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On the Hermite-Hadamard-Fejér type integral inequality for s-convex function

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Abstract

In this paper, we extend some estimates of the right hand side of a Hermite- Hadamard-Fejér type inequality for functions whose first derivatives absolute values are s-convex.The results presented here would provide extensions of those given in earlier works.

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1. Introduction

The function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.
The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [6], [9]):

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f:I\subset\mathbb{R}\to\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$.
In [5], Dragomir and Agarwal proved the following results connected with the right part of (1.1).

Lemma 1. *Let $f:I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)dt. \tag{1.2}$$

Theorem 1. *Let $f:I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)}{8}\left(|f'(a)|+|f'(b)|\right). \tag{1.3}$$

Definition 1. *[1] Let s be a real numbers, $s\in(0,1]$. A function $f:[0,\infty)\to[0,\infty)$ is said to be $s-$convex (in the second sense), or that f belongs to the class K_s^2, if f*

$$f(\alpha x+(1-\alpha)y)\leq\alpha^s f(x)+(1-\alpha)^s f(y)$$

for all $x,y\in[0,\infty)$ and $\alpha\in[0,1]$.

In [4], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for $s-$convex functions in the second sense.

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Theorem 2. *[4] Suppose that $f:[0,\infty)\to[0,\infty)$ is an s-convex function in the second sense, where $s\in(0,1)$, and let $a,b\in[0,\infty)$, $a<b$. If $f\in L^1([a,b])$, then the following inequalities hold:*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int\limits_a^b f(x)dx\leq\frac{f(a)+f(b)}{s+1}. \tag{1.4}$$

The constant $k=\frac{1}{s+1}$ is the best possible in the second inequality in (1.4).
The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities (see, [8], [11]-[16], [19], [20]). In [7], Fejer gave a weighted generalization of the inequalities (1.1) as the following:

Theorem 3. *$f:[a,b]\to\mathbb{R}$, be a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\leq\frac{1}{b-a}\int_a^b f(x)w(x)dx\leq\frac{f(a)+f(b)}{2}\int_a^b w(x)dx \tag{1.5}$$

holds, where $w:[a,b]\to\mathbb{R}$ is nonnegative, integrable, and symmetric about $x=\frac{a+b}{2}$.

In [13], some inequalities of Hermite-Hadamard-Fejer type for differentiable convex mappings were proved using the following lemma.

Lemma 2. *Let $f:I^\circ\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$, and $w:[a,b]\to[0,\infty)$ be a differentiable mapping. If $f'\in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2}\int_a^b w(x)dx-\int_a^b f(x)w(x)dx=\frac{(b-a)^2}{2}\int_0^1 p(t)f'(ta+(1-t)b)dt \tag{1.6}$$

for each $t\in[0,1]$, where

$$p(t)=\int_t^1 w(as+(1-s)b)ds-\int_0^t w(as+(1-s)b)ds.$$

The main result in [13] is as follows:

Theorem 4. *Let $f:I^\circ\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$, and $w:[a,b]\to[0,\infty)$ be a differentiable mapping and symmetric to $\frac{a+b}{2}$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds: for $p>1$,*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(x)dx-\int_a^b f(x)w(x)dx\right|\leq\frac{b-a}{2}\left[\int_0^1(g(t))^p\,dt\right]^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}} \tag{1.7}$$

where $g(t)=\left|\int_{a+(b-a)t}^{b-(b-a)t}w(x)dx\right|$ for $t\in[0,1]$.

Definition 2. *Let $f\in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha>0$ with $a\geq0$ are defined by*

$$J_{a^+}^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\ \ x>a$$

and

$$J_{b^-}^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt,\ \ x<b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a^+}^0 f(x)=J_{b^-}^0 f(x)=f(x)$.

Meanwhile, Sarikaya et al.[10] presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha>0$.

Lemma 3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]=\frac{b-a}{2}\int_0^1\left[(1-t)^\alpha-t^\alpha\right]f'(ta+(1-t)b)\,dt. \tag{1.8}$$

It is remarkable that Sarikaya et al.[10] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\leq\frac{f(a)+f(b)}{2} \tag{1.9}$$

with $\alpha>0$.

