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Research Article

Geometric Brownian Motion Based on Stochastic Differential Equation Modeling Considering the Change Point Estimation for the Fluctuation of the Turkish Lira Against the US Dollar

Sevda Özdemir Çalıkuşu[1*], Fevzi Erdoğan[2] and Nihal İnce[3]

[1] *Department of Accounting and Tax, Ozalp Vocational School, Van Yuzuncu Yil University, Van, Türkiye*
[2] *Department of Econometrics , Faculty of Economics and Administrative Sciences, Van Yuzuncu Yil University, Van, Türkiye*
[3] *Department of Statistics, Faculty of Science, Eskisehir Technical University, Eskisehir, Türkiye*
[*] *Corresponding author*

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Abstract

In this study, the data showing the fluctuation of the Turkish Lira (TL) against the US Dollar (USD) between 19.06.2017 and 19.06.2022 were examined with Geometric Brownian Motion Stochastic Differential Equation Modeling (GBM SDEM). The study aims to get the GBM stochastic differential equation that best fits USD/TL data by considering the change point estimation (CP). Considering CP when working with abruptly changing datasets has a positive effect on the performance of the constructed model. In addition, there may be more than one CP in the dataset, and as the number of CP increases, more suitable models can be obtained for the dataset. The results are supported by graphs that show the proposed SDE model fits the dataset.

1. Introduction

Ordinary differential equations (ODEs) do not account for stochastic influences when modeling dynamic systems. Differential equations become "stochastic differential equations (SDEs)" when random components are added to them, and the stochastic part is referred as "noise". SDEs make it possible to connect probability theory with the older and more complex domains of ordinary and partial differential equations [1]. As a result, stochastic calculus uses concepts from probability theory extensively. An SDE is a differential equation in which one or more elements are stochastic processes, and the solution is a stochastic process as well [2].
Let $\{X_t, t \geq 0\}$ be a one-dimensional diffusion process defined by the Itô SDE

$$dX_t = f(t, X_t, \theta)\, dt + g(t, X_t, \theta)\, dW_t \tag{1.1}$$

where W_t is a Brownian Motion (BM). For a solution of (1.1) to exist, in addition to the initial condition X_0, it is anticipated that sufficient regularity conditions are valid for the drift function $f(\cdot)$ and the diffusion coefficient $g(\cdot)$. The functions $f(\cdot)$ and $g(\cdot)$ may or may not depend on t or a statistical parameter $\theta \in \theta \subset \mathbb{R}^d, d \geq 1$ [3].
In modeling the financial market, especially the stock market, the BM is often used to describe the movement of time series variables, and the movement of asset prices in corporate finance plays a vital role in constructing a statistical model [4, 5].
Particularly, one of the most significant concepts in constructing a financial model is the Geometric Brownian Motion (GBM), a special case of the BM process [4]. Considering the definitions above the GBM has the following SDE:

$$dX_t = \theta_1 X_t dt + \theta_2 X_t dW_t, \quad X_0 = x_0 \tag{1.2}$$

and $\theta_2 > 0$. Since solving (1.2) with the initial condition $X_0 = x_0$, where X_t, by applying Itô's lemma the exact solution of (1.2) yields

$$X_t = x_0 e^{\left(\theta_1 - \frac{1}{2}\theta_2^2\right)t + \theta_2 W_t}$$

Email addresses: **sevdaozdemir@yyu.edu.tr, ferdogan@yyu.edu.tr, nihalince23@gmail.com**
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Here, θ_1 denotes the return that the stock would earn over a brief period and is known as the percentage drift. θ_2 denotes the percentage volatility, which is a stochastic process [3]. Therefore, when taking into account a BM trajectory that represents this differential equation, the right-hand side term $\theta_1 X_t dt$ checks the "trend" of this trajectory whereas the term $\theta_2 X_t dW_t$ checks the "random noise" effect in the trajectory [4,5]. GBM is a widely used mathematical model for predicting the future price of financial assets, including stocks [6]. So, to help investors in forecasting share prices for a brief period of investment time, a mathematical model as basic as GBM is necessary [7].In this work, instead of models such as Ornstein-Uhlenbeck SDE, Cox-Ingersoll-Ross SDE, and Vasicek SDE used in the literature, the GBM SDE model was preferred. Because, unlike other models, transition probabilities and the exact solution of the GBM SDE can be calculated more easily, so that parameter estimation of the established model can be done with the MLE method [3]. While mean-reverting or stochastic volatility models such as OU, CIR, or Vasicek SDE can capture more complex dynamics, they typically require more intricate calibration procedures and do not offer the same closed-form likelihood expressions [3,8,9]. This is important for shorter and easier computer coding.

Furthermore, this study draws methodological inspiration from the YUIMA Project [3], a well-established R framework developed by Iacus, Yoshida, and collaborators, offering validated tools for SDE modeling, including change point analysis [10,11]. YUIMA provides a reliable foundation for implementing the estimation techniques used in this paper. GBM remains a cornerstone model in modern quantitative finance, supported by extensive literature and robust integration into modeling frameworks such as the YUIMA Project, which we also utilize in this work. Its mathematical simplicity and empirical adaptability further justify its selection in modeling volatile exchange rate contexts.

