Konuralp Journal of Mathematics, 6 (1) (2018) 26-34

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

Some Fixed Point Results for A Multivalued Generalization of Generalized Hybrid Mappings in $CAT(\kappa)$-Spaces

Emirhan Hacıoğlu[1*] and Vatan Karakaya[2]

[1] *Department of Mathematics, Yildiz Technical University, Davutpasa Campus, Esenler, 34220 Istanbul, Turkey*
[2] *Department of Mathematical Engineering, Yildiz Technical University, Davutpasa Campus, Esenler, 34210 Istanbul,Turkey*
[*] *Corresponding author E-mail: emirhanhacioglu@hotmail.com*

Abstract

The studies about hybrid mappings are mainly focused on single-valued mappings. Now, we give definition of multivalued generalization of generalized hybrid mappings which is defined in $CAT(0)$ spaces and also studied on $CAT(\kappa)$ spaces. This new definition is general than multivalued nonexpansive mappings, multivalued hybrid mappings and multivalued nonspreading mappings. Under suitable conditions, we prove some existence and stability results. We also study some convergence of multivalued proximal version of Thianwan iteration scheme for non-self multivalued generalized hybrid mappings in $CAT(\kappa)$-spaces.

Keywords: *Multivalued hybrid mapping; Fixed point; Convergence; $CAT(\kappa)$ spaces.*
2010 Mathematics Subject Classification: *47H10*

1. Introduction and Preliminaries

The studies of fixed point theory on nonlinear structures is very important since most events in the real world have nonlinear structure and the fixed point theory is very useful tool when the real life events reduced to mathematical modelings. Geodesic spaces are well defined examples of these nonlinear structures. Because of as a geodesic space $CAT(0)$ spaces and Hilbert spaces have similar structures, the fixed point studies on the Hilbert and Banach spaces has been studied on these spaces in parallel. These works mainly done for single and multivalued nonexpansive mappings. Although some of these studies moved on $CAT(\kappa)$ spaces for single valued mappings, there are just few studies for multivalued mappings. In this study, we generalize generalized hybrid mappings which is defined in $CAT(0)$ spaces to multivalued case which is general than multivalued nonexpansive, hybrid and nonspreading mapping and under suitable conditions, we prove some existence and convergence results in $CAT(\kappa)$-spaces. As well as this new defined multivalued mapping class is general than the most of multivalued mappings in literature, it is also multivalued generalization of many mapping classes of single valued hybrid mappings whose multivalued generalizations do not exist in the literature. Therefore, this study is very comprehensive. Let H be a Hilbert space and $K \subseteq H,\ K \neq \emptyset$. Let us take T as a single valued mapping from K to H. If T satisfies

$$||Tx - Ty|| \leq ||x - y||,$$

$$2||Tx - Ty||^2 \leq ||Tx - y||^2 + ||Ty - x||^2$$

and

$$3||Tx - Ty||^2 \leq ||x - y||^2 + ||Tx - y||^2 + ||Ty - x||^2$$

for all $x, y \in K$ then it is called nonexpansive, nonspreading[1] and hybrid[2], respectively. None of these classes of mappings is included in the other. In 2010, Aoyama et al.[3] defined $\lambda-$hybrid as follows;

$$(1+\lambda)||Tx - Ty||^2 - \lambda||x - Ty||^2 \leq (1-\lambda)||x - y||^2 + \lambda||Tx - y||^2$$

where $x, y \in K$ and λ is fixed real number. $\lambda-$hybrid mappings are general than nonexpansive mappings, nonspreading mappings and hybrid mappings. In 2011, Aoyama and Kohsaka[4] introduced $\alpha-$nonexpansive mappings in Banach spaces as follows;

$$||Tx - Ty||^2 \leq (1-2\alpha)||x - y||^2 + \alpha||Tx - y||^2 + \alpha||x - Ty||^2$$

Email addresses: **emirhanhacioglu@hotmail.com (Emirhan Hacıoğlu), vkkaya@yahoo.com (Vatan Karakaya)**

where $x,y \in K$ and $\alpha < 1$ is fixed. They showed that $\alpha-$nonexpansive and $\lambda-$hybrid are equivalent in Hilbert spaces for $\lambda < 2$.

Many iterative processes to approximate a fixed point of multivalued mappings have been introduced in metric and Banach spaces. The well known one is defined by Nadler as generalization of Picard as follows;

$$x_{n+1} \in Tx_n.$$

A multivalued version of Mann and Ishikawa fixed point procedures goes as follow;

$$x_{n+1} \in (1-\zeta_n)x_n + \zeta_n Tx_n$$

and

$$\begin{aligned} x_{n+1} &\in (1-\zeta_n)x_n + \zeta_n Ty_n, \\ y_n &\in (1-\varsigma_n)x_n + \varsigma_n Tx_n \end{aligned}$$

where $\{\zeta_n\}$ and $\{\varsigma_n\}$ are sequences in $[0,1]$. Also In 2008, Thianwan introduced two step iteration as follows;

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)y_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)x_n + \beta_n Tx_n \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Now, we give the definition of the multivalued proximal version of Thianwan iteration as follows;

$$\begin{aligned} x_{n+1} &= P_K((1-\alpha_n)y_n \oplus \alpha_n u_n), \\ y_n &= P_K((1-\beta_n)x_n \oplus \beta_n v_n) \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ and $u_n \in Ty_n$, $v_n \in Tx_n$.

Let (X,d) be a metric space and $K \subseteq X$, $K \neq \emptyset$. In rest of this paper, we will use following notations; $C(X)$ for all nonempty, closed subsets of X, $CC(X)$ for all nonempty closed and convex subsets of X, $KC(X)$ for nonempty, compact and convex subsets of X and $CB(X)$ for all nonempty, closed and convex subsets of X. Let H_d be Hausdorff metric on $CB(X)$ defined by

$$H_d(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{x\in B} d(x,A)\}$$

where $d(x,B) = \inf\{d(x,y) : y \in B\}$. A point p is called fixed point of multivalued mapping T if $p \in Tp$ and the set of all fixed points of T is denoted by $F(T)$.