For some recent results connected with fractional integral inequalities see [2], [3], [17], [18], [21], [22].
In this article, using functions whose derivatives absolute values are s-convex, we obtained new inequalities of Hermite-Hadamard-Fejer type and Hermite-Hadamard type involving fractional integrals. The results presented here would provide extensions of those given in earlier works.

2. Main Results

We will establish some new results connected with the right-hand side of (1.5) and (1.1) involving fractional integrals used the following Lemma. Now, we give the following new Lemma for our results (see, [12]):

Lemma 4. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^\circ$ with $a < b$ and let $w : [a,b] \to \mathbb{R}$. If $f', w \in L[a,b]$, then, for all $x \in [a,b]$, the following equality holds:*

$$\begin{aligned} \int_a^b \left(\int_a^t w(u)du\right)^{\alpha} f'(t)dt - \int_a^b \left(\int_t^b w(u)du\right)^{\alpha} f'(t)dt &= \left(\int_a^b w(u)du\right)^{\alpha} [f(a)+f(b)] \\ &\quad -\alpha \int_a^b \left(\int_a^t w(u)du\right)^{\alpha-1} w(t)f(t)dt - \alpha \int_a^b \left(\int_t^b w(u)du\right)^{\alpha-1} w(t)f(t)dt \end{aligned} \tag{2.1}$$

where $\alpha > 1$.

Now, by using the above lemma, we prove our main theorems:

Theorem 6. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^\circ$ with $a < b$ and let $w : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. If $|f'|$ is $s-$convex on $[a,b]$, then the following inequality holds:*

$$\left| \left(\int_a^b w(u)du\right)^{\alpha} [f(a)+f(b)] \quad -\alpha \int_a^b \left(\int_a^t w(u)du\right)^{\alpha-1} w(t)f(t)dt - \alpha \int_a^b \left(\int_t^b w(u)du\right)^{\alpha-1} w(t)f(t)dt \right| \tag{2.2}$$

$$\leq \quad \|w\|^{\alpha}_{[a,b],\infty} (b-a)^{\alpha+1} A(\alpha,s) \left(\left|f'(a)\right| + \left|f'(b)\right|\right) \tag{2.3}$$

where

$$A(\alpha,s) = \left[\frac{1}{\alpha+s+1}\left(1-\frac{1}{2^{\alpha+s}}\right) + B_{\frac{1}{2}}(s+1,\alpha+1) - B_{\frac{1}{2}}(\alpha+1,s+1)\right],$$

$\alpha > 0$, $\|w\|_{[a,b],\infty} = \sup_{t\in[a,b]} |w(t)|$ *and B_x is the incomplete beta function defined as follows*

$$B_x(m,n) = \int_0^x t^{m-1}(1-t)^{n-1}dt, \;\; m,n > 0, \; 0 < x \leq 1.$$

Proof. We take absolute value of (2.1), we find that

$$\begin{aligned} &\left| \left(\int_a^b w(u)du\right)^{\alpha} [f(a)+f(b)] - \alpha \int_a^b \left(\int_a^t w(u)du\right)^{\alpha-1} w(t)f(t)dt - \alpha \int_a^b \left(\int_t^b w(u)du\right)^{\alpha-1} w(t)f(t)dt \right| \\ \leq \quad & \int_a^b \left| \left(\int_a^t w(u)du\right)^{\alpha} - \left(\int_t^b w(u)du\right)^{\alpha} \right| \left|f'(t)\right| dt \\ \leq \quad & \|w\|^{\alpha}_{[a,b],\infty} \int_a^b \left|(t-a)^{\alpha} - (b-t)^{\alpha}\right| \left|f'(t)\right| dt \\ = \quad & \|w\|^{\alpha}_{[a,b],\infty} \left\{ \int_a^{\frac{a+b}{2}} \left[(b-t)^{\alpha} - (t-a)^{\alpha}\right] \left|f'(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b)\right| dt + \int_{\frac{a+b}{2}}^b \left[(t-a)^{\alpha} - (b-t)^{\alpha}\right] \left|f'(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b)\right| dt \right\}. \end{aligned}$$