In recent years, the USD/TRY exchange rate has experienced abrupt fluctuations due to economic and political instabilities. Modeling such stochastic behaviors necessitates techniques that can accommodate sudden structural changes, motivating the use of GBM SDE models with CP.In this study, we consider a change point problem for the fluctuation of a process solution to an SDE when the observations are collected at separate times, where SDE is the equation covering the GBM process.Building on this framework, the study incorporates recent advances in change point detection and estimation. The statistical study of change point detection and estimation has garnered a lot of attention recently. In response to the need for updated and relevant literature on SDE and change point analysis in financial contexts, this study incorporates several recent academic contributions published between 2020 and 2025.Notably, Sun et al. [12] propose a novel interval-based approach for detecting structural shifts and estimating change-points in time series, particularly suited for high-frequency financial data. Their interdisciplinary method integrates elements of signal processing and econometrics, offering strong theoretical support for the CP framework employed in this study. Malinowski [13] focuses on the mathematical well-posedness of symmetric set-valued SDEs under relaxed Lipschitz conditions. This contribution provides foundational backing to the choice of GBM as a tractable and robust model, ensuring analytical consistency in stochastic environments.These works highlight the practical relevance of CP methodologies in volatile markets and validate their use in currency exchange rate modeling, such as the USD/TRY dynamics examined in this paper. Together, these works reinforce the theoretical grounding and methodological relevance of the current study, aligning it with recent advancements in the intersection of stochastic modeling and empirical finance.With this theoretical foundation and updated literature in place, the broader relevance of change point issues across disciplines becomes evident. Basically, change point issues arise in a variety of disciplines, including literature, geology, health, finance, and even everyday life. A change point is a location or a point in time when observations follow one distribution up to that point and a different distribution moving forward in terms of statistics [14].

Chen et al. [15] formed a theory for counting the change points in conditional variance (volatility) of a nonparametric model where the regression and conditional variance functions are unknown. Lee et al. [16] assessed the problem of testing for a parameter change using the CUSUM test based on one-step estimators in diffusion processes. Gregorio and Iacus [17] dealt with a change-point problem for the volatility of a diffusion process observed at discrete times. Iacus and Yoshida [11] dealt with a change-point problem for the volatility of a process solution to a SDE in which observations were gathered at discrete times. They determined the moment the volatility regime changed with the MLE approach that using the approximated likelihood, retroactively. They took into account the CP issue for the drift for continuous time observations of diffusion processes. Oliveira et al. [18] aimed to accomplish three purposes in their article. These include, firstly, identifying change points (regime-switching) in the Brazilian energy spot price time series; secondly, selecting the most appropriate SDE to model Brazilian energy spot prices; and finally, pricing five distinct types of options used to manage electricity price risk in Brazil. In this way, they modeled the Brazilian energy spot prices with the SDE model they obtained by considering the change point. Tonaki et al. [19] treated the change point problem in ergodic diffusion processes from discrete observations. They proposed a technique to predict the change point of the parameter in two cases: one in which the diffusion parameter changes, and the other in which the diffusion parameter does not change but the drift parameter does. They also presented rates of convergence and distributional results of the change point estimators.

Emerging market economies such as Türkiye are those that have not yet completed their capital accumulation. Such economies have structural current account deficits. This situation may cause the upward movements in exchange rates to turn into a systematic risk for the economies of such countries. Therefore, modeling such sharp movements in exchange rates is important for the implementation of effective monetary and financial policies in such countries' economies. So, considering the studies mentioned above, in this study, the data showing the fluctuation of the Turkish Lira (TL) against the US Dollar (USD) between 19.06.2017 and 19.06.2022 were examined by GBM SDE modeling. As a matter of fact, the application of stochastic models to the USD/TL exchange rate in the national literature [20–22] is an interesting subject. However, in this study, unlike other studies, sudden changes in the USD/TL exchange rate were modeled with SDE, considering the CP. For this reason, the study aims to get the GBM SDE that best fits USD/TL data by considering the CP. In this research, Iacus and Yoshida's method, which they conducted in 2012 for CP, was applied [11]. For this purpose, the study was conducted in four steps. In the first step, the GBM SDE model was established without taking into account the CP using the available dataset. Then, the change point estimators of the data were calculated for GBM SDE. Here, only a single CP was accounted for first, and then double CPs were accounted for according to the CP values. The dataset was divided into two and then three regions, respectively, and GBM SDE was re-established for the new regions. In other words, three different GBM SDEs were obtained. In the second step, the parameters of each GBM SDE model were estimated using the MLE method based on the data. In the third step, by applying the Euler-Maruyama (EM) approximation method, the approximate trajectories of the stochastic process, which is the solution of the mentioned SDEs, were acquired for each GBM SDE. Two solutions obtained by taking the mean and median of these trajectories are presented as optimum solutions. In the fourth step, RMSE and MAPE values were calculated for each GBM SDE model. The results were supported by graphs that the proposed SDE model fitted the dataset.

