

$$\lambda_1=\frac{(2+3k)+\sqrt{9k^2-8}}{2} \quad \text{and} \quad \lambda_2=\frac{(2+3k)-\sqrt{9k^2-8}}{2} \tag{4.4}$$

$$\text{i.e.} \quad \lambda_1=1+r_1 \quad \text{and} \quad \lambda_2=1+r_2, \tag{4.5}$$

that plays an important role in setting the explicit formula for a sequence.

Theorem 4.3 (Binet type formula)**.** *For the binomial transforms $\{BM_{k,n}\}$ and $\{Bm_{k,n}\}$, we have*

$$BM_{k,n}=\frac{\lambda_1^n-\lambda_2^n}{\sqrt{9k^2-8}} \quad \textit{and} \quad Bm_{k,n}=\lambda_1^n+\lambda_2^n.$$

Proof. Using (4.4) and performing some necessary calculations proves the result. □

Theorem 4.4. *If $M_{k,n}=\frac{r_1^n-r_2^n}{\sqrt{9k^2-8}}$ and $m_{k,n}=r_1^n+r_2^n$, then the nth term of the corresponding binomials transform are given by*

$$BM_{k,n}=\frac{(r_1+1)^n-(r_2+1)^n}{\sqrt{9k^2-8}} \quad \textit{and} \quad Bm_{k,n}=(r_1+1)^n+(r_2+1)^n.$$

Proof. Using Theorem 4.3 and relation (4.5), the result can be established. □

Thus, after getting the Binet type formula for a sequence, one can easily derive and prove several identities. For example, for sequences $M_{k,n}$ and $m_{k,n}$ sum of the first n terms of the corresponding binomial transform are given as the follows.

Theorem 4.5. *For $n \geq 0$, we have*

$$\sum_{a=0}^{n} BM_{k,a} = \frac{1}{2}\Big[(3+3k)BM_{k,n} - BM_{k,n+1} + 1\Big]$$

and

$$\sum_{a=0}^{n} Bm_{k,a} = \frac{1}{2}\Big[(3+3k)Bm_{k,n} - Bm_{k,n+1} - 3k\Big].$$

Proof. Using Theorem 4.3, we have

$$\begin{aligned}
\sum_{a=0}^{n} BM_{k,a} &= \frac{1}{\sqrt{9k^2-8}}\Big(\sum_{a=0}^{n}\lambda_1^a - \sum_{a=0}^{n}\lambda_2^a\Big)\\
&= \frac{1}{\sqrt{9k^2-8}}\Big(\frac{\lambda_1^{n+1}-1}{\lambda_1-1} - \frac{\lambda_2^{n+1}-1}{\lambda_2-1}\Big)\\
&= \frac{1}{\sqrt{9k^2-8}}\Big[\frac{(\lambda_2-1)(\lambda_1^{n+1}-1)-(\lambda_1-1)(\lambda_2^{n+1}-1)}{(\lambda_1-1)(\lambda_2-1)}\Big]\\
&= \frac{\lambda_2\lambda_1^{n+1}-\lambda_1\lambda_2^{n+1}-\lambda_1^{n+1}+\lambda_2^{n+1}}{2\sqrt{9k^2-8}} + \frac{1}{2} \qquad \text{(using (1.2) and (4.4))}\\
&= \frac{\lambda_1\lambda_2(\lambda_1^n-\lambda_2^n)-(\lambda_1^{n+1}-\lambda_2^{n+1})}{2\sqrt{9k^2-8}} + \frac{1}{2}\\
&= \frac{1}{2}\Big[(3+3k)BM_{k,n} - BM_{k,n+1} + 1\Big].
\end{aligned}$$

Argument for the second identity is very similar to the above. □

5. Conclusion

This study demonstrates the transformative effect of various binomial and Catalan transforms on k-Mersenne and k-Mersenne-Lucas numbers, generating novel integer sequences, some of which are identified in OEIS. Various generating and exponential generating functions, alongside Binet-type formulas and new recurrence relations are obtained for associated binomial transforms, which provides a foundation for further investigation. This is useful because number patterns show up in many places – like in computer science for writing code, in understanding natural patterns, or even in designing secure systems. Finding these new patterns and their rules gives scientists more tools to use in these areas.

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