Since $\left|f'\right|$ is s-convex on $[a,b]$, it follows that

$$\begin{aligned} &\left| \left(\int_a^b w(u)du\right)^{\alpha} [f(a)+f(b)] - \alpha \int_a^b \left(\int_a^t w(u)du\right)^{\alpha-1} w(t)f(t)dt - \alpha \int_a^b \left(\int_t^b w(u)du\right)^{\alpha-1} w(t)f(t)dt \right| \\ \leq \quad & \|w\|^{\alpha}_{[a,b],\infty} \left\{ \int_a^{\frac{a+b}{2}} \left[(b-t)^{\alpha} - (t-a)^{\alpha}\right] \left[\left(\frac{b-t}{b-a}\right)^s \left|f'(a)\right| + \left(\frac{t-a}{b-a}\right)^s \left|f'(b)\right|\right] dt \right. \\ & \left. + \int_{\frac{a+b}{2}}^b \left[(t-a)^{\alpha} - (b-t)^{\alpha}\right] \left[\left(\frac{b-t}{b-a}\right)^s \left|f'(a)\right| + \left(\frac{t-a}{b-a}\right)^s \left|f'(b)\right|\right] dt \right\} \end{aligned} \tag{2.4}$$

$$= \|w\|_{[a,b],\infty}^{\alpha}\left[\frac{\left|f'(a)\right|}{(b-a)^s}\int_a^{\frac{a+b}{2}}\left[(b-t)^{\alpha+s}-(t-a)^{\alpha}(b-t)^{s}\right]dt+\frac{\left|f'(b)\right|}{(b-a)^s}\int_a^{\frac{a+b}{2}}\left[(b-t)^{\alpha}(t-a)^{s}-(t-a)^{\alpha+s}\right]dt\right.$$
$$\left.+\frac{\left|f'(a)\right|}{(b-a)^s}\int_{\frac{a+b}{2}}^{b}\left[(t-a)^{\alpha}(b-t)^{s}-(b-t)^{\alpha+s}\right]dt+\frac{\left|f'(b)\right|}{(b-a)^s}\int_{\frac{a+b}{2}}^{b}\left[(t-a)^{\alpha+s}-(b-t)^{\alpha}(t-a)^{s}\right]dt\right].$$

By simple computation,

$$\int_a^{\frac{a+b}{2}}\left[(b-t)^{\alpha+s}-(t-a)^{\alpha}(b-t)^{s}\right]dt=\frac{(b-a)^{\alpha+s+1}}{\alpha+s+1}-\frac{(b-a)^{\alpha+s+1}}{2^{\alpha+s+1}(\alpha+s+1)}-(b-a)^{\alpha+s+1}B_{\frac{1}{2}}(\alpha+1,s+1),\tag{2.5}$$

$$\int_a^{\frac{a+b}{2}}\left[(b-t)^{\alpha}(t-a)^{s}-(t-a)^{\alpha+s}\right]dt=(b-a)^{\alpha+s+1}B_{\frac{1}{2}}(s+1,\alpha+1)-\frac{(b-a)^{\alpha+s+1}}{2^{\alpha+s+1}(\alpha+s+1)},\tag{2.6}$$

$$\int_{\frac{a+b}{2}}^{b}\left[(t-a)^{\alpha}(b-t)^{s}-(b-t)^{\alpha+s}\right]dt=(b-a)^{\alpha+s+1}B_{\frac{1}{2}}(s+1,\alpha+1)-\frac{(b-a)^{\alpha+s+1}}{2^{\alpha+s+1}(\alpha+s+1)},\tag{2.7}$$

$$\int_{\frac{a+b}{2}}^{b}\left[(t-a)^{\alpha+s}-(b-t)^{\alpha}(t-a)^{s}\right]dt=\frac{(b-a)^{\alpha+s+1}}{\alpha+s+1}-\frac{(b-a)^{\alpha+s+1}}{2^{\alpha+s+1}(\alpha+s+1)}-(b-a)^{\alpha+s+1}B_{\frac{1}{2}}(\alpha+1,s+1).\tag{2.8}$$