The remainder of this paper is organized as follows: Section 2 describes the material and methodology, including the theoretical background and estimation techniques. Section 3 presents the empirical findings and model results. Section 4 concludes the study with key insights and implications.

2. Material Method

It is advantageous for the researcher to be able to approximate the solution because it is typically challenging to acquire the exact solution of an SDE. The Euler-Maruyama approximation method (EM), which is based on discretizing the SDE with provided (1.1) on a regular grid of times $t_i = i\cdot\Delta, i = 0,\ldots,n$ where Δ is a given time lag, is a straightforward numerical method that has been utilized in the literature for this purpose. When applied to (1.1), the EM scheme has the form

$$\mathrm{X_{t_{i+1}} = X_{t_i} + f(t_i, X_{t_i}, \theta)\Delta + g(t_i, X_{t_i}, \theta)\Delta W_i, i = 1,\ldots,n-1}$$

where $\mathrm{\Delta W_i = W_{t_{i+1}} - W_{t_i}} \sim \sqrt{\Delta}\mathrm{N}(0,1)$ is the sequence of independent increments of the Wiener process [3]. By using an EM technique, the extreme values of approximation trajectories $\mathrm{X_{t_{1+1}}}$ are attained in this study.
When a physical problem is aimed to be modeled with SDE, first of all, it is decided which SDE model to use. Then, before proceeding to solve the proposed SDE, the θ vector of the parameters in equation (1.1) needs to be estimated with the help of the real $\mathrm{N}+1$ data points. For this, there are several available methods in the literature such as the maximum likelihood estimation method (MLE), the quasi-maximum likelihood estimation (QMLE) method, the pseudo-maximum likelihood estimation method, the generalized moments method, the Bayesian estimation method, the nonparametric estimation method and so on [3,9]. In this study, the MLE method has been used to estimate the parameters of the proposed SDE models for the available data, and the QMLE method was used to estimate the change point. For this reason, these two estimation methods were discussed in this section from the real data.
Consider the (1.1) Itô SDE. Given that the observation data of $\mathrm{X(t)}$, which is the solution of equation (1.1) is $\mathrm{x_0, x_1, x_2, \ldots, x_N}$. Given the $\mathrm{N}+1$ observation data in question, the issue at hand is how to estimate the θ vector of the parameters in equation (1.1). For this reason, for a given vector θ, starting from (t_{k-1}, x_{k-1}) the transition probability density function of the point (t_k, x_k) is assumed as $p(t_k, x_k \mid t_{k-1}, x_{k-1};\theta)$. Suppose the initial probability density is $p_0(x_0 \mid \theta)$. Then, the joint probability density function in the MLE of the vector θ is expressed as

$$\mathrm{D}(\theta) = \mathrm{p_0}(\mathrm{x_0} \mid \theta)\prod_{\mathrm{k=1}}^{\mathrm{N}} \mathrm{p}(\mathrm{t_k}, \mathrm{x_k} \mid \mathrm{t_{k-1}}, \mathrm{x_{k-1}};\theta). \tag{2.1}$$

In order to estimate the vector θ, it is necessary to maximize the equation (2.1) for θ or avoid small numbers in computer calculations minimize the function $L(\theta) = \ln(D(\theta))$ which is the logarithmic expression of the equation (2.1). In other words, it is necessary to minimize the equation (2.2)

$$\mathrm{L}(\theta) = -\ln(\mathrm{p_0}(\mathrm{x_0} \mid \theta)) - \sum_{\mathrm{k=1}}^{\mathrm{N}} \ln(\mathrm{p}(\mathrm{t_k}, \mathrm{x_k} \mid \mathrm{t_{k-1}}, \mathrm{x_{k-1}};\theta)). \tag{2.2}$$

Let θ^* be the value θ which minimizes $\mathrm{L}(\theta)$ [9]. It is quite difficult to find the optimal θ^* value since equation (2.2) contains a transition probability density function and these transition densities are not generally known for diffusion processes. To eliminate this problem, these transition probabilities must be estimated. For this, when the Euler approach is applied to (2.2), if it is taken into account that the observed values of the X_t process at the time $t = t_{k-1}$ are $X_{t_{k-1}} = x_{k-1}$,

$$\mathrm{X_{t_k} \approx x_{k-1} + f(t_{k-1}, x_{k-1};\theta)\Delta t + g(t_{k-1}, x_{k-1};\theta)\sqrt{\Delta t}\eta_k}$$

is reached, where $\eta_\mathrm{k} \sim \mathrm{N}(0,1)$. With $\mathrm{X_{t_i}} \sim \mathrm{N}(\mu_\mathrm{i}, \sigma_\mathrm{i}^2)$, the transition probability density function is obtained as

$$\mathrm{p}(\mathrm{t_k}, \mathrm{x_k} \mid \mathrm{t_{k-1}}, \mathrm{x_{k-1}};\theta) \approx \frac{1}{\sqrt{2\pi\sigma_\mathrm{k}^2}}\exp\left(\frac{-(\mathrm{x_k}-\mu_\mathrm{k})^2}{2\sigma_\mathrm{k}^2}\right)$$

where $\mu_\mathrm{k} = \mathrm{x_{k-1}} + \mathrm{f}(\mathrm{t_{k-1}}, \mathrm{x_{k-1}};\theta)\Delta\mathrm{t}$ and $\sigma_\mathrm{k} = \mathrm{g}(\mathrm{t_{k-1}}, \mathrm{x_{k-1}};\theta)\sqrt{\Delta\mathrm{t}}$. In this procedure, it is assumed that the transition probability densities approximate the normal distribution using the Euler formula. From here, the θ^* estimator is obtained by minimizing the equation (2.2). In other words, θ^* provides the equality (2.3) [23,24].

$$\theta^* = \underset{\theta}{\mathrm{argmin}}\mathrm{L}(\theta). \tag{2.3}$$

However, it is mostly unknown what the transition probabilities are for the diffusion term in an SDE model. Therefore, developing an estimation technique for equation parameters that does not require previous knowledge of the transition probabilities of the diffusion term is crucial. So, the QMLE method introduced below is designed to accomplish this purpose [25].
Take into account a multidimensional diffusion process

$$dX_t = f(t, X_t;\theta_2)\,dt + g(t, X_t;\theta_1)\,dW_t, \quad X_0 = x_0 \tag{2.4}$$

where W is an r-dimensional standard Wiener process independent of the initial variable x_0. Furthermore, $\theta_1 \in \theta_1 \subset \mathbb{R}^p$, $\theta_2 \in \theta_2 \subset \mathbb{R}^\mathrm{q}$, $\mathrm{f}: \mathbb{R}^\mathrm{d} \times \theta_2 \to \mathbb{R}^\mathrm{d}$ and $\mathrm{g}: \mathbb{R}^\mathrm{d} \times \theta_1 \to \mathbb{R}^\mathrm{d} \otimes \mathbb{R}^\mathrm{r}$. As shown below, given the optimal convergence rates of the estimators for these parameters, the entitling of θ_2 and θ_1 is natural in theory. Given sampled data $X_n = (X_{t_1})_{i=0,\ldots,n}$ with $\mathrm{t_i = i\Delta_n, \Delta_n} \to 0$ as $\mathrm{n} \to \infty$, for multidimensional diffusions, the QMLE estimator utilizes the following approach of the genuine log-likelihood

$$\ln(X_n;\theta) = -\frac{1}{2}\sum_{i=1}^{n}\left\{\log\det(S_{i-1}(\theta_1)) + \frac{1}{\Delta_n}S_{i-1}^{-1}(\theta_1)\left[(\Delta X_i - \Delta_n f_{i-1}(\theta_2))^{\otimes 2}\right]\right\}$$

where $\theta=(\theta_1,\theta_2), A^{-1}$ the inverse of $A, S=g^{\otimes 2}, A^{\otimes 2}=AA^T, A[B]=\mathrm{tr}(AB)$ and $\Delta X_i=X_{t_{i-1}}-X_{t_i}, S_i(\theta_1)=\mathrm{S}(\theta_1;\mathrm{X_{t_i}}), \mathrm{f_i}(\theta_2)=\mathrm{f}(\mathrm{X_{t_i}};\theta_2)$. In this situation, the QMLE of θ is an estimator that provides

$$\hat{\theta}=\underset{\theta}{\operatorname{argmax}} \ln(\mathrm{X_n};\theta)$$

exactly or approximately [3].

When considering the changes in the markets it should be paid attention to the change point analysis. This includes determining the date when stochastic model parameters vary owing to exogenous factors. In this environment, the most significant parameter is the changes in the volatility process in (2.4), i.e., the spread maturity, as in the financial markets in general. In fact, the term drift is typically treated as an ambiguous or offensive term in the statistical model. However, it is probable to estimate consistently if $T\to\infty$ [10, 18, 26]. Take into account a multidimensional SDE of the form

$$dY_t=f(X_t)\,dt+g(X_t,\theta)\,dW_t,\quad t\in[0,T]$$

where $\theta\in\theta\subset\mathbb{R}^p, f$ and X_t are multidimensional processes, $g:\mathbb{R}^d\times\theta\to\mathbb{R}^d\otimes\mathbb{R}^r$ is the diffusion coefficient (volatility) matrix and $\mathrm{W_t}$ is an r-dimensional Wiener process [3]. Let $\tau^*\in[0,\ \mathrm{T}]$ be the change point in question is not known. Now, it will be determined retrospectively whether, and if so, when a change in the value of the parameter θ has happened, and the parameter θ will be consistently estimated before and after the change point. The following equation formalizes the change point problem for the volatility:

$$Y_t=\begin{cases} Y_0+\int_0^t f_s ds+\int_0^t g\left(X_s,\theta_0^*\right)dW_s & \text{for } t\in[0,\tau^*) \\ Y_{\tau^*}+\int_{\tau^*}^t f_s ds+\int_{\tau^*}^t g\left(X_s,\theta_1^*\right)dW_s & \text{for } t\in[\tau^*,T]. \end{cases}$$

Unknown instant change point τ^*, along with θ_0^* and θ_1^*, must be inferred from the observations drawn along the trajectory of (X, Y) [3]. The next expression by Iacus and Yoshida implements how the QMLE technique is used in this work [11]. Let $\Delta_\mathrm{i}\mathrm{Y}=\mathrm{Y_{t_i}}-\mathrm{Y_{t_{i-1}}}$ and define

$$\Phi_\mathrm{n}(\mathrm{t};\theta_0,\theta_1)=\sum_{\mathrm{i=1}}^{\lfloor \mathrm{nt/T}\rfloor}\mathrm{G_i}(\theta_0)+\sum_{\mathrm{i=[nt/T]+1}}^{\mathrm{n}}\mathrm{G_i}(\theta_1)$$

with

$$\mathrm{G_i}(\theta)=\log\det\mathrm{S}\left(\mathrm{X_{t_{i-1}}},\theta\right)+\Delta_\mathrm{n}^{-1}(\Delta_\mathrm{i}\mathrm{Y})'\,\mathrm{S}\left(\mathrm{X_{t_{i-1}}},\theta\right)^{-1}(\Delta_\mathrm{i}\mathrm{Y})$$

and $S=g^{\otimes 2}$. Given that an estimator $\hat{\theta}_k$ is available for each $\theta_k, k=0,1$. The definition of $\hat{\theta}_k$ is the same as $\hat{\theta}_k=\theta_k^*$, if θ_k^* are known. The change point estimator of τ^* is [3]

$$\hat{\tau}=\arg\min_{t\in[0,T]}\Phi_n\left(t;\hat{\theta}_0,\hat{\theta}_1\right)$$

Choosing the best model among the proposed SDE models for the available data is a result that researchers want to achieve. There are various criteria for this in statistical sciences. Each criterion has its advantages and disadvantages. For this reason, the two most used criteria in the literature are included in this study. These are Mean Absolute Percent Error (MAPE) and Root Mean Square Error (RMSE). For different models or at different times, the MAPE is used to gauge how accurate an estimation is and is calculated with the following formula

$$\text{MAPE }=\frac{1}{n}\left(\sum_{i=1}^{n}\frac{|X_i-Y_i|}{X_i}\right)\cdot 100$$

where n is the number of observations, X_i is the observed values, and Y_i is the predicted values from the model [27]. In numerous statistical research, the RMSE has been employed as a common metric to assess the performance of models. The RMSE is calculated with the following equation

$$\text{RMSE }=\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-Y_i)^2}$$

where n is the number of observations, $\mathrm{X_i}$ is the observed values, and $\mathrm{Y_i}$ is the predicted values from the model like the MAPE [27].

All stages of model implementation, including the simulation of GBM SDE trajectories, parameter estimation via MLE and QMLE, and change point detection, were carried out in RStudio using version 4.0.5 and the validated functions provided in the 'yuima' package [3, 9].

3. The Research Findings and Discussion

In this study, the data showing the fluctuation of the TL against the USD between 19.06.2017 and 19.06.2022 (see Figure 3.1) were examined by GBM SDEM. This specific time window was selected to capture significant exchange rate instability and potential regime shifts, especially during the Turkish currency crises of 2018 and 2021. The volatility pattern observed in this period strongly motivated the application of change point detection methods within an SDE framework. In contrast, the post-2022 period is characterized by high but stable volatility, which is generally considered suitable for modeling with standard SDEs without explicitly accounting for structural breaks. This period also includes the introduction of financial instruments such as the foreign exchange–linked deposit protection scheme ('KKM') and monetary policy adjustments made amid ongoing macroeconomic pressures. These dynamics influenced exchange rate movements and investor behavior, further justifying the use of structural break detection.The historical exchange rate data used in this study were retrieved from Yahoo Finance [28]. Daily closing prices were used to ensure consistency and alignment with standard financial modeling practices.

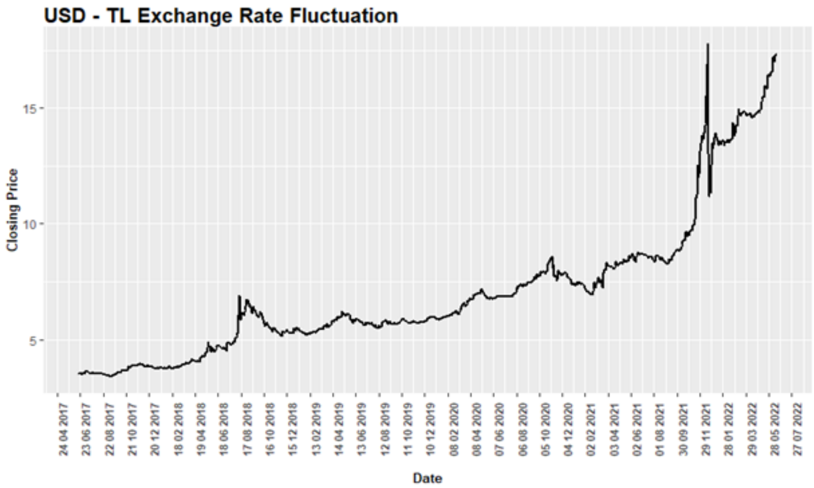

Figure 3.1: The exchange rate fluctuation of USD/TL between 19.06.2017 and 19.06.2022

The study aims to get the GBM SDE that best fits USD/TL data by considering the CP. Before proceeding to the modeling, change point estimators for the GBM SDE of the mentioned data were calculated. The purpose of calculating the change point is to identify the points where the data suddenly changes and find the GBM SDE that best models the data by considering these points. Here, only a single change point was estimated first, and then two change points were estimated. Accordingly, the first CP is $\hat{\tau}_1 = 14.11.2021$, the second CP is $\hat{\tau}_2 = 30.12.2021$. Afterward, according to the determined CP, the dataset was divided into two and then three regions, respectively. The data before and after these CP values were modeled separately with GBM SDE and RMSE and MAPE values were calculated for each model. Among these models, the model that best explained the data was selected according to both criteria. These results were also supported by graphical representation.

First of all, with the help of the USD/TL data, the parameters of the GBM SDE model given by equation (1.2) were estimated as $\tilde{\theta}_1 = 1.5932, \tilde{\theta}_2 = 0.5524$ using the MLE method without considering CP. Accordingly, if the parameters are put in (1.2),

$$dX_t = 1.5932X_t dt + 0.5524X_t dW_t, \quad X_0 = x_0, t \in [0,\ T] \tag{3.1}$$

was obtained, where $0 = t_0 = 19.06.2017$ and $T = 19.06.2022$. Then, with the number of data $N = 1827$; when the EM schema was applied using N steps, 1000 trajectories of the stochastic process, which is the solution of (3.1) GBM SDE, were obtained as in Figure 3.2. Because the solutions sought for SDEs are random, it is difficult to find a model that directly fits the data. So, the more simulations are performed when searching for EM solutions, the closer the solution will be to the existing dataset. For this reason, in this study, 1000 simulations were made. In order to obtain the optimum solution that fits the dataset, the mean and median of 1000 trajectories mentioned here were taken and these two solutions were presented as possible solutions for (3.1) GBM SDE. The results were given in Figure 3.3; in which the black line represents the dataset, the red line is the mean of the EM trajectories of (3.1) GBM SDE, and the blue line is the median of the EM trajectories of the (3.1) GBM SDE.

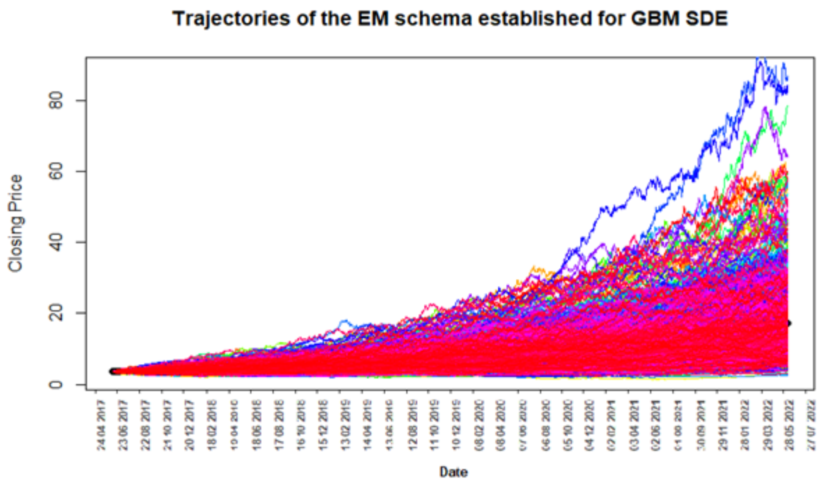

Figure 3.2: Trajectories of the EM schema of (3.1) GBM SDE using N steps

Considering Figure 3.3, the presented optimum solutions cannot reflect the fluctuation in the dataset. Because the dataset contains fluctuations that show sudden changes. For this reason, the change point of the dataset has been estimated according to GBM SDE and this estimate is $\hat{\tau}_1 = 14.11.2021$. Accordingly, this dataset is divided into two regions from the change point $\hat{\tau}_1$. Considering the data before $\hat{\tau}_1$ and after $\hat{\tau}_1$, two different GBM SDE models have been created. For this, the procedures performed in (3.1) GBM SDE modeling have been followed respectively. The MLE of the parameter obtained for GBM SDE, taking into account the $\hat{\tau}_1$ change point, was given in Table 3.1.

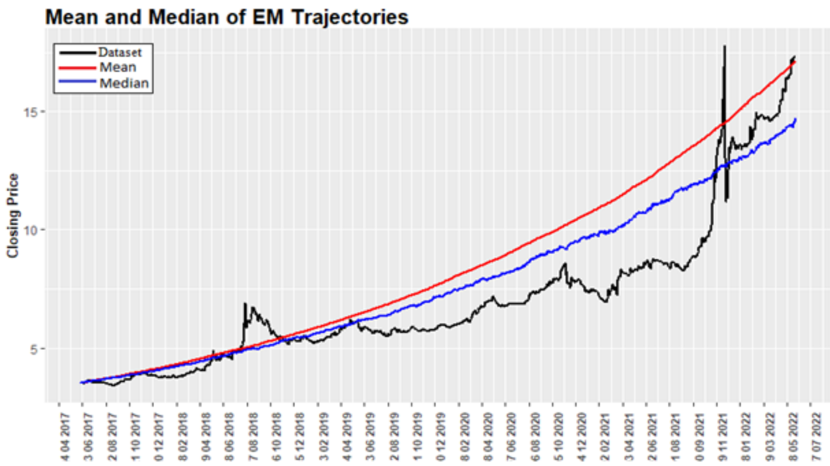

Figure 3.3: Mean and Median of EM Trajectories of (3.1) GBM SDE

$\tilde{\theta}_1$		$\tilde{\theta}_2$	
$t \in [0, \hat{\tau}_1)$	$t \in [\hat{\tau}_1, T]$	$t \in [0, \hat{\tau}_1)$	$t \in [\hat{\tau}_1, T]$
1.0453	0.5499	0.3727	0.4077

Table 3.1: MLE of the parameter obtained for GBM SDE considering the change point $\hat{\tau}_1$

Accordingly, when the initial value at the change point $\hat{\tau}_1$ is considered as the first value at $\hat{\tau}_1$; in other words, since the moment $\widehat{\tau}_1$ corresponds to the 1610th data, the obtained GBM SDE equation is obtained as

$$dX_t = \begin{cases} 1.0453X_t dt + 0.3727X_t dW_t, & X_0 = x_0, \quad t \in [0, \hat{\tau}_1) \\ 0.5499X_t dt + 0.4077X_t dW_t, & X_0{}^* = x_{1611}, t \in [\hat{\tau}_1, T] \end{cases} \tag{3.2}$$

where $0 = t_0 = 19.06.2017$ and $T = 19.06.2022$. Similar to the previous modeling, the number of data in the interval $[\,0, \hat{\tau}_1\,)$ is $N_1 = 1610$ and the number of data in the interval $[\,\hat{\tau}_1,\ T\,]$ is $N_2 = 217$ ($N = N_1 + N_2$); using N_1 steps for the interval $[0, \hat{\tau}_1)$ and N_2 steps for the interval $[\hat{\tau}_1, T]$, the 1000 trajectories EM schema of the stochastic process, which is the solution of (3.2) GBM SDE, was obtained as in Figure 3.4 and Figure 3.5, respectively.

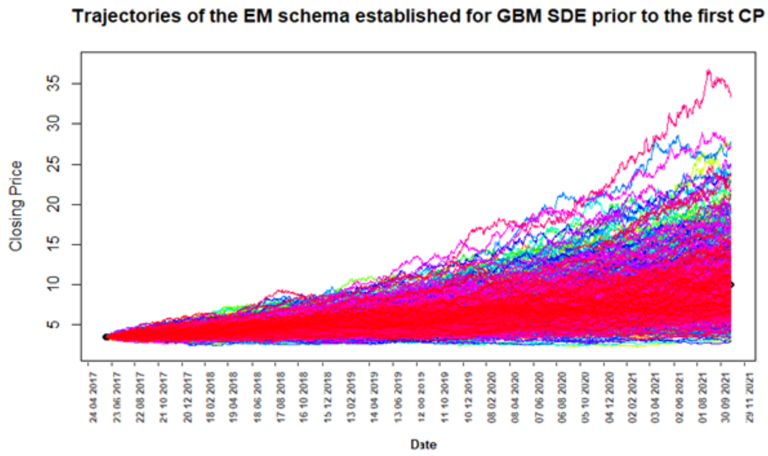

Figure 3.4: Trajectories of the EM schema of (3.2) GBM SDE for $[0, \hat{\tau}_1)$ using N_1 steps

For each interval, the mean and median of the 1000 trajectories specified here were taken by performing similar operations as in finding the optimum solutions of (3.1) GBM SDE, and these two solutions were presented as optimum solutions for (3.2) GBM SDE. The results are given in Figure 3.6 for the interval $[0, \hat{\tau}_1)$ and in Figure 3.7 for the interval $[\hat{\tau}_1, T]$. In each figure, in the regions indicated above, black lines represent the dataset, red lines represent the mean of (3.2) GBM SDE's EM trajectories and blue lines represent the median of (3.2) GBM SDE's EM trajectories. When these solutions are combined, the optimum solution of (3.2) GBM SDE is obtained, which is shown in Figure 3.8; where the black line represents the dataset, the green line the mean of the EM trajectories of (3.1) GBM SDE, the yellow line the median of the EM trajectories of the (3.1) GBM SDE and the pink line the mean of the EM trajectories of (3.2) GBM SDE, the purple line the median of the EM trajectories of the (3.2) GBM SDE. Considering Figure 3.8, it can be said that the optimum solutions proposed for (3.2) GBM SDE are closer to the dataset than the optimum solutions proposed for (3.1) GBM SDE. Thus, with the CP, relatively more suitable GBM SDE EM approximations are obtained for the available dataset. Apart from visually, the closeness of the solutions to the dataset was be given in Table 3.3 by calculating the RMSE and MAPE values.

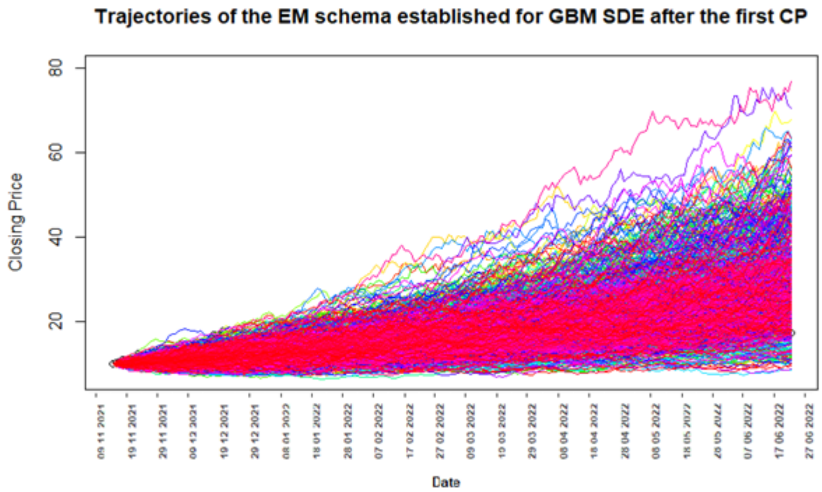

Figure 3.5: Trajectories of the EM schema of (3.2) GBM SDE for $[\hat{\tau}_1, T]$ using N_2 steps

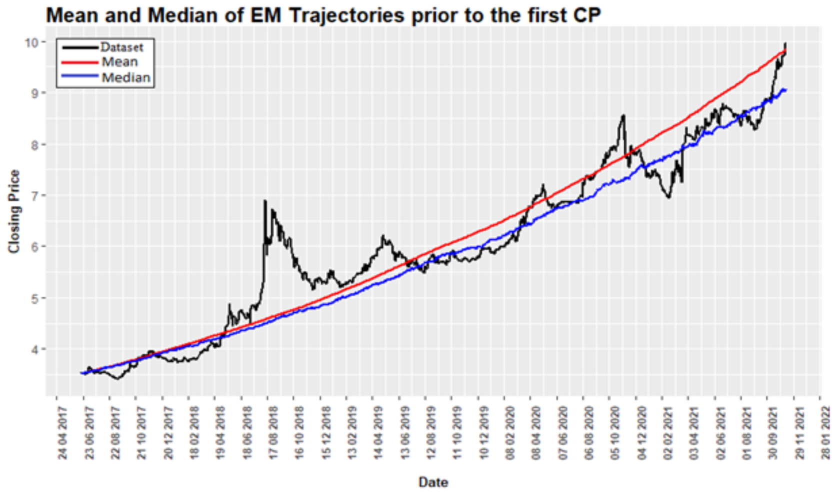

Figure 3.6: Mean and Median of EM Trajectories of (3.2) GBM SDE for $[0, \hat{\tau}_1)$

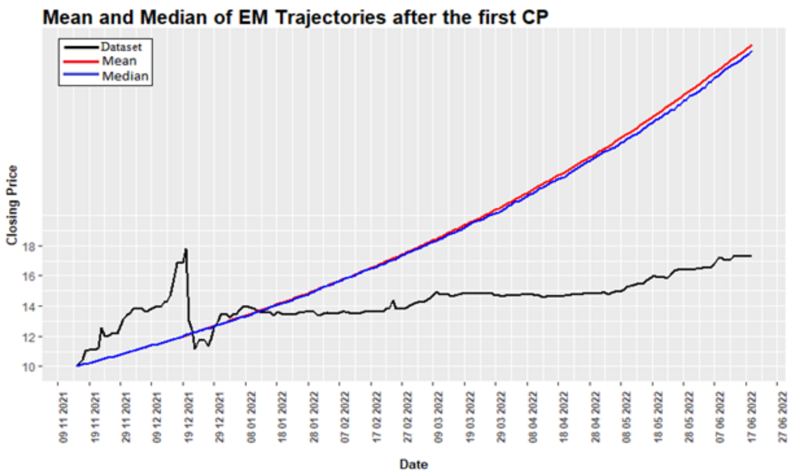

Figure 3.7: Mean and Median of EM Trajectories of (3.2) GBM SDE for $[\hat{\tau}_1, T]$

However, looking at Figure 3.8 it seems that the right tail of the optimum solutions is well above the dataset For this reason, it was deemed appropriate to make one more change point estimation on the dataset in the interval $[\hat{\tau}_1, T]$. Thus, the expression "the first CP (1st CP)" was used for the moment $\hat{\tau}_1$. Therefore, the other change point in the interval $[\hat{\tau}_1$, T] was expressed as "the second CP (2nd CP)" and denoted by $\hat{\tau}_2$. So, in this interval, the change point of the dataset was estimated according to GBM SDE and this estimate is $\hat{\tau}_2 = 30.12.2021$. Accordingly, the mentioned interval was divided into two regions from the change point $\hat{\tau}_2$. Considering the data from $\hat{\tau}_1$ to $\hat{\tau}_2$ and after $\hat{\tau}_2$, two different GBM SDE models were created. For this, the procedures performed in (3.2) GBM SDