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A Novel Generalized Difference Spaces Constructed by the Modulus Function

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Abstract

A major role of this document is to present a generalized difference spaces denoted by $w(\Delta^r,\hat{A},p,f,q,s)$, $w_0(\Delta^r,\hat{A},p,f,q,s)$, and $w_\infty(\Delta^r,\hat{A},p,f,q,s)$, of which arguments are defined as follows, and also to investigate some algebraic and topological characteristics of the spaces. Here; $\hat{A}$ is an infinite matrix, $p=(p_k)$ is a bounded sequence of strictly positive real numbers, f is any modulus function, q is a semi norm, and s is any non-negative real number. Besides these, the relationship between the spaces obtained by various values of those arguments is going to be considered. Finally, the newly obtained results are going to be compared with those of other studies.

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1. Fundamental Facts

Although the matrix domain has certainly been the most common tool in recent times used in defining a new sequence space, many other methods and techniques have been utilized to built a new sequence space. These new sequence spaces, which will be explained in more detail below, are built using the modulus function. Let us start by giving the basic concepts that will be used in this study. Firstly we want to give the definition of a sequence. A sequence is a collection of numbers in a particular order. With a more formal statement, a sequence of numbers is a function whose domain is the set of positive integers $\mathbb{Z}^+$. To signal the fact that the domains are restricted to the set of positive integers, it is conventional to use a letter like n from the middle of the alphabet for the independent variable, instead of the x, y, z and t used so widely in other contexts. The number $a(n)$ is called the n^{th} term of the sequence, or the term with index n. To describe sequence, we often write the first few terms as well as a formula for the n^{th} term. We refer to the sequence whose n^{th} term is a_n as "the sequence (a_n)." Here, the curly braces ( ) indicate we have in mind all the terms of the sequence, not just a single term. We note here that the sequences are named according to the range set of a given function. Because of all these, if the range is the set of real numbers $\mathbb{R}$, then the sequence is called real-valued sequence. The space consisting of a collection of all real-valued sequences is indicated by notation w. When a set is a subspace of w, it is said that a sequence space. The symbols ℓ_∞, c and c_0 denote spaces whose members consist of all bounded, convergent and null sequences, respectively. In addition to these, when $1<p<\infty$ the symbols cs, bs, ℓ_1 and ℓ_p; denote spaces such that its members consist of all convergent, bounded, absolutely and $p-$absolutely convergent series, respectively. Also, $bv=\{x=(x_k):\sum_k|\Delta x_k|<\infty\}$ in which $\Delta x=(x_k-x_{k+1})$ for all sequences $x=(x_k)$ and $bv_0=bv\cap c_0$. Moreover, when $p\geq 1$ the notations w_0^p, w^p and w_∞^p denote spaces whose members consist of are strongly summable to zero, summable and bounded of by the Cesàro method of order 1.

An important case occurs in such conditions that X is a linear space according to the coordinate-wise addition and scalar multiplication of sequence when X is equal to any one these sequence spaces ℓ_∞, c, c_0, ℓ_p, bs, cs, bv, bv_0, w_0^p, w^p and w_∞^p.

Now, let us give very short historical knowledge and brief developments about the space of *almost convergent sequences*. There are two different notations, which are f and $\widehat{c}$, of the space of the related space. Since the modulus function is also denoted by f, in order to avoid any confusion, throughout the article notation $\widehat{c}$ will be used for the aforementioned space. Now, let us explain the concept of Banach limit, which is the basis for the formation of this idea. The shift by 1 operator φ is defined on ω by the rule $\varphi_n(x)=x_{n+1}$ for all $n\in\mathbb{N}$. A Banach limit L is defined on ℓ_∞, as a non–negative linear functional, such that $L(\varphi x)=L(x)$ and $L(e)=1$. A sequence $x=(x_k)\in\ell_\infty$ is said to be almost convergent to the generalized limit α if all Banach limits of x is α [1] , and denoted by $\widehat{c}-\lim x_k=\alpha$. For more comprehensive information about the Banach limit, the reader can consult to Çolak and Çakar [5], and Das [6]. By utilizing the opinion of the Banach limits, G.G.Lorentz [11] presented and after then examined some properties of the almost convergent sequence spaces $\widehat{c}$.

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Let φ^j be the composition of φ with itself j times and define $t_{mn}(x)$ for any sequence $x=(x_k)$ by

$$t_{mn}(x) := \frac{1}{m+1}\sum_{j=0}^{m}\varphi_n^j(x) \ \text{ for all } \ m,n\in\mathbb{N}.$$

It has been proved by Lorentz in [11] that $\widehat{c}-\lim x_k=\alpha$ iff $\lim_{m\to\infty} t_{mn}(x)=\alpha$ when it is uniformly in n. The fact that a convergent sequence is almost convergent and its both ordinary and generalized limits are equal is a widely known fact. By $\widehat{c}_0$ and $\widehat{c}$, we denote the space of all almost null and almost convergent sequences, in other words

$$\widehat{c}_0 := \left\{x=(x_k)\in\omega : \lim_{m\to\infty}\sum_{k=0}^{m}\frac{x_{n+k}}{m+1}=0 \text{ uniformly in } n\right\},$$

$$\widehat{c} := \left\{x=(x_k)\in\omega : \exists\alpha\in\mathbb{C}\ni \lim_{m\to\infty}\sum_{k=0}^{m}\frac{x_{n+k}}{m+1}=\alpha \text{ uniformly in } n\right\}.$$

It is obvious that the following inclusion relations $c\subset\widehat{c}\subset\ell_\infty$ are valid.
Also, notations by $\widehat{c}_0$, $\widehat{c}$ and $\widehat{cs}$ denote the spaces of almost null and almost convergent sequences and series, respectively.
Maddox [14,15] defined a complex sequence x to be strongly almost convergent to a number l iff $\frac{1}{m+1}\sum\limits_{k=0}^{m}|x_{n+k}-l|\to 0 \ (m\to\infty,$ uniformly in $n)$.
It leads to the concept of what we shall call strong almost convergence. By $[\hat{c}]$ he denotes the space of all strongly almost convergence sequences, i.e.,

$$[\hat{c}] : = \left\{x=(x_k)\in\omega : \lim_{m\to\infty}\frac{1}{m+1}\sum_{k=0}^{m}|x_{n+k}-l|=0, \text{ uniformly in } n\right\}.$$

It is immediate that $[\hat{c}]\subset\hat{c}$, and it is easy to see that the inclusion is strict. Also $[\hat{c}]$ is a closed subspace of l_∞ and with strict inclusions we have $c\subset[\hat{c}]\subset\hat{c}\subset l_\infty$.
Now, we begin by recalling basic definitions involved in paranorm and others, which will be used in the next sections. Let X be a linear topological space. If a function g having its domain X and range $\mathbb{R}$ and satisfies the following four conditions it is said be a paranorm function

i) $g(\theta)=0$,
ii) $g(x)=g(-x)$
iii) $g(x+y)=g(x)+g(y)$
iv) $|\alpha_n-\alpha|\to 0$ and $g(x_n-x)\to 0$ imply $g(\alpha_n x_n-\alpha x)\to 0$

for all $\alpha\in\mathbb{R}$ and all $x\in X$, where θ is the zero vector in the linear space X.
In this paragraph, we briefly describe some concepts involved in $K-$ space, $FK-$ space, $BK-$ space, $AK-$ space. These concepts aren't fairly easy. For a sequence space X having a linear topology, we recall that X is called a $K-$space if and only if each of the functions $p_n : X\to\mathbb{R}$ described by $p_n(x)=x_n$ is continuous for every $n\in\mathbb{N}$. This is in fact true if it is assumed that X is a $K-$ space, then the space X is said to be an $FK-$ space if and only if X is a complete linear metric space. If the definition is examined closely one sees that an FK-space is a complete total paranormed space. At this point, it is convenient to recall the fact about $FK-$ space. Specifically, if a topology of the $FK-$ space is normable then a $FK-$space is said to be a $BK-$space, that is we can say that a BK-space is a normed FK-space. For example; when $1\le p<\infty$ the space ℓ_p having the norm $\|x\|_p=(\sum_k|x_k|^p)^{\frac{1}{p}}$ is a $BK-$ space and each classical sequence space c_0, c and ℓ_∞ having the norm $\|x\|_\infty=sup_k|x_k|$ is a $BK-$ space. By assuming that e^k is a sequence in which only non-zero term is in its k^{th} place for each $k\in\mathbb{N}$ and $\phi=span\{e^k\}$ being the set of all finitely non-zero sequences we can say that an $FK-$ space X has the AK property if $\phi\subset X$ and $\{e^{(k)}\}$ is a basis for X. When ϕ is dense in X, in that case X is said to be an $AD-$space whereby AK implies AD. When $1\le p<\infty$, it is known that the spaces c_0, cs and ℓ_p are AK-spaces.
In this paragraph, we are going to deal with the definition of the $\alpha-$ and $\beta-$ duals for any sequence spaces. Let X and Y be sequence spaces. In that case, the following set

$$S(X,Y)=\{z=(z_k)\in w : xz=(x_k z_k)\in Y \text{ for all } x=(x_k)\in X\} \tag{1.1}$$

is known the multiplier space for X and Y. First of all, for arbitrary sequence space V following two inclusion relations $S(X,Y)\subset S(V,Y)$ when $V\subset X$ and $S(X,Y)\subset S(X,V)$ when $Y\subset V$ are valid. By making special choices in the multiplier space, we reach the $\alpha-$ and $\beta-$ duals for any sequence space Ω. More clearly, if we choose $Y=\ell_1$ and cs in the notation of (1.1) we get $\Omega^\alpha=S(\Omega,\ell_1)$, $\Omega^\beta=S(\Omega,cs)$ respectively. In literature, the $\alpha-$dual and $\beta-$dual are known as *Köthe-Toeplitz dual,* and *generalized Köthe-Toeplitz dual* respectively *[2]*.
The modulus function has fundamental importance for this work. Thus, the definition of it is presented in this paragraph. If a function f has its domain and range as $[0,\infty)$ and satisfies the following four conditions it is said to be a modulus function

i) $f(x)=0$ iff $x=0$,
ii) $f(x+y)\le f(x)+f(y)$,
iii) f is increasing,
iv) f is continuous from the right at 0 [16].

Actually the conditions clearly show that the function is continuous onto $[0,\infty)$ and its another one properties is either bounded or unbounded, due to well-known results from elementary analysis. If f_1 and f_2 are modulus function then $f_1\circ f_2$ and f_1+f_2 are modulus functions. For more detail see [39,40].
In 1953, Nakano [16] introduced the concept of modulus function and it has been utilized to answer some of the constructional problems related to the theory of FK-spaces. Among others one question; "is there an $FK-$space in which the sequence of coordinate vectors is

bounded", put forward by A. Wilansky, has been answered by W. H. Ruckle [19] by a negative one. This problem has been tackled by building a collection of scalar $FK-$spaces $L(f)$ in which f is a modulus function. In fact, $L(f)$ is a generalization of the spaces ℓ_p $(0<p\leq 1)$. When considered in term of a positive real sequence $r=(r_k)$; another expansion of the space ℓ_p if $(p>0)$ has been presented by Simons [22]. The reader may refer to the reference [19].
Ruckle [19] proved that, for arbitrary modulus f, $L(f)\subset l_1$ and $L(f)^{\alpha}=l_{\infty}$ in which

$$L(f)^{\alpha} \quad = \quad \left\{ y=(y_k)\in w : \sum_k |y_k x_k| < \infty \text{ for all } x\in L(f) \right\}$$

is the α-dual of $L(f)$.
A sequence $a=(a_k)$ is called to be summable $(C,1)$ only when $\lim_n \frac{1}{n}\sum_{i=1}^{n} a_i \in c$. Kuttner [9] have introduced the spaces of strongly Cesàro summable sequences, and later on Maddox [12] and others have more generalized the concept. Strongly Cesàro summable sequences in term of a modulus function has been introduced by Maddox [13] as a new expansion of the definition of strongly Cesàro summable. Moreover, Connor [10] has brought a new insight to his description of summability method by writing a non-negative regular matrix A in place of Cesàro matrix. Then, in [20], he has presented strongly almost $A-$summability concept following Connor [10], however, his definition does not seen satisfactory and natural. Following the definition in [20] and specialising the infinite matrix don't result in strongly almost convergent sequences in terms of a modulus function. After that, Savaş has presented an alternative way of defining strongly almost $A-$summability in terms of a modulus function in his new study [21]. This new definition is more natural and involves definition of strongly almost convergence in terms of a modulus function as a particular case. The sets $w_0\left(\hat{A},f,p\right)$, $w\left(\hat{A},f,p\right)$ and $w_{\infty}\left(\hat{A},f,p\right)$ will called the spaces of strongly almost summable to zero, strongly almost summable and strongly almost bounded with respect to the modulus f respectively [21].
The parameter s in the factor k^{-s} has been utilized by Bulut and Çakar [3], in order to generalize the Maddox sequence space $l(p)$ in which $p=(p_k)$ is a bounded sequence consisting of positive real numbers and $s\geq 0$. Its function is extension. For instance, the space $l(p,s)=\left\{x\in w : \sum_{k=1}^{\infty} k^{-s}|x_k|^{p_k}<\infty\right\}$ involves $l(p)$ as a subspace for $s>0$, and in that case it coincides with $\ell(p)$ only when $s=0$.

2. Difference Sequence Spaces

In this part of the document, many of the sequence spaces we discuss will be defined by means of a difference sequence or generalized difference sequence. To study such sequence spaces we shall need to understand the concept of difference sequence of or generalized difference sequence of a sequence.
Motivation for Kızmaz's [24] introduction of the difference sequence space in his 1981 paper to be the notion difference operator. Let's now explain this concept. Let $\lambda\in\{\ell_{\infty},c,c_0\}$. Then, $\lambda(\Delta)$ which formed the sequences $x=(x_k)$ is called the *difference sequence spaces* if the sequence (x_k-x_{k+1}) obtained by using $x=(x_k)$ is member of the sequence space λ.
In order to effectively deal with concrete situations, we have briefly considered several important articles [23,25,26] for difference sequence space.
Perhaps the most basic article for our document is Çolak and Et's [8] article entitled "On generalized difference sequence spaces." They first defined the spaces $\Delta^m\lambda$ for $\lambda\in\{c,\ell_{\infty},c_0\}$ and after then examined some of its algebraic and topological properties.
Çolak and Et [8] took further and generalized the Kızmaz's idea [24] such that

$$\Delta^m\lambda=\left\{x=(x_k)\in\omega : \Delta^m x\in\lambda\right\},$$

in which $\Delta^1 x=(x_k-x_{k+1})$ and $\Delta^m x=\Delta(\Delta^{m-1}x)$ for $m\in\{1,2,3,\ldots\}$.
Sarıgöl [29] after Kızmaz [24], defined the spaces $\lambda(\Delta_r)$ which is expanded the difference spaces $\lambda(\Delta)$ in a different way. Now, let us explain this expansion. In 1987 Sarıgöl [29] defined following generalized difference sequence space

$$\lambda(\Delta_r): \quad = \quad \{x=(x_k)\in\omega : \Delta_r x=\{k^r(x_k-x_{k+1})\}\in\lambda \;\; \text{for } r<1\}$$

after then he determined the *Köthe-Toeplitz dual, generalized Köthe-Toeplitz dual,* in addition to these *Garling dual,* (see *[2]* definition of *Garling dual*) respectively of the mentioned space $\lambda(\Delta_r)$, in which $\lambda\in\{\ell_{\infty},c,c_0\}$. It is fairly easy to see that $\lambda(\Delta_r)\subset\lambda(\Delta)$, when $0<r<1$ and $\lambda(\Delta)\subset\lambda(\Delta_r)$, when $r<0$.
Ahmad and Mursaleen [27] have expanded those spaces to $\lambda(p,\Delta)$ and investigated the related problems in 1987. Köthe-Toeplitz duals for the set $\ell_{\infty}(p,\Delta)$ and $c_0(p,\Delta)$ have been described by Malkowsky [30] and, new proofs of the properties of the matrix transformations in [27] have been presented. Choudhary and Mishra [31] investigated some characteristics of the sequence space $c_0(\Delta_r)$ when $r\geq 1$ in 1993. Again in that year, Mishra [32] has given a characterization of a $BK-$spaces containing subspace which is isomorphic to $sc_0(\Delta)$ in view of matrix maps and also a sufficient condition in order that a matrix map from $s\ell_{\infty}(\Delta)$ into a $BK-$ space is a compact operator. He also demonstrated that any matrix defined from $s\ell_{\infty}(\Delta)$ to a $BK-$space involving any subspace which is isomorphic to $s\ell_{\infty}(\Delta)$ is a compact one where

$$s\lambda(\Delta)=\{x=(x_k)\in\omega : (\Delta x_k)\in\lambda,\; x_1=0 \;\text{ for }\; \lambda=\ell_{\infty} \;\text{ or }\; c_0\}.$$

In 1996, Mursaleen et al. [33] described and investigated the sequence space

$$\ell_{\infty}(p,\Delta_r)=\{x=(x_k)\in\omega : \Delta_r x\in\ell_{\infty}(p)\},\;\; (r>0).$$

Gnanaseelan and Srivastava [34] have described and investigated the spaces $\lambda(u,\Delta)$ for any sequence $u=(u_k)$ consisting of non-complex numbers in such a way that

(i) $\dfrac{|u_k|}{|u_{k+1}|}=1+O(1/k)$ for each $k\in\mathbb{N}_1=\{1,2,3,\ldots\}$.
(ii) $k^{-1}|u_k|\sum_{i=0}^{k}|u_i|^{-1}=O(1)$.

(iii) $\left(k\left|u_k^{-1}\right|\right)$ is a sequence composed of positive numbers which are increasing monotonically toward infinity.

Malkowsky [35] defined the spaces $\lambda(u,\Delta)$ for any fixed sequence $u=(u_k)$ without imposing any restriction onto u in 1986. He also showed that the sequence spaces $\lambda(u,\Delta)$ are $BK-$spaces having the norm which is described by

$$\|x\| = \sup_{k\in\mathbb{N}} \left|u_{k-1}\left(x_{k-1}-x_k\right)\right| \text{ with } u_0 = x_0 = 1.$$

When $r\geq 1$; the spaces

$$S_r(p,\Delta) = \{x=(x_k)\in\omega : (k^r|\Delta x_k|)\in c_0(p)\}$$

defined by Gaur and Mursaleen [36] a expanded the space $S_r(\Delta)$, again they determined the characterization of $(S_r(p,\Delta):\ell_\infty)$ and $(S_r(p,\Delta):\ell_1)$ matrix classes. Almost simultaneously Malkowsky et al. [37] and Asma and Çolak [28] firstly introduced the sequence spaces $\ell_\infty(p,u,\Delta)$, $c(p,u,\Delta)$ and $c_0(p,u,\Delta)$ after then they determined Köthe-Toeplitz duals of those spaces. More Recently; the characterization the matrix classes $(\Delta\lambda:\mu)$ and $(\Delta\lambda:\Delta\mu)$ are determined by Malkowsky and Mursaleen [38] in which $\lambda=c_0(p)$, $c(p)$, $\ell_\infty(p)$ and $\mu=c_0(q)$, $c(q)$, $\ell_\infty(q)$.

3. The Sequence Spaces $w_0(\Delta^r,\hat{A},p,f,q,s)$, $w(\Delta^r,\hat{A},p,f,q,s)$ and $w_\infty(\Delta^r,\hat{A},p,f,q,s)$

This section plays a major role in this document. Using the Δ^r generalized difference operator for $r\in\mathbb{N}$, the infinite matrix $A=(a_{mk})$ of non-negative real numbers, the $p=(p_k)$ sequence of non-negative real numbers, the function f any modulus, the q seminorm function and $s\in\mathbb{R}$ are the starting point of this document and each is in itself a vast and complicated subject.
Assume that X is a complex linear space having zero element θ, $X=(X,q)$ is a seminormed space having the seminorm q. The set of all $X-$valued sequences $x=(x_k)$ which is the linear space commonly used coordinate-wise operations is denoted by $S(X)$. Assuming $\lambda=(\lambda_k)$ is an arbitrary sequence and also $x\in S(X)$ will allow us to write $\lambda x=(\lambda_k x_k)$ when $X=\mathbb{C}$ is taken, we get $w(X)$ is denoted briefly by w, which is the space of all complex-valued sequences. We call this condition as scalar-valued one.
If we assume that $A=(a_{mk})$ is any nonnegative matrix, $p=(p_k)$ is a sequence consisting of positive real numbers and f is a modulus function the sequence spaces on the complex field $\mathbb{C}$ is given as follows

$$w_0(\Delta^r,\hat{A},p,f,q,s) = \left\{x\in S(X) : \lim_{m\to\infty}\sum_k \frac{a_{mk}}{k^s}[f(q(\Delta^r x_{k+n}))]^{p_k} = 0, \text{uniformly in } n, s\geq 0\right\}$$

$$w(\Delta^r,\hat{A},p,f,q,s) = \left\{x\in S(X) : \lim_{m\to\infty}\sum_k \frac{a_{mk}}{k^s}[f(q(\Delta^r x_{k+n}-le))]^{p_k} = 0, \text{uniformly in } n, \exists l\in\mathbb{C}, s\geq 0\right\}$$

$$w_\infty(\Delta^r,\hat{A},p,f,q,s) = \left\{x\in S(X) : \sup_m\sum_k \frac{a_{mk}}{k^s}[f(q(\Delta^r x_{k+n}))]^{p_k} < \infty,\ s\geq 0\right\}$$

When $\phi(X)$ is given as the space of finite sequences in X, we have the following valid relation $\phi(X)\subseteq w(\hat{A},p,f,q,s)$.
In the literature, those spaces are reduced to some sequence spaces. For instance, if we take $(X,q)=(\mathbb{C},|,|)$, $A=(C,1)$, the Cesàro matrix, $p_k=1$, for each k and $r=s=0$, we obtain the spaces $[\hat{c}_0(f)]$, $[\hat{c}(f)]$ and $[\hat{c}(f)]_\infty$ which are presented by Pehlivan [18]. Furthermore, the spaces involve in [4, 7, 14, 15, 17, 21] as derived a particular case.
The lemma that will now be presented is fundamental, but even here it will help us while proving some of the following theorem.

Lemma 3.1. *If $a_k,\ b_k\in\mathbb{C}$ and $0<p_k\leq\sup p_k=H$ for all $k\in\mathbb{N}$. When $C=\max\left(1,2^{H-1}\right)$, following famous inequality*

$$|a_k+b_k|^{p_k}\leq C\left(|a_k|^{p_k}+|b_k|^{p_k}\right)$$

is valid, see Maddox [12].

Now, we begin to serve and establish some basic theorems related to the sequence spaces introduced in this section.

Theorem 3.2. *Let $r\geq 1$, then the following inclusion relations are strictly valid.*

(i) $w_0(\Delta^{r-1},\hat{A},p,f,q,s)\subset w_0(\Delta^r,\hat{A},p,f,q,s)$,
(ii) $w(\Delta^{r-1},\hat{A},p,f,q,s)\subset w(\Delta^r,\hat{A},p,f,q,s)$,
(iii) $w_\infty(\Delta^{r-1},\hat{A},p,f,q,s)\subset w_\infty(\Delta^r,\hat{A},p,f,q,s)$.

Proof. Since the proof does not have any difficulty and even it is very simple, here, we will only show that $w_\infty(\Delta^{r-1},\hat{A},p,f,q,s)\subset w_\infty(\Delta^r,\hat{A},p,f,q,s)$, the validity of other inclusions can also be shown in a similar way. Let us $x\in w_\infty(\Delta^{r-1},\hat{A},p,f,q,s)$, that is

$$\sup_m\sum_k a_{mk}k^{-s}\left[f\left(q(\Delta^{r-1}x_{k+n})\right)\right]^{p_k}<\infty.$$

Since q is a seminorm, modulus function f is increasing, $A=(a_{mk})$ is a nonnegative matrix and $a_{mk}k^{-s}>0$ is valid for each m and k;

$$\frac{a_{mk}}{k^s}\{f[q(\Delta^r x_{k+n})]\}^{p_k} < C\frac{a_{mk}}{k^s}\{f[q(\Delta^{r-1}x_{k+n})]\}^{p_k} + C\frac{a_{mk}}{k^s}\{f[q(\Delta^{r-1}x_{k+n+1})]\}^{p_k}$$

is valid from Lemma 3.1. Taking the summation after extending our index from 1 to ∞, if the supremum is taken over m and n, we get desired inclusion relation.

□

Corollary 3.3. *We have*

(i) $w_0(\Delta^i,\hat{A},p,f,q,s) \subset w_0(\Delta^r,\hat{A},p,f,q,s)$,
(ii) $w(\Delta^i,\hat{A},p,f,q,s) \subset w(\Delta^r,\hat{A},p,f,q,s)$,
(iii) $w_\infty(\Delta^i,\hat{A},p,f,q,s) \subset w_\infty(\Delta^r,\hat{A},p,f,q,s)$

for $i = 0,1,2,...,r-1$.

Theorem 3.4. *When* $p = (p_k) \in \ell_\infty$, *therefore the sets* $w_0(\Delta^r,\hat{A},p,f,q,s)$, $w(\Delta^r,\hat{A},p,f,q,s)$ *and* $w_\infty(\Delta^r,\hat{A},p,f,q,s)$ *are linear spaces over the complex field* $\mathbb{C}$.

Proof. We choose to prove just $w_0(\Delta^r,\hat{A},p,f,q,s)$, since the other two can be done in a similar way. Now, let us take both $x,y \in w_0(\Delta^r,\hat{A},p,f,q,s)$ and $a,b \in \mathbb{C}$, in that case, there exist two positive integers T_a and T_b, depend on a and b respectively such that $|a| \leq T_a$ and $|b| \leq T_b$. Due to the fact that the modulus function f is subadditive, q is a seminorm and Δ^r having a linear property, we have following inequality

$$\frac{a_{mk}}{k^s}\left\{f\left(q\left(\Delta^r ax_{k+n}+\Delta^r by_{k+n}\right)\right)\right\}^{p_k} \leq C\tfrac{a_{mk}}{k^s}T_a^H\left\{f\left(q\left(\Delta^r x_{k+n}\right)\right)\right\}^{p_k} + C\tfrac{a_{mk}}{k^s}T_b^H\left\{f\left(q\left(\Delta^r y_{k+n}\right)\right)\right\}^{p_k}.$$

To actually check that by summing over from $k=1$ to ∞ in the last inequality requires $ax+by \in w_0(\Delta^r,\hat{A},p,f,q,s)$ a little more care. This, in fact, concludes the proof. □

Theorem 3.5. *The spaces* $w_0(\Delta^r,\hat{A},p,f,q,s)$ *is paranormed space by G defined by*

$$G(x) = \sup_m \left\{\sum_k a_{mk}k^{-s}[f(q(\Delta^r x_{k+n}))]^{p_k}\right\}^{\frac{1}{M}}$$

where $M = max(1, sup_k p_k)$.

Proof. According to the definition of the sequence spaces $w_0(\Delta^r,\hat{A},p,f,q,s)$, we ensure that the existence of $G(x) \in \mathbb{R}$, for all $x \in w_0(\Delta^r,\hat{A},p,f,q,s)$. Now, let us trace a standard type procedure in this proof.
It can easily be controlled that the conditions $G(\theta) = 0$, $G(x) = G(-x)$ and $G(x+y) \leq G(x)+G(y)$ by Minkowski's inequality are valid. For showing the continuity of scalar multiplication let us assume that (μ^t) be a sequence of scalars such that $|\mu^t - \mu| \to 0$ and $G(x^t - x) \to 0$ for arbitrary sequence $(x^t) \in w_0(\Delta^r,\hat{A},p,f,q,s)$. We are going to demonstrate that $G(\mu^t x^t - \mu x) \to 0$ as $t \to \infty$. When $N \in \mathbb{N}$ such that $N > 1$, say $\tau_t = |\mu^t - \mu|$ then

$$\left\{\sum_k a_{mk}k^{-s}[f(q(\Delta^r\lambda^t x^t_{k+n} - \Delta^r\lambda^0 x^0_{k+n}))]^{p_k}\right\}^{\frac{1}{M}} \leq \left\{\sum_k a_{mk}^{\frac{1}{M}}k^{-\frac{s}{M}}\left\{(Nf(q(\Delta^r x^t_{k+n} - \Delta^r x^0_{k+n})))^{\frac{p_k}{M}} + (f(q(\lambda^t - \lambda^0)\Delta^r x^0_{k+n}))^{\frac{p_k}{M}}\right\}^M\right\}^{\frac{1}{M}}$$

where $|\lambda^t| \leq N$. Thus, we get

$$G\left(\lambda^t x^t - \lambda^0 x^0\right) \leq N^{\frac{H}{M}}G(x^t - x^0) + \sup_m\left\{\sum_k a_{mk}k^{-s}[f(q(\lambda^t - \lambda^0)\Delta^r x^0_{k+n}]^{p_k}\right\}^{\frac{1}{M}} \tag{3.1}$$

Because of $G\left(x^t - x^0\right) \to 0$ $(t \to \infty)$ from the assumption what is to be proved that

$$\sup_m\left\{\sum_k a_{mk}k^{-s}[f(q(\lambda^t - \lambda^0)\Delta^r x^0_{k+n})]^{p_k}\right\}^{\frac{1}{M}} \to 0 \ \ (m \to \infty).$$

Since $\lambda^t \to \lambda^0$ as $t \to \infty$, we can find a $D > 0$ such that $|\lambda^t - \lambda^0| \leq D$ for all $t \in \mathbb{N}$. It is trivial that $Dx^0 = (Dx^0_k) \in w_0(\Delta^r,\hat{A},p,f,q,s)$ because $w_0(\Delta^r,\hat{A},p,f,q,s)$ is a linear space. Therefore, when $\varepsilon > 0$ is given, we can find a unique a positive integer m_0 depend on ε such that

$$\left\{\sum_k a_{mk}k^{-s}[f(q(D\Delta^r x^0_{k+n})]^{p_k}\right\}^{\frac{1}{M}} < \frac{\varepsilon}{2} \tag{3.2}$$

for all $m > m_0$, it follows

$$\left\{\sum_k a_{mk}k^{-s}[f(q(D\Delta^r x^0_{k+n})]^{p_k}\right\}^{\frac{1}{M}} < \infty$$

Additionally, for every t and $m \leq m_0$, by considering that

$$\left\{\sum_k a_{mk}k^{-s}[f(q(\lambda^t - \lambda^0)\Delta^r x^0_{k+n})]^{p_k}\right\}^{\frac{1}{M}} < \infty$$

is valid, and at the same time there exists at least one k_0 such that for each t and $m \leq m_0$

$$\left\{\sum_{k>k_0} a_{mk}k^{-s}[f(q(\lambda^t-\lambda^0)\Delta^r x^0_{k+n})]^{p_k}\right\}^{\frac{1}{M}} < \frac{\varepsilon}{4} \tag{3.3}$$

is still valid. Again, for $\lambda^t \to \lambda^0$ as $t \to \infty$ when $m \leq m_0$, since

$$\lim_{t\to\infty}\left\{\sum_{k=1}^{k_0} a_{mk}k^{-s}[f(q(\lambda^t-\lambda^0)\Delta^r x^0_{k+n})]^{p_k}\right\}^{\frac{1}{M}} = \left\{\sum_{k=1}^{k_0} a_{mk}k^{-s}\left[f\left(\lim_{t\to\infty}|\lambda^t-\lambda^0|q(\Delta^r x^0_{k+n})\right)\right]^{p_k}\right\}^{\frac{1}{M}} = 0$$

for every $m \leq m_0$ and for the same $\varepsilon > 0$ there exists at least one such that for $t > t_0$

$$\left\{\sum_{k=1}^{k_0} a_{mk}k^{-s}\left[f\left(q((\lambda^t-\lambda^0)\Delta^r x^0_{k+n})\right)\right]^{p_k}\right\}^{\frac{1}{M}} < \frac{\varepsilon}{4} \tag{3.4}$$

is also valid. Thus, from equations (3.3) and (3.4), for every $\varepsilon > 0$ and $m \leq m_0$ there is at least one to such that when $t > t_0$,

$$\left\{\sum_{k} a_{mk}k^{-s}\left[f\left(q((\lambda^t-\lambda^0)\Delta^r x^0_{k+n})\right)\right]^{p_k}\right\}^{\frac{1}{M}} < \frac{\varepsilon}{2}$$

is obtained. From this last equation and (3.2) for every m and $\varepsilon > 0$, when $t > t_0$,

$$\left\{\sum_{k} a_{mk}k^{-s}\left[f\left(q((\lambda^t-\lambda^0)\Delta^r x^0_{k+n})\right)\right]^{p_k}\right\}^{\frac{1}{M}} < \frac{\varepsilon}{2}$$

is obtain, from which

$$\sup_m\left\{\sum_{k} a_{mk}k^{-s}\left[f\left(q((\lambda^t-\lambda^0)\Delta^r x^0_{k+n})\right)\right]^{p_k}\right\}^{\frac{1}{M}} \to 0(t\to\infty)$$

is obviously seen. It is seen that the second part in (3.1) also approaches to zero. Thus, G is a paranorm function, $(w_0(\Delta^r,\hat{A},p,f,q,s),G)$ is a paranormed space. When the definitions of f and q are taken into consideration, it can also be seen that G paranorm is not total. □

Theorem 3.6. *The inclusion*

$$w(\Delta^r,\hat{A},p,q,s) \subset w(\Delta^r,\hat{A},p,f,q,s)$$

is valid, for the infinite matrix $A = (a_{mk})$ has a non-negative regular matrix and $\inf p_k > 0$.

Proof. Let us assume that $\inf p_k = h > 0$, $x \in w(\Delta^r,\hat{A},p,q,s)$ and $0 < \varepsilon < 1$. First, take the case $0 \leq u \leq \delta$ for every u, When $f(u) < \varepsilon$ we can choose $0 < \delta < 1$. Therefore, by the basic rule

$$\begin{aligned}
\sum_k \frac{a_{mk}}{k^s}\{f[q(\Delta^r x_{k+n}-l)]\}^{p_k} &= \sum_{[q(\Delta^r x_{k+n}-l)]\leq\delta} \frac{a_{mk}}{k^s}\{f[q(\Delta^r x_{k+n}-l)]\}^{p_k} \\
&\quad + \sum_{[q(\Delta^r x_{k+n}-l)]>\delta} \frac{a_{mk}}{k^s}\{f[q(\Delta^r x_{k+n}-l)]\}^{p_k} \\
&\leq \sum_k \frac{a_{mk}}{k^s}[\varepsilon]^{p_k} + \sum_k \frac{a_{mk}}{k^s}\left[\frac{2f(1)}{\delta}q(\Delta^r x_{k+n}-l)\right]^{p_k} \\
&\leq \varepsilon^H \sum_k \frac{a_{mk}}{k^s} + max(d_1,d_2)\sum_k \frac{a_{mk}}{k^s}[q(\Delta^r x_{k+n}-l)]
\end{aligned}$$

where $d_1 = \left[\frac{2f(1)}{\delta}\right]^h$ and $d_2 = \left[\frac{2f(1)}{\delta}\right]^H$. Letting $m \to \infty$ in the last inequality, we have desired result. □

Definition 3.7. *Let q_1 and q_2 be two seminorm on X. q_1 is stronger than q_2 if and only if there exists a constant M such that*

$$q_2(u) \leq Mq_1(u)$$

for all u.

Theorem 3.8. *The following inclusion relations*

(i) If $s > 1$, then $w(\Delta^r,\hat{A},p,f,q,s) \subset w(\Delta^r,\hat{A},p,f\circ f_1,q,s)$,
(ii) $w(\Delta^r,\hat{A},p,f_1,q,s) \cap w(\Delta^r,\hat{A},p,f_2,q,s) \subset w(\Delta^r,\hat{A},p,f_1+f_2,q,s)$,
(iii) $w(\Delta^r,\hat{A},p,f,q_1,s) \cap w(\Delta^r,\hat{A},p,f,q_2,s) \subset w(\Delta^r,\hat{A},p,f,q_1+q_2,s)$,
(iv) If q_1 stronger than q_2 then $w(\Delta^r,\hat{A},p,f,q_1,s) \subset w(\Delta^r,\hat{A},p,f,q_2,s)$,
(v) If $\limsup_{t\to\infty}\frac{f_1(t)}{f_2(t)} < \infty$ then $w_\infty(\Delta^r,\hat{A},p,f_2(t),q,s) \subset w_\infty(\Delta^r,\hat{A},p,f_1(t),q,s)$,
(vi) If $s_1 \leq s_2$ then $w(\Delta^r,\hat{A},p,f,q,s_1) \subset w(\Delta^r,\hat{A},p,f,q,s_2)$.

are valid, for the infinite matrix $A=(a_{mk})$ has a non-negative regular matrix, the modulus functions f, f_1, f_2, the seminorm functions q, q_1, q_2 and the real numbers $s, s_1, s_2 \geq 0$.

Proof. (i) We can write the general choices analogous to the above, i.e., first, take the choose δ such that $0<\delta<1$ and $f(t)<\varepsilon$ with $0 \leq t \leq \delta$ for all $\varepsilon>0$ Now, let us suppose that $s>1$ and $x \in w(\Delta^r, \hat{A}, p, f, q, s)$, therefore it is seen that from Lemma 3.1

$$f[f_1(q(\Delta^r x_{k+n}))] \leq \frac{2f(1)}{\delta} f_1[q(\Delta^r x_{k+n}-l)]$$

for $f_1[q(\Delta^r x_{k+n}-l)]>\delta$. In that case, we have

$$\begin{aligned}
\sum_k a_{mk} k^{-s}[f \circ f_1(q(\Delta^r x_{k+n}))]^{p_k} &\leq \sum_k a_{mk} k^{-s}[\varepsilon]^{p_k}+\sum_k a_{mk} k^{-s}\left\{\frac{2f(1)}{\delta} f_1[q(\Delta^r x_{k+n}-l)]\right\}^{p_k} \\
&\leq \varepsilon^H \sum_k a_{mk} k^{-s}+\max(d_1, d_2) \sum_k a_{mk} k^{-s}[f_1(q(\Delta^r x_{k+n}))]^{p_k} \\
&< \infty .
\end{aligned}$$

where $d_1=\left[\frac{2f(1)}{\delta}\right]^h$ and $d_2=\left[\frac{2f(1)}{\delta}\right]^H$. This, in fact, concludes the proof.
(ii)When the following simple calculations derived from Lemma 3.1 are considered, the correctness of the inclusion is understandable.

$$\frac{a_{mk}}{k^s}\{(f_1+f_2)[q(\Delta^r x_{k+n}-l)]\}^{p_k}<C \frac{a_{mk}}{k^s}\{f_1[q(\Delta^r x_{k+n}-l)]\}^{p_k}+C \frac{a_{mk}}{k^s}\{f_2[q(\Delta^r x_{k+n}-l)]\}^{p_k}$$

from Lemma 3.1.
(iii) We have already proven in (ii), therefore, to prove this part of the theorem, it would be enough to examine the following inequality

$$\frac{a_{mk}}{k^s}\{f[(q_1+q_2)(\Delta^r x_{k+n}-l)]\}^{p_k}<C \frac{a_{mk}}{k^s}\{f[q_1(\Delta^r x_{k+n}-l)]\}^{p_k}+C \frac{a_{mk}}{k^s}\{f[q_2(\Delta^r x_{k+n}-l)]\}^{p_k}$$

from Lemma 3.1. Since (iv), (v) and (vi) may be easily established we omit the detail. □

Corollary 3.9. *Let the infinite non-negative a non-negative matrix $A=(a_{mk})$ be a regular matrix. We have that*

(i) If $s>1$, then $w(\Delta^r, \hat{A}, p, q, s) \subset w(\Delta^r, \hat{A}, p, f, q, s)$,
(ii) If $q_1 \equiv q_2$, then $w(\Delta^r, \hat{A}, p, f, q_1, s) \equiv w(\Delta^r, \hat{A}, p, f, q_2, s)$,
(iii) $w(\Delta^r, \hat{A}, p, f, q) \subset w(\Delta^r, \hat{A}, p, f, q, s)$,
(iv) $w(\Delta^r, \hat{A}, p, q) \subset w(\Delta^r, \hat{A}, p, q, s)$,
(v) $w(\Delta^r, \hat{A}, f, q) \subset w(\Delta^r, \hat{A}, f, q, s)$.

for f modulus functions, q_1, q_2 seminorm functions and $s \geq 0$.

Theorem 3.10. *When; arbitrary regular infinite matrix $A=(a_{mk})$ of positive real numbers, $f \in \ell_\infty$, and any real numbers $s>0$; The series, $\sum_k a_k x_k$ convergent necessary and sufficiency condition $a_k \in \Phi$.*

Proof. Only a small fraction of this proof is sufficiency of it, because the convergence of the series $\sum_k a_k x_k$ is extremely clear from the definition of Φ. For obtaining the necessity, first we begin in connection with $a \notin \Phi$ and then introduce a function and show that convergence of the series defined above for this function is not possible. Under this assumption, we know that there can be a strictly increasing (m_k) sequence such that $m_k \in \mathbb{Z}^+$ for all $k \in \mathbb{N}$, in the same sense $m_1<m_2<\cdots$ and also $|a_{m_k}|>0$. Now we need to define a function as follows

$$y_k=\begin{cases}\frac{u}{q_u a_{m_k}} & , \quad k=m_k, \\ \Theta & , \quad k \neq m_k,\end{cases}$$

for $u \in X$ and $q(u)>0$. Due to the fact that the infinite non-negative matrix $A=(a_{mk})$ is regular, $f \in \ell_\infty$ and the real number $s>0$, we get

$$\lim_{m \to \infty} \sum_k a_{mk} k^{-s}[f(q(\Delta^r y_{k+n}))]=0$$

hence $y \in w_0(\Delta^r, \hat{A}, p, f, q, s)$ but $\sum_k a_k y_k=\infty$. This is a contradiction to $\sum_k a_k y_k$ convergent. So the aim is achieved for part of the proof. □

Corollary 3.11. *Let the infinite matrix $A=(a_{mk})$ be any regular matrix such that $a_{mk} \in \mathbb{R}^+$, $s>0$, $f \in \ell_\infty$. Then, the following*

$$[w_0(\Delta^r, \hat{A}, p, f, q, s)]^\beta=\Phi .$$

is valid.

Definition 3.12. *The space $M[E]$ is defined as follows,*

$$M[E]=\{\alpha=(\alpha_k):(\alpha_k x_k) \in E \text{ for every } x \in E\}$$

for all non-empty subset E of $S(X)$.

Theorem 3.13. *Let the infinite matrix $A=(a_{mk})$ be any regular matrix such that $a_{mk} \in \mathbb{R}^+$, and the function f be a modulus and $0<p_k \leq 1$. If $x \in w_\infty(\hat{A}, p, f, q, s)$, then*

$$\ell_\infty \subset M[w_\infty(\Delta^r, \hat{A}, p, f, q, s)].$$

Proof. Let $a = (a_k) \in \ell_\infty$. Therefore, we can find a positive integer K as $|a_k| \le K$. Since q is a seminorm and the function f is a modulus, it is not difficult, the following calculations can be made

$$\begin{aligned} a_{mk}k^{-s}\{f[q(\Delta^r a_{k+n}x_{k+n})]\}^{p_k} &= a_{mk}k^{-s}\left\{f\left[q\left(\sum_{v=0}^{r}(-1)^v\binom{r}{v}a_{k+n+v}x_{k+n+v}\right)\right]\right\}^{p_k} \\ &\le a_{mk}k^{-s}\left\{K\sum_{v=0}^{r}\binom{r}{v}f[q(x_{k+n+v})]\right\}^{p_k} \\ &\le CK^H\binom{r}{v}^H\sum_{v=0}^{r}a_{mk}k^{-s}[f(q(x_{k+n+v}))]^{p_k}. \end{aligned}$$

From $k=1$ to ∞ we replace k onto the last inequality, the desired result $\ell_\infty \subset M[w_\infty(\Delta^r, \hat{A}, p, f, q, s)]$ is obtained. □

Theorem 3.14. *If* $0 < p_k \le r_k < \infty$, *then*

$$w_0(\Delta^r, \hat{A}, p, f, q, s) \subset w_0(\Delta^r, \hat{A}, r, f, q, s).$$

Proof. Let $x \in w_0(\Delta^r, \hat{A}, p, f, q, s)$ and $0 < p_k \le r_k < \infty$, for all $k \in \mathbb{N}$. Under these assumptions, we can choose $m_0 \in \mathbb{N}$ for all $0 < \varepsilon < 1$ such that

$$\sum_k a_{mk}k^{-s}[f(q(\Delta^r x_{k+n}))]^{p_k} < \varepsilon < 1$$

for all $m > m_0$. By using the definitions of parameters, we get

$$a_{mk}k^{-s}[f(q(\Delta^r x_{k+n}))]^{p_k} < \varepsilon < 1$$

and

$$a_{mk}k^{-s}[f(q(\Delta^r x_{k+n}))]^{r_k} \le a_{mk}k^{-s}[f(q(\Delta^r x_{k+n}))]^{p_k} < \varepsilon < 1$$

for all $m > m_0$. Again, adding after applying the last inequality from $k=1$ to ∞, letting $m \to \infty$, this implies the desired result. □

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References

[1] S. Banach, *Théorie des Opérations Linéaires*, Monografie Matematyczne, vol. 1, Warszawa, 1932.
[2] F. Başar, *Summability Theory and Its Applications,* İstanbul 2011.
[3] E. Bulut, Ö. Çakar, *The sequence space $l(p,s)$ and related matrix tranformations*, Commun. Fac. Sci. Univ. Ankara, Ser. A1, Math. Stat. 28, (1979), 33–44.
[4] M. Candan, *Some new sequence spaces defined by a modulus function and infinite matrix in a seminormed space,* J. Math. Anal., 3(2) (2012), 1–9.
[5] R. Çolak, Ö. Çakar. *Banach limits and related matrix transformations,* Stud. Sci. Math. Hung. 24 (1989), 429–436.
[6] G. Das, *Banach and other limits,* J. London Math. Soc. 7 (1973), 501–507.
[7] A. Esi, *Modülüs fonksiyonu yardımıyla tanımlanmış bazı yeni dizi uzayları ve istatistiksel yakınsaklık,* Doktora tezi, Fırat Üniversitesi Elazığ, 1995.
[8] M. Et, R. Çolak, *On generalized difference sequence spaces,* Soochow J. Math. 21 (4) (1995) 377–386.
[9] B. Kuttner, *Note on strong summability*, J. London Math. Soc. 21 (1946), 118–122.
[10] J. Connor, *On strong Matrix summability with respect to a modulus and statistical convergence*, Canad. Mat. Bull. 32 (2) (1989), 194–198.
[11] G.G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. 80 (1948) 167–190.
[12] I. J. Maddox, *Spaces of strongly summable sequences*, *Quart. J. Math.* ***18****(1967), 345–355.*
[13] I. J. Maddox, *Sequence spaces defined by a modulus*, *Math. Proc. Camb. Philos Soc.* ***100****(1986), 161–166.*
[14] I. J. Maddox, *A New Type of Convergence*, Math. Proc. Cambridge Philos. Soc 83 (1978), 61–64.
[15] I. J. Maddox, *On Strong Almost Convergence*, Math. Proc. Cambridge Philos. Soc. 85 (1979), 345–350.
[16] H. Nakano, *Concave modulares*, J. Math. Loc. Japan, Vol.5, (1953), 29–49.
[17] S. Nanda, *Strongly Almost Convergent Sequences*, Bull Calcutta Math. Soc. 76 (1984) 236–40.
[18] S. Pehlivan, *A sequence space defined by a modulus*, *Erciyes Univ. Journal of Science,* ***5****(1-2)(1989), 875–880.*
[19] W. H. Ruckle, *FK spaces in which the sequence of coordinate vectors is bounded*, *Canad. J. Math.* ***25****(1973), 973–978.*
[20] E. Savaş, *On Strong Almost A-Summability with respect to a Modulus and Statistical Convergence*, Indian J. Pure Appl. Math. 23 (1992), 217–22.
[21] E. Savaş, *On Some Generalized Sequence Spaces Defined by a Modulus*, Indian J. Pure Appl. Math., 30 (5) (1999) 459–464.
[22] S.Simons, *The sequence spaces $l(p_v)$ and $m(p_v)$*, Proc. London Math. Soc. (3) 15.
[23] B. Altay, F. Başar, *The matrix domain and the fine spectrum of the difference operator Δ on the sequence space ℓ_p,* $(0 < p < 1)$, Commun. Math. Anal. **2**(2)(2007), 1–11.
[24] H. Kızmaz, *On certain sequence spaces.* Canad. Math. Bull. **24**(2)(1981), 169–176.
[25] F. Başar, B. Altay, *On the space of sequences of p-bounded variation and related matrix mappings,* Ukrainian Math. J. **55**(1)(2003), 136–147.
[26] R. Çolak, M. Et, E. Malkowsky, *Some Topics of Sequence Spaces, in : Lecture Notes in Mathematics,* Fırat Univ. Press, 2004, pp. 1-63. ISBN:975–394–0386–6.
[27] Z.U. Ahmad, Mursaleen, *Köthe-Toeplitz duals of some new sequence spaces and their matrix maps*, Publ. Inst. Math. (Beograd) **42**(1987), 57–61.
[28] Ç. Asma, R. Çolak, *On the Köthe-Toeplitz duals of some generalized sets of difference sequences*, Demonstratio Math. **33**(2000), 797–803.
[29] M.A. Sarıgöl, *On difference sequence spaces*, J. Karadeniz Tech. Univ. Fac. Arts Sci. Ser. Math.-Phys. **10**(1987), 63–71.
[30] E. Malkowsky, *Absolute and ordinary Köthe-Toeplitz duals of some sets of sequences and matrix transformations*, Pulb. Inst. Math (Beograd), (NS), **46**(60)(1989), 97–103.
[31] B. Choudhary, S.K. Mishra, *A note on certain sequence spaces*, J. Anal. **1**(1993), 139–148.
[32] S.K. Mishra, *Matrix maps involving certain sequence spaces*, Indian J. Pure Appl. Math. **24**(2)(1993), 125–132.

[33] M. Mursaleen, A.K. Gaur, A.H. Saifi, *Some new sequence spaces and their duals and matrix transformations,* Bull. Calcutta Math. Soc. **88**(3)(1996), 207–212.
[34] C. Gnanaseelan, P.D. Srivastava, *The* $\alpha-,\beta-$ *and* $\gamma-$ *duals of some generalised difference sequence spaces*, Indian J. Math. **38**(2)(1996), 111–120.
[35] E. Malkowsky, *A note on the Köthe-Toeplitz duals of generalized sets of bounded and convergent difference sequences,* J. Anal. **4**(1996), 81–91.
[36] A.K. Gaur, M. Mursaleen, *Difference sequence spaces*, Int. J. Math. Sci. **21**(4)(1998), 701–706.
[37] E. Malkowsky, M. Mursaleen, Qamaruddin, *Generalized sets of difference sequences, their duals and matrix transformations, in : Sequence Spaces and Applications,* Narosa, New Delhi, 1999, pp. 68–83.
[38] E. Malkowsky, M. Mursaleen, *Some matrix transformations between the difference sequence spaces* $\Delta c_0(p)$*,* $\Delta c(p)$ *and* $\Delta \ell_\infty(p)$, Filomat **15**(2001), 353–363.
[39] T. Bilgin, *The sequence spaces* $\ell(p,f,q,s)$ *on seminormed spaces,* Bul. of Calcutta Math. Soc. **86**(4)(1994), 295–304.
[40] A Şahiner, *Some new paranormed space defined by modulus function,* Indian J. Pure Appl. Math. **33**(12)(2002), 1877–1888.