Writing (2.5)-(2.8) in (2.4), we obtain (2.2) which the proof of theorem is completed. □

Corollary 1. *Under the same assumptions of Theorem 6 with $w(u)=1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(x)\right]\right|\le\frac{(b-a)}{2}A(\alpha,s)\left(\left|f'(a)\right|+\left|f'(b)\right|\right).\tag{2.9}$$

Remark 1. *If we take $s=1$ in (2.9), the inequality (2.9) reduces to*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(x)\right]\right|\le\frac{(b-a)}{2(\alpha+2)}\left[\left(1-\frac{1}{2^{\alpha+1}}\right)+\frac{1}{(\alpha+1)}\left(1-\frac{\alpha+3}{2^{\alpha+1}}\right)\right]\left(\left|f'(a)\right|+\left|f'(b)\right|\right).\tag{2.10}$$

Remark 2. *If we take $\alpha=1$ in (2.10), the inequality (2.10) reduces to (1.3).*

Corollary 2. *Under the same assumptions of Theorem 6 with $s=1$, then the following inequality holds:*

$$\left|\left(\int_a^b w(u)du\right)^{\alpha}[f(a)+f(b)]-\alpha\int_a^b\left(\int_a^t w(u)du\right)^{\alpha-1}w(t)f(t)dt-\alpha\int_a^b\left(\int_t^b w(u)du\right)^{\alpha-1}w(t)f(t)dt\right|$$
$$\le \|w\|_{[a,b],\infty}^{\alpha}\frac{(b-a)^{\alpha+1}}{(\alpha+2)}\left[\left(1-\frac{1}{2^{\alpha+1}}\right)+\frac{1}{(\alpha+1)}\left(1-\frac{\alpha+3}{2^{\alpha+1}}\right)\right]\left(\left|f'(a)\right|+\left|f'(b)\right|\right).$$

Theorem 7. *Let $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$ and let $w:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$. If $|f'|^q$ is s-convex on $[a,b]$, $q>1$, then the following inequality holds:*

$$\left|\left(\int_a^b w(u)du\right)^{\alpha}[f(a)+f(b)]-\alpha\int_a^b\left(\int_a^t w(u)du\right)^{\alpha-1}w(t)f(t)dt-\alpha\int_a^b\left(\int_t^b w(u)du\right)^{\alpha-1}w(t)f(t)dt\right|\tag{2.11}$$
$$\le \|w\|_{[a,b],\infty}^{\alpha}\frac{2^{\frac{1}{p}}(b-a)^{\alpha+1}}{(\alpha p+1)^{\frac{1}{p}}}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{s+1}\right)^{\frac{1}{q}}$$

where $\alpha>0$, $\frac{1}{p}+\frac{1}{q}=1$, and $\|w\|_{[a,b],\infty}=\sup_{t\in[a,b]}|w(t)|$.

Proof. We take absolute value of (2.1). Using Holder's inequality, we find that

$$\begin{aligned}
&\left|\left(\int_a^b w(u)du\right)^{\alpha}[f(a)+f(b)]-\alpha\int_a^b\left(\int_a^t w(u)du\right)^{\alpha-1}w(t)f(t)dt-\alpha\int_a^b\left(\int_t^b w(u)du\right)^{\alpha-1}w(t)f(t)dt\right|\\
\leq\ &\left(\int_a^b\left|\left(\int_a^t w(u)du\right)^{\alpha}-\left(\int_t^b w(u)du\right)^{\alpha}\right|^p dt\right)^{\frac{1}{p}}\left(\int_a^b\left|f'(t)\right|^q dt\right)^{\frac{1}{q}}\\
\leq\ &\|w\|_{[a,b],\infty}^{\alpha}\left(\int_a^b\left|(t-a)^{\alpha}-(b-t)^{\alpha}\right|^p dt\right)^{\frac{1}{p}}\left(\int_a^b\left|f'(t)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

By simple computation, since $(M-N)^q\leq M^q-N^q$ for any $M\geq N\geq 0$ and $q\geq 1$, then

$$\left|(t-a)^{\alpha}-(b-t)^{\alpha}\right|^p\leq\begin{cases}(b-t)^{p\alpha}-(t-a)^{p\alpha}, & \text{for } t\in\left[a,\frac{a+b}{2}\right]\\[2ex] (t-a)^{p\alpha}-(b-t)^{p\alpha}, & \text{for } t\in\left[\frac{a+b}{2},b\right].\end{cases}$$

Hence, it follows that

$$\int_a^b\left|(t-a)^{\alpha}-(b-t)^{\alpha}\right|^p dt\leq\int_a^{\frac{a+b}{2}}\left[(b-t)^{p\alpha}-(t-a)^{p\alpha}\right]dt+\int_{\frac{a+b}{2}}^b\left[(t-a)^{p\alpha}-(b-t)^{p\alpha}\right]dt=\frac{2(b-a)^{\alpha p+1}}{(\alpha p+1)}\left(1-\frac{1}{2^{\alpha p}}\right).$$

Since $\left|f'(t)\right|^q$ is $s-$convex on $[a,b]$

$$\left|f'\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|^q\leq\left(\frac{b-t}{b-a}\right)^s\left|f'(a)\right|^q+\left(\frac{t-a}{b-a}\right)^s\left|f'(b)\right|^q. \tag{2.12}$$

From (2.12), it follows that

$$\begin{aligned}
&\left|\left(\int_a^b w(u)du\right)^{\alpha}[f(a)+f(b)]-\alpha\int_a^b\left(\int_a^t w(u)du\right)^{\alpha-1}w(t)f(t)dt-\alpha\int_a^b\left(\int_t^b w(u)du\right)^{\alpha-1}w(t)f(t)dt\right|\\
\leq\ &\|w\|_{[a,b],\infty}^{\alpha}\frac{2^{\frac{1}{p}}(b-a)^{\alpha+\frac{1}{p}}}{(\alpha p+1)^{\frac{1}{p}}}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left(\int_a^b\left[\left(\frac{b-t}{b-a}\right)^s\left|f'(a)\right|^q+\left(\frac{t-a}{b-a}\right)^s\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\\
=\ &\|w\|_{[a,b],\infty}^{\alpha}\frac{(b-a)^{\alpha+\frac{1}{p}}}{(\alpha p+1)^{\frac{1}{p}}}\left(2-\frac{1}{2^{\alpha p-1}}\right)^{\frac{1}{p}}\left(\left(\frac{\left|f'(a)\right|^q}{(b-a)^s}\right)\int_a^b(b-t)^s dt+\left(\frac{\left|f'(b)\right|^q}{(b-a)^s}\right)\int_a^b(t-a)^s dt\right)^{\frac{1}{q}}\\
=\ &\|w\|_{[a,b],\infty}^{\alpha}\frac{(b-a)^{\alpha+1}}{(\alpha p+1)^{\frac{1}{p}}}\left(2-\frac{1}{2^{\alpha p-1}}\right)^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}
\end{aligned}$$

which this completes the proof. □

Corollary 3. *Under the same assumptions of Theorem 7 with $w(u)=1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(x)\right]\right|\leq\frac{(b-a)}{2(\alpha p+1)^{\frac{1}{p}}}\left(2-\frac{1}{2^{\alpha p-1}}\right)^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}. \tag{2.13}$$

Corollary 4. *Let the conditions of Theorem 7 hold. If we take $\alpha=1$ in (2.11), then the following inequality holds:*

$$\left|\left(\int_a^b w(u)du\right)\frac{f(a)+f(b)}{2}-\int_a^b w(t)f(t)dt\right|\leq\|w\|_{[a,b],\infty}\frac{(b-a)^2}{2(p+1)^{\frac{1}{p}}}\left(2-\frac{1}{2^{p-1}}\right)^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}. \tag{2.14}$$

Remark 3. *If we take $\alpha=s=1$ in (2.13) or $w(u)=s=1$ in (2.14), we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left(2-\frac{1}{2^{p-1}}\right)^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}.$$

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