Let (X,d) be bounded metric space and take $x,y \in X$ and $K \subseteq X$, $K \neq \emptyset$. A geodesic path (or shortly a geodesic) joining x and y is a map $c : [0,t] \subseteq \mathbb{R} \to X$ such that $c(0) = x$, $c(t) = y$ and $d(c(r),c(s)) = |r-s|$ for all $r,s \in [0,t]$. In fact, c is an isometry and $d(c(0),c(t)) = t$. The image of c, $c([0,t])$ is called geodesic segment from x to y and it is not necessarily be unique. If it is unique then it is denoted by $[x,y]$. $z \in [x,y]$ if and only if there exists $t \in [0,1]$ such that $d(z,x) = (1-t)d(x,y)$ and $d(z,y) = td(x,y)$. The point z is denoted by $z = (1-t)x \oplus ty$. For fixed $r > 0$, the space (X,d) is called r-geodesic space if any two point $x,y \in X$ with $d(x,y) < r$ there is a geodesic joining x to y. if for every $x,y \in X$, there is a geodesic path then (X,d) called geodesic space and uniquely geodesic space if that geodesic path is unique for any pair x,y. We call a subset $K \subseteq X$ as a convex subset if it contains all geodesic segment joining any pair of points in it.

Definition 1.1. (see:[5]) Let $\kappa \in \mathbb{R}$.

i) if $\kappa = 0$, *then* M_κ^n *is Euclidean space* $\mathbb{E}^n$,
ii) if $\kappa > 0$, *then* M_κ^n *is obtained from the sphere* $\mathbb{S}^n$ *by multiplying distance function by* $\frac{1}{\sqrt{\kappa}}$,
iii) if $\kappa < 0$, *then* M_κ^n *is obtained from hyperbolic space* $\mathbb{H}^n$ *by multiplying distance function by* $\frac{1}{\sqrt{-\kappa}}$.

In a geodesic metric space (X,d), a geodesic triangle $\Delta(x,y,z)$ consist of three point x,y,z as vertices and three geodesic segments of any pair of these points, that is, $q \in \Delta(x,y,z)$ means that $q \in [x,y] \cup [x,z] \cup [y,z]$. The triangle $\overline{\Delta}(\bar{x},\bar{y},\bar{z})$ in M_κ^2 is called comparison triangle for the triangle $\Delta(x,y,z)$ such that $d(x,y) = d(\bar{x},\bar{y})$, $d(x,z) = d(\bar{x},\bar{z})$ and $d(y,z) = d(\bar{y},\bar{z})$ and such a comparison triangle always exists provided that the perimeter $d(x,y)+d(y,z)+d(z,x) < 2D_\kappa$ ($D_\kappa = \frac{\pi}{\sqrt{K}}$ if $\kappa > 0$ and ∞ otherwise) in M_κ^2 (Lemma 2.14 in [5]). A point point $\bar{z} \in [\bar{x},\bar{y}]$ called comparison point for $z \in [x,y]$ if $d(x,z) = d(\bar{x},\bar{z})$. A geodesic triangle $\Delta(x,y,z)$ in X with perimeter less than $2D_\kappa$ (and given a comparison triangle $\overline{\Delta}(\bar{x},\bar{y},\bar{z})$ for $\Delta(x,y,z)$ in M_κ^2) satisfies $CAT(\kappa)$ inequality if $d(p,q) \le d(\overline{p},\overline{q})$ for all $p,q \in \Delta(x,y,z)$ where $\overline{p},\overline{q} \in \overline{\Delta}(\bar{x},\bar{y},\bar{z})$ are the comparison points of p,q respectively. The D_κ-geodesic metric space (X,d) is called $CAT(\kappa)$ space if every geodesic triangle in X with perimeter less than $2D_\kappa$ satisfies the $CAT(\kappa)$ inequality.

If for every $x,y,z \in X$, there is an $R \in (0,2]$ satisfying CN* inequality

$$d^2(x,(1-\lambda)y \oplus \lambda z) \le (1-\lambda)d^2(x,y) + \lambda d^2(x,z) - \frac{R}{2}\lambda(1-\lambda)d^2(y,z)$$

then (X,d) is called $R-$*convex* space [6]. Hence, (X,d) is a $CAT(0)$ space if and only if it is a $2-$*convex* space.

Lemma 1.1. *(see:[7]) Let* $\kappa > 0$ *and* (X,d) *be a* $CAT(\kappa)$ *space with* $diam(X) < \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ *for some* $\varepsilon \in (0,\frac{\pi}{2})$. *Then* (X,d) *is a* $R-$*convex space for* $R = (\pi-2\varepsilon)\tan(\varepsilon)$.

Proposition 1.1. *(see:[5]) Let X be* $CAT(\kappa)$ *space. Then any ball of radius smaller than* $\frac{\pi}{2\sqrt{\kappa}}$ *is convex.*

Proposition 1.2. *(See: Exercise 2.3(1) in [5]) Let $\kappa > 0$ and (X,d) be a $CAT(\kappa)$ space with $diam(X) < \frac{D_\kappa}{2} = \frac{\pi}{2\sqrt{\kappa}}$. Then, for any $x,y,z \in X$ and $t \in [0,1]$, we have*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$

Let $\{x_n\}$ be a bounded sequence in a $CAT(\kappa)$ space X, $x \in X$ and

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius of $\{x_n\}$ is defined by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\},$$

the asymptotic radius of $\{x_n\}$ with respect to $K \subseteq X$ is defined by

$$r_K(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in K\},$$

and the asymptotic center of $\{x_n\}$ is defined by

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}$$

and let $\omega_w(x_n) := \cup A(\{x_n\})$ where union is taken on all subsequences of $\{x_n\}$.

Definition 1.2. (see:[8]) A sequence $\{x_n\} \subset X$ is said to be $\Delta-$ convergent to $x \in X$ if x is the unique asymptotic center of all subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_n x_n = x$ and read as x is the $\Delta-$limit of $\{x_n\}$.

Proposition 1.3. *(see:[8]) Let X be a complete $CAT(\kappa)$ space, $K \subseteq X$ nonempty, closed and convex, $\{x_n\}$ be a sequence in X. If $r_K(\{x_n\}) < \frac{\pi}{2\sqrt{\kappa}}$, then $A_K(\{x_n\})$ consists of exactly one point.*

Lemma 1.2. *(see:[11])*

i) Every bounded sequence in X has a Δ-convergent subsequence,
ii) If K is a closed and convex subset of X and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.

Lemma 1.3. *(see:[11]) If $\{x_n\}$ is a bounded sequence in X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = u$ and the sequence $\{d(x_n,u)\}$ converges, then $x = u$.*

Lemma 1.4. *(see:[8]) Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$. Let K be a nonempty, closed and convex subset of X. Then*

i) the metric projection $P_K(x)$ of x onto K is a singleton,
ii) if $x \notin K$ and $y \in K$ with $y \neq P_K(x)$, then $\angle_{P_K(x)}(x,y) \geq \frac{\pi}{2}$,
iii) for each $y \in K$, $d(P_K(x),P_K(y)) \leq d(x,y)$.

Definition 1.3. T is called generalized multivalued hybrid mapping from X to $CB(X)$ if

$$\begin{aligned} H^2(Tx,Ty) &\leq a_1(x)d^2(x,y) + a_2(x)d^2(Tx,y) + a_3(x)d^2(x,Ty) \\ &\quad + k_1(x)d^2(Tx,x) + k_2(x)d^2(Ty,y) \end{aligned}$$

is satisfied for all $x,y \in X$ where $a_1, a_2, a_3, k_1, k_2 : X \to [0,1]$ with $a_1(x)+a_2(x)+a_3(x) < 1$, $2k_1(x) < 1-a_2(x)$ and $2k_2(x) < 1-a_3(x)$ for all $x \in X$.

2. Existence and Stability of Fixed Point Sets

Proposition 2.1. *Let $\kappa > 0$ and K be a nonempty, closed and convex subset of complete $CAT(\kappa)$ space X with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$ and $T : K \to C(K)$ be a generalized multivalued hybrid mapping with $F(T) \neq \emptyset$, then $F(T)$ is closed.*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ and $x_n \to x \in X$.

$$\begin{aligned} d^2(Tx,x_n) &\leq H^2(Tx,Tx_n) \\ &\leq a_1(x)d^2(x,x_n) + a_2(x)d^2(Tx,x_n) + a_3(x)d^2(x,Tx_n) \\ &\quad + k_1(x)d^2(Tx,x) + k_2(x)d^2(Tx_n,x_n) \\ &\leq a_1(x)d^2(x,x_n) + a_2(x)d^2(Tx,x_n) + a_3(x)d^2(x,x_n) \\ &\quad + k_1(x)d^2(Tx,x) + k_2(x)d^2(x_n,x_n) \end{aligned}$$

implies that

$$d^2(Tx,x_n) \leq d^2(x,x_n) + \frac{k_1(x)}{1-a_2(x)} d^2(Tx,x)$$

then taking limit on n we have

$$(1 - \frac{k_1(x)}{1-a_2(x)})d(Tx,x) \leq 0.$$

□

Theorem 2.1. *Let $\kappa > 0$ and K be a nonempty, compact and convex subset of complete $CAT(\kappa)$ space X with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$ and $T : K \to C(X)$ be a generalized multivalued hybrid mapping with $k_1(x) = 0$ for all $x \in K$. Then, there is a sequence $\{x_n\}$ in K such that $\lim_{n\to\infty} d(x_n,Tx_n) = 0$ if and only if $F(T) \neq \emptyset$.*

Proof. Assume that $\{x_n\}$ is a sequence in K with $\lim_{n\to\infty} d(x_n,Tx_n) = 0$ Then since K is compact, there is convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$, say $x_{n_i} \to z \in K$. Then, by Lemma 1.4, we can find a sequence $\{y_n\}$ such that $d(x_n,y_n) = d(x_n,Tx_n)$ for all $n \in \mathbb{N}$. Then since $d(x_{n_i},Tz) \leq d(x_{n_i},y_{n_i}) + d(y_{n_i},Tz)$ and $d(y_{n_i},Tz) \leq d(x_{n_i},y_{n_i}) + d(x_{n_i},Tz)$ we have that $\limsup_{i\to\infty} d(x_{n_i},Tz) = \limsup_{i\to\infty} d(y_{n_i},Tz)$. Then using properties of T, we have

$$\begin{aligned}
d^2(Tz,y_{n_i}) &\leq H^2(Tz,Tx_{n_i})\\
&\leq a_1(z)d^2(z,x_{n_i}) + a_2(z)d^2(Tz,x_{n_i}) + a_3(z)d^2(z,Tx_{n_i})\\
&\quad +k_2(z)d^2(Tx_{n_i},x_{n_i})\\
&\leq a_1(z)d^2(z,x_{n_i}) + a_2(z)d^2(Tz,x_{n_i})\\
&\quad +a_3(z)[d(z,x_{n_i}) + d(x_{n_i},Tx_{n_i})]^2 + k_2(z)d^2(Tx_{n_i},x_{n_i})
\end{aligned}$$

so we get that

$$\limsup_{i\to\infty} d(Tz,x_{n_i}) \leq \limsup_{i\to\infty} d(x_{n_i},z) = 0$$

then

$$d(z,Tz) \leq d(z,x_{n_i}) + d(x_{n_i},Tz)$$

implies that

$$d(z,Tz) \leq \limsup_{i\to\infty} d(z,x_{n_i}) + \limsup_{i\to\infty} d(x_{n_i},Tz) = 0$$

Hence we get that $z \in Tz$. □

Theorem 2.2. *Let $\kappa > 0$ and K be a nonempty, closed and convex subset of complete $CAT(\kappa)$ space X with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$ and $T : K \to C(K)$ be a generalized multivalued hybrid mapping satisfying either*

i) $a_2(x) = 0$, $\frac{2k_2(x)}{1-a_3(x)} < \frac{R}{2}$ *and* $k = \sqrt{\sup \frac{a_1(x)+k_2(x)}{1-k_1(x)}} < 1$

ii) $a_3(x) = 0$, $\frac{2k_1(x)}{1-a_2(x)} < \frac{R}{2}$ *and* $k = \sqrt{\sup \frac{a_1(x)+k_1(x)}{1-k_2(x)}} < 1$

for all $x \in K$, where $R = (\pi - 2\varepsilon)tan(\varepsilon)$. Then $F(T) \neq \emptyset$.

Proof. Let $x_0 \in K$ and $x_{n+1} \in Tx_n$ such that $d(x_{n+1},x_n) = d(Tx_n,x_n)$ for all $n \in \mathbb{N}$. Assume that $a_2(x) = 0$. Then

$$\begin{aligned}
d^2(x_{n+1},x_n) &= d^2(Tx_n,x_n) \leq H^2(Tx_n,Tx_{n-1})\\
&\leq a_1(x_n)d^2(x_n,x_{n-1}) + a_3(x)d^2(Tx_{n-1},x_n)\\
&\quad +k_1(x_n)d^2(Tx_n,x_n) + k_2(x_n)d^2(Tx_{n-1},x_{n-1})\\
&\leq a_1(x_n)d^2(x_n,x_{n-1}) + k_1(x_n)d^2(Tx_n,x_n)\\
&\quad +k_2(x_n)d^2(Tx_{n-1},x_{n-1})
\end{aligned}$$

implies that

$$d^2(x_{n+1},x_n) \leq \frac{a_1(x_n)+k_2(x_n)}{1-k_1(x_n)} d^2(x_n,x_{n-1})$$

hence we have

$$\begin{aligned}
d(x_{n+1},x_n) &\leq \sqrt{\frac{a_1(x_n)+k_2(x_n)}{1-k_1(x_n)}} d(x_n,x_{n-1})\\
&\leq kd(x_n,x_{n-1})\\
&\leq k^n d(x_1,x_0).
\end{aligned}$$

Let $n < m$,then

$$\begin{aligned}
d(x_m,x_n) &\leq \textstyle\sum_{i=n}^{i=m-1} d(x_{i+1},x_i)\\
&\leq \textstyle\sum_{i=n}^{i=m-1} k^i d(x_1,x_0)\\
&\leq d(x_1,x_0)\textstyle\sum_{i=n}^{i=m-1} k^i.
\end{aligned}$$

Since $k < 1$ then the sequence (x_n) is Cauchy sequence and since space is complete then $x_n \to z \in X$ and since K is closed then $z \in K$

$$\begin{aligned}
d^2(x_n,Tz) &\leq H^2(Tx_{n-1},Tz)\\
&\leq a_1(z)d^2(x_{n-1},z) + a_3(z)d^2(x_{n-1},Tz)\\
&\quad +k_1(z)d^2(Tx_{n-1},x_{n-1}) + k_2(z)d^2(Tz,z).
\end{aligned}$$

On the other hand, for any $u \in Tz$

$$d^2(x_n, \frac{1}{2}u \oplus \frac{1}{2}z) \leq \frac{1}{2}d^2(x_n,u) + \frac{1}{2}d^2(x_n,z) - \frac{R}{8}d^2(z,u)$$

which implies that

$$d^2(z,u) \leq \frac{4}{R}d^2(x_n,u) + \frac{4}{R}d^2(x_n,z)$$

and by taking infimum on u, we have

$$d^2(z,Tz) \leq \frac{4}{R}d^2(x_n,Tz) + \frac{4}{R}d^2(x_n,z).$$

We get that

$$\begin{aligned} d^2(x_n,Tz) &\leq a_1(z)d^2(x_{n-1},z) + a_3(z)d^2(x_{n-1},Tz) \\ &\quad + k_1(z)d^2(Tx_{n-1},x_{n-1}) + k_2(z)d^2(Tz,z) \\ &\leq a_1(z)d^2(x_{n-1},z) + a_3(z)d^2(x_{n-1},Tz) + k_1(z)d^2(Tx_{n-1},x_{n-1}) \\ &\quad + k_2(z)(\frac{4}{R}d^2(x_n,Tz) + \frac{4}{R}d^2(x_n,z)) \end{aligned}$$

which implies that

$$(1 - k_2(z)\frac{4}{R} - a_3(z)) \lim_{n\to\infty} d^2(x_n,Tz) \leq 0$$

so $\lim_{n\to\infty} d^2(x_n,Tz) = 0$. Hence $d^2(z,Tz) \leq d^2(x_n,z) + d^2(x_n,Tz) \to 0$ implies that $z \in Tz$. □

Theorem 2.3. *Let $\kappa > 0$ and K be a nonempty, closed and convex subset of complete CAT(κ) space X with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$ and $T_1,T_2 : K \to C(K)$ be two generalized multivalued hybrid mappings satisfying either*

i) $a_2(x) = 0$, $\frac{2k_2(x)}{1-a_3(x)} < \frac{R}{2}$ *and* $k = \sqrt{\sup \frac{a_1(x)+k_2(x)}{1-k_1(x)}} < 1$ *or*
ii) $a_3(x) = 0$, $\frac{2k_1(x)}{1-a_2(x)} < \frac{R}{2}$ *and* $k = \sqrt{\sup \frac{a_1(x)+k_1(x)}{1-k_2(x)}} < 1$

for all $x \in K$, where $R = (\pi - 2\varepsilon)tan(\varepsilon)$. Then

$$H(F(T_1),F(T_2)) \leq \frac{1}{1-k} \sup_{x\in K} H(T_1x,T_2x).$$

Proof. Assume that $a_2(x) = 0$. Let $x_0 \in F(T_1)$ and $x_{n+1} \in T_2x_n$ such that $d(x_{n+1},x_n) = d(T_2x_n,x_n)$ for all $n \geq 0$. Then the (x_n) is convergent to $z \in T_2z$ and also

$$\begin{aligned} d(x_0,z) &\leq \textstyle\sum_{i=0}^{\infty} d(x_i,x_{i+1}) \\ &\leq \textstyle\sum_{i=0}^{\infty} k^i d(x_0,x_1) \\ &= d(x_0,x_1)\textstyle\sum_{i=0}^{\infty} k^i \\ &\leq d(x_0,x_1)\frac{1}{1-k} \end{aligned}$$

and since $d(x_0,x_1) = d(x_0,T_2x_0) \leq H(T_1x_0,T_2x_0) \leq \sup_{x\in K} H(T_1x,T_2x)$, $d(x_0,z) \leq \sup_{x\in K} H(T_1x,T_2x)\frac{1}{1-k}$. So for all $x_0 \in F(T_1)$ we can find $z \in F(T_2)$ and similarly for all $x_0^{'} \in F(T_2)$ we can find $z^{'} \in F(T_1)$. Hence

$$H(F(T_1),F(T_2)) \leq \frac{1}{1-k} \sup_{x\in K} H(T_1x,T_2x)$$

holds. □

Lemma 2.1. *Let $\kappa > 0$ and K be a nonempty, closed and convex subset of complete CAT(κ) space X with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$ and $\{T_n : K \to C(K)\}$ be a sequence of generalized multivalued hybrid mappings with same coefficient functions satisfying either*

i) $a_2(x) = 0$, $\frac{2k_2(x)}{1-a_3(x)} < \frac{R}{2}$ *and* $k = \sqrt{\sup \frac{a_1(x)+k_2(x)}{1-k_1(x)}} < 1$ *or*
ii) $a_3(x) = 0$, $\frac{2k_1(x)}{1-a_2(x)} < \frac{R}{2}$ *and* $k = \sqrt{\sup \frac{a_1(x)+k_1(x)}{1-k_2(x)}} < 1$

for all $x \in K$, where $R = (\pi - 2\varepsilon)tan(\varepsilon)$. If $\{T_n\}$ is uniformly convergent to a multivalued map $T : K \to C(K)$, then T is generalized multivalued hybrid mapping.

Proof. Since for all $n \geq 0$,

$$
\begin{aligned}
H^2(T_n x, T_n y) &\leq a_1(x)d^2(x,y) + a_2(x)d^2(T_n x, y) + a_3(x)d^2(x, T_n y) \\
&\quad + k_1(x)d^2(T_n x, x) + k_2(x)d^2(T_n y, y)
\end{aligned}
$$

is satisfied, taking limit on n we get that

$$
\begin{aligned}
H^2(Tx, Ty) &\leq a_1(x)d^2(x,y) + a_2(x)d^2(Tx, y) + a_3(x)d^2(x, Ty) \\
&\quad + k_1(x)d^2(Tx, x) + k_2(x)d^2(Ty, y).
\end{aligned}
$$

□

Theorem 2.4. *Let $\kappa > 0$ and K be a nonempty, closed and convex subset of complete CAT(κ) space X with diam$(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$ and $\{T_n : K \to C(K)\}$ be a sequence of generalized multivalued hybrid mappings with same coefficient functions satisfying either*

i) $a_2(x) = 0$, $\frac{2k_2(x)}{1-a_3(x)} < \frac{R}{2}$ and $k = \sqrt{\sup \frac{a_1(x)+k_2(x)}{1-k_1(x)}} < 1$ or
ii) $a_3(x) = 0$, $\frac{2k_1(x)}{1-a_2(x)} < \frac{R}{2}$ and $k = \sqrt{\sup \frac{a_1(x)+k_1(x)}{1-k_2(x)}} < 1$

for all $x \in K$, where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$. If $\{T_n\}$ is uniformly convergent to a multivalued map $T : K \to C(K)$, then $F(T_n)$ converges to $F(T)$.

Proof. By above Theorem 2.3 and Lemma 2.1, we have

$$
H(F(T_n), F(T)) \leq \frac{1}{1-k} \sup_{x \in K} H(T_n x, Tx)
$$

for all $n \in \mathbb{N}$ and taking limit on n, we conclude that $\lim_{n\to\infty} H(F(T_n), F(T)) = 0$. □

Example 2.1. Let $X = [3,9]$ with usual metric and $T : X \to C(X)$ be multivalued mapping defined by

$$
Tx = \begin{cases} \{3\}, & x \in [3,5], \\ [3, \frac{4x^3+1}{x^3+1}], & x \in (5,9]. \end{cases}
$$

We will show that T is generalized multivalued hybrid mapping with $a_1(x) = k_1(x) = k_2(x) = 0$, $a_2(x) = \frac{x}{x+2}$, $a_3(x) = \frac{2}{x+2}$ for all $x \in X$.
Case 1: if $x, y \in [3,5]$, it is obvious.
Case 2: if $x \in [3,5]$, $y \in (5,9]$, then we have that $H^2(Tx, Ty) \leq 1$, $4 < d^2(Tx, y)$, $0 \leq d^2(x, Ty)$ and therefore

$$
\begin{aligned}
H^2(Tx, Ty) &\leq \frac{x}{x+2}4 + \frac{2}{x+2}d^2(x, Ty) \\
&\leq \frac{x}{x+2}d^2(Tx, y) + \frac{2}{x+2}d^2(x, Ty).
\end{aligned}
$$

Case 3: if $x, y \in (5,9]$, then we have that $H^2(Tx, Ty) \leq 1$, $1 < d^2(Tx, y)$, $1 < d^2(x, Ty)$ and therefore

$$
\begin{aligned}
H^2(Tx, Ty) &\leq \frac{x}{x+2} + \frac{2}{x+2} \\
&\leq \frac{x}{x+2}d^2(Tx, y) + \frac{2}{x+2}d^2(x, Ty).
\end{aligned}
$$

Thus, T is a generalized multivalued hybrid mapping with fixed point 3, $T(3) = \{3\}$. However, it is not nonexpansive. Since $T(5) = 3$ and $T(5.1) = [3, 3.978...]$ satisfies $H(T(5), T(5.1)) = 0.978... > d(5, 5.1) = 0.1$.

3. Convergence Results

Theorem 3.1. *Let $\kappa > 0$ and K be a nonempty, closed and convex subset of complete CAT(κ) space X with diam$(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$ and $T : K \to KC(X)$ be a generalized multivalued hybrid mapping with $\frac{2k_1(x)}{1-a_2(x)} < \frac{R}{2}$ for all $x \in K$ where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$. If $\{x_n\}$ is a sequence in K with $\Delta - \lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ then $z \in K$ and $z \in T(z)$.*

Proof. By Lemma 1.2, $z \in K$. We can find a sequence $\{y_n\}$ such that $y_n \in Tx_n$, $d(x_n, y_n) = d(x_n, Tx_n)$, so we have $\lim_{n\to\infty} d(x_n, y_n) = 0$ and since Tz is compact we can find a sequence $\{z_n\}$ in Tz such that $d(y_n, z_n) = d(y_n, Tz)$. Then there is a convergent subsequence $\{z_{n_i}\}$ of $\{z_n\}$, say $\lim_{i\to\infty} z_{n_i} = u \in Tz$.

$$
\begin{aligned}
d(x_{n_i}, u) &\leq d(x_{n_i}, y_{n_i}) + d(y_{n_i}, z_{n_i}) + d(z_{n_i}, u) \\
&\leq d(x_{n_i}, y_{n_i}) + d(y_{n_i}, Tz) + d(z_{n_i}, u) \\
&\leq d(x_{n_i}, y_{n_i}) + H(Tx_{n_i}, Tz) + d(z_{n_i}, u) \\
&\leq d(x_{n_i}, y_{n_i}) + H(Tx_{n_i}, Tz) + d(z_{n_i}, u)
\end{aligned}
$$

implies that $\limsup_{i\to\infty} d(x_{n_i},u) \leq \limsup_{i\to\infty} H(Tx_{n_i},Tz)$ and $\Delta-\lim_{i\to\infty} x_{n_i} = z$. Because of T is generalized multivalued hybrid mapping,

$$\begin{aligned}
H^2(Tz,Tx_{n_i}) &\leq a_1(z)d^2(x_{n_i},z)+a_2(z)d^2(Tz,x_{n_i})+a_3(z)d^2(Tx_{n_i},z)\\
&\quad +k_1(z)d^2(Tz,z)+k_2(z)d^2(Tx_{n_i},x_{n_i})\\
&\leq a_1(z)d^2(x_{n_i},z)+a_2(z)[d(x_{n_i},Tx_{n_i})+H(Tx_{n_i},Tz)]^2\\
&\quad +a_3(z)[d(Tx_{n_i},x_{n_i})+d(x_{n_i},z)]^2+k_1(z)d^2(Tz,z)+k_2(z)d^2(Tx_{n_i},x_{n_i})
\end{aligned}$$

which implies that

$$\begin{aligned}
\limsup_{i\to\infty} H^2(Tx_{n_i},Tz) &\leq \limsup_{i\to\infty} d^2(x_{n_i},z)+\frac{k_1(x)}{1-a_2(x)}d^2(z,Tz)\\
&\leq \limsup_{i\to\infty} d^2(x_{n_i},z)+\frac{k_1(x)}{1-a_2(x)}d^2(z,u).
\end{aligned}$$

By CN* inequality we have

$$d^2(x_{n_i},\frac{1}{2}z\oplus\frac{1}{2}u) \leq \frac{1}{2}d^2(x_{n_i},z)+\frac{1}{2}d^2(x_{n_i},u)-\frac{R}{8}d^2(z,u)$$

and combining all of these we get

$$\begin{aligned}
\limsup_{i\to\infty} d^2(x_{n_i},\frac{1}{2}z\oplus\frac{1}{2}u) &\leq \frac{1}{2}\limsup_{i\to\infty} d^2(x_{n_i},z)+\frac{1}{2}\limsup_{i\to\infty} d^2(x_{n_i},u)-\frac{R}{8}d^2(z,u)\\
&\leq \frac{1}{2}\limsup_{n\to\infty} d^2(x_{n_i},z)+\frac{1}{2}\limsup_{n\to\infty} H(Tx_{n_i},Tz)-\frac{R}{8}d^2(z,u).\\
&\leq \frac{1}{2}\limsup_{i\to\infty} d^2(x_{n_i},z)+\frac{1}{2}\limsup_{i\to\infty} d^2(x_{n_i},z)\\
&\quad +\frac{k_1(x)}{2(1-a_2(x))}d^2(z,u)-\frac{R}{8}d^2(z,u)\\
&= \limsup_{i\to\infty} d^2(x_{n_i},z)+(\frac{k_1(x)}{2(1-a_2(x))}-\frac{R}{8})d^2(z,u)
\end{aligned}$$

which implies that

$$(\frac{R}{8}-\frac{k_1(x)}{2(1-a_2(x))})d^2(z,u) \leq \limsup_{i\to\infty} d^2(x_{n_i},z)-\limsup_{i\to\infty} d^2(x_{n_i},\frac{1}{2}z\oplus\frac{1}{2}u) \leq 0$$

and by assumptions, we have $z = u \in Tz$. □

Corollary 3.1. *Let K be a nonempty, closed and convex subset of complete $CAT(0)$ space X and $T : K \to KC(X)$ be a generalized multivalued hybrid mapping. If $\{x_n\}$ be a bounded sequence in K with $\Delta-\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} d(x_n,Tx_n) = 0$ then $z \in K$ and $z \in T(z)$.*

Lemma 3.1. *Let $\kappa > 0$ and K be a nonempty, closed and convex subset of complete $CAT(\kappa)$ space X with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$ and $T : K \to KC(X)$ be a generalized multivalued hybrid mapping with $\frac{2k_1(x)}{1-a_2(x)} < R$ for all $x \in K$ where $R = (\pi-2\varepsilon)tan(\varepsilon)$. Let $\{x_n\}$ be a sequence in K with $\lim_{n\to\infty} d(x_n,Tx_n) = 0$ and $\{d(x_n,p)\}$ converges for all $p \in F(T)$. Then $\omega_w(x_n) \subseteq F(T)$ and $\omega_w(x_n)$ include exactly one point.*

Proof. Let take $u \in \omega_w(x_n)$ then there exist subsequence $\{u_n\}$ of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$. Then, by Lemma 1.2 there exist subsequence $\{v_n\}$ of $\{u_n\}$ with $\Delta-\lim_{n\to\infty} v_n = v \in K$ and by Theorem 3.1 we have $v \in F(T)$ and by Lemma 1.3 we conclude that $u = v$, hence we get $\omega_w(x_n) \subseteq F(T)$. Let take subsequence $\{u_n\}$ of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. Because of $v \in \omega_w(x_n) \subseteq F(T)$, $\{d(x_n,u)\}$ converges, so by Lemma 1.3 we have $x = u$, this means that $\omega_w(x_n)$ include exactly one point. □

Theorem 3.2. *Let $\kappa > 0$ and K be a nonempty, closed and convex subset of complete $CAT(\kappa)$ space X with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$ and $T : K \to CC(X)$ be a generalized multivalued hybrid mapping with $\frac{2k_1(x)}{1-a_2(x)} < \frac{R}{2}$ for all $x \in K$ where $R = (\pi-2\varepsilon)\tan\varepsilon$), $F(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$. If $\{x_n\}$ is a sequence in K defined by iteration scheme 1.1 with $\liminf_{n\to\infty} \beta_n[(1-\beta_n)\frac{R}{2}-\frac{2k_2(x)}{1-a_3(x)}] > 0$ then $\lim_{n\to\infty} d(x_n,Tx_n) = 0$ and $\{d(x_n,p)\}$ converges for all $p \in F(T)$.*

Proof. Let $p \in F(T)$ then for any $x \in K$, we have that

$$H^2(Tx,Tp) \leq d^2(x,p)+\frac{k_2(p)}{1-a_3(p)}d^2(Tx,x)$$

since metric projection P_K is nonexpansive by Lemma 1.4, and $P_K(p) = \{x \in K : d(p,x) = d(p,K)\} = \{p\}$ we have

$$\begin{aligned}
d^2(y_n,p) &= d^2(P_K((1-\beta_n)x_n \oplus \beta_n v_n), P_K(p)) \\
&\leq d^2((1-\beta_n)x_n \oplus \beta_n v_n, p) \\
&\leq (1-\beta_n)d^2(x_n,p) + \beta_n d^2(v_n,p) \\
&\quad -\frac{R}{2}(1-\beta_n)\beta_n d^2(x_n,v_n) \\
&\leq (1-\beta_n)d^2(x_n,p) + \beta_n d^2(v_n,Tp) \\
&\quad -\frac{R}{2}(1-\beta_n)\beta_n d^2(x_n,Tx_n) \\
&\leq (1-\beta_n)d^2(x_n,p) + \beta_n H^2(Tx_n,Tp) \\
&\quad -\frac{R}{2}(1-\beta_n)\beta_n d^2(x_n,Tx_n) \\
&\leq (1-\beta_n)d^2(x_n,p) + \beta_n(d^2(x_n,p) \\
&\quad +\frac{k_2(p)}{1-a_3(p)}d^2(Tx_n,x_n)) - \frac{R}{2}(1-\beta_n)\beta_n d^2(x_n,Tx_n) \\
&\leq d^2(x_n,p) + \beta_n(\frac{k_2(p)}{1-a_3(p)} - \frac{R}{2}(1-\beta_n))d^2(Tx_n,x_n) \\
&\leq d^2(x_n,p)
\end{aligned}$$

and

$$\begin{aligned}
d^2(x_{n+1},p) &= d^2(P_K((1-\alpha_n)y_n \oplus \alpha_n u_n), P_K(p)) \\
&\leq d^2((1-\alpha_n)y_n \oplus \alpha_n u_n), p) \\
&\leq (1-\alpha_n)d^2(y_n,p) + \alpha_n d^2(u_n,p) \\
&\quad -\frac{R}{2}(1-\alpha_n)\alpha_n d^2(y_n,u_n) \\
&\leq (1-\alpha_n)d^2(y_n,p) + \alpha_n d^2(u_n,Tp) \\
&\quad -\frac{R}{2}(1-\alpha_n)\alpha_n d^2(y_n,Ty_n) \\
&\leq (1-\alpha_n)d^2(y_n,p) + \alpha_n H^2(Ty_n,Tp) \\
&\quad -\frac{R}{2}(1-\alpha_n)\alpha_n d^2(y_n,Ty_n) \\
&\leq (1-\alpha_n)d^2(y_n,p) + \alpha_n(d^2(y_n,p) \\
&\quad +\frac{k_2(p)}{1-a_3(p)}d^2(Ty_n,y_n)) - \frac{R}{2}(1-\alpha_n)\alpha_n d^2(y_n,Ty_n) \\
&\leq (1-\alpha_n)d^2(y_n,p) + \alpha_n(d^2(y_n,p)) \\
&\quad +\alpha_n(\frac{k_2(p)}{1-a_3(p)} - \frac{R}{2}(1-\alpha_n))d^2(Ty_n,y_n)) \\
&\leq (1-\alpha_n)d^2(y_n,p) + \alpha_n(d^2(y_n,p)) \\
&\quad +\alpha_n(\frac{k_2(p)}{1-a_3(p)} - \frac{R}{2}(1-\alpha_n))d^2(Ty_n,y_n) \\
&\leq d^2(y_n,p) + \alpha_n(\frac{k_2(p)}{1-a_3(p)} - \frac{R}{2}(1-\alpha_n))d^2(Ty_n,y_n) \\
&\leq d^2(y_n,p) \\
&\leq d^2(x_n,p).
\end{aligned}$$

Here we have $d^2(x_{n+1},p) \leq d^2(x_n,p)$ implies that $\lim_{n\to\infty} d(x_n,p)$ exists and since $d(x_{n+1},p) \leq d(y_n,p) \leq d(x_n,p)$ so we have $\lim_{n\to\infty}[d(x_n,p) - d(y_n,p)] = 0$. Since $\beta_n(\frac{k_2(p)}{1-a_3(p)} - \frac{R}{2}(1-\beta_n))d^2(Tx_n,x_n)) \leq d^2(x_n,p) - d^2(y_n,p)$, by assumption we have that $\lim_{n\to\infty} d^2(Tx_n,x_n) = 0$, thus $\lim_{n\to\infty} d(Tx_n,x_n) = 0$. □

Theorem 3.3. *Let $\kappa > 0$ and K be a nonempty, closed and convex subset of complete CAT(κ) space X with diam$(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$ and $T : K \to KC(X)$ be a generalized multivalued hybrid mapping with $\frac{2k_1(x)}{1-a_2(x)} < \frac{R}{2}$ for all $x \in K$ where $R = (\pi-2\varepsilon)\tan\varepsilon)$, $F(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$. If $\{x_n\}$ is a sequence in K defined by iteration scheme 1.1* with $\liminf_{n\to\infty}\beta_n[(1-\beta_n)\frac{R}{2} - \frac{2k_2(x)}{1-a_3(x)}] > 0$ *then $\{x_n\}$ have a $\Delta-$limit which in $F(T)$.*

Proof. Since $\lim_{n\to\infty} d(x_n,Tx_n) = 0$ and $\{d(x_n,p)\}$ converges for all $p \in F(T)$ from Theorem 3.2, then desired result follows from Lemma 3.1. □

Theorem 3.4. *Let $\kappa > 0$ and K be a nonempty, compact and convex subset of complete $CAT(\kappa)$ space X with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$ and $T : K \to CC(X)$ be a continuous generalized multivalued hybrid mapping with $T(p) = \{p\}$ for all $p \in F(T)$. If $\{x_n\}$ is a sequence in K defined by iteration scheme 1.1 with $\liminf_{n\to\infty} \beta_n[(1-\beta_n)\frac{R}{2} - \frac{2k_2(x)}{1-a_3(x)}] > 0$ then $\{x_n\}$ strongly converges to a point of $F(T)$.*

Proof. By Theorem 3.2, we have that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ and $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. Since K is compact there is a convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$, say $\lim_{i\to\infty} x_{n_i} = z$. Then we have

$$d(z,Tz) \leq d(z,x_{n_i}) + d(x_{n_i},Tx_{n_i}) + H(Tx_{n_i},Tz)$$

and taking limit on i, continuity of T implies that $z \in Tz$. □

4. Conclusion

In this paper we defined multivalued version of generalized hybrid mappings in $CAT(\kappa)$ spaces. This definition is contains definition of various multivalued mappings in $CAT(\kappa)$ spaces, $CAT(0)$ spaces and Hilbert spaces. We have showed that this new class have fixed point under different conditions. We also defined multivalued proximal version of Thianwan iteration procedure and proved that this iteration produce $\Delta-$convergent and strongly convergent sequence to fixed point of multivalued generalized hybrid mappings in non-self case with end point condition.

Acknowledgement

We thank the referee(s) for the time spent in reviewing our paper and making very helpful comments which improved the paper.

References

[1] F. Kohsaka and W. Takahashi, Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces, Arch. Math. (Basel), 91(2008), 166-177.
[2] W. Takahashi, Fixed point theorems for new nonlinear mappings in a Hilbert space, J.Nonlinear Convex Anal., 11 (2010), 79-88.
[3] K. Aoyama, S. Iemoto, F. Kohsaka and W. Takahashi, Fixed point and ergodic theorems for λ-hybrid mappings in Hilbert spaces, J. Nonlinear Convex Anal., 11 (2010), 335-343.
[4] K. Aoyama and F. Kohsaka, Fixed point theorem for α-nonexpansive mappings in Banach spaces, Nonlinear Anal., 74 (2011), 4387-4391.
[5] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer-Verlag, BerlinU, Heidelberg, 1999.
[6] S. Ohta, Convexities of metric spaces, Geom. Dedic. 125 (2007), 225-250
[7] Panyanak, Bancha, Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces, Computers & Mathematics with Applications 54.6 (2007), 872-877.
[8] Espínola, Rafa, and Aurora Fernández-León, $CAT(\kappa)$-spaces, weak convergence and fixed point, Journal of Mathematical Analysis and Applications, 353.1 (2009), 410-427.
[9] F. Gursoy and V. Karakaya, A Picard-S hybrid type iteration method for solving a differential equation with retarded argument , ArXiv preprint, (2014), arXiv:1403.2546.
[10] F. Gursoy, A Picard-S Iterative Method for Approximating Fixed Point of Weak-Contraction Mappings, Filomat, 30 (2016), 2829-2845.
[11] S. Dhompongsa, B. Panyanak, On 4-convergence theorems in $CAT(0)$ spaces, Comput. Math. Appl.56 (2008), 2572-2579.
[12] E. Hacioğlu and V. Karakaya, Existence and convergence for a new multivalued hybrid mapping in $CAT(\kappa)$ spaces, Carpathian Journal of Mathematics, 33(3) (2017), 319-326.
[13] R. A. Rashwan and S. M. Altwqi, On the convergence of SP-iterative scheme for three multivalued nonexpansive mappings in $CAT(\kappa)$ spaces, Palestine Journal of Mathematics, Vol. 4(1) (2015) , 73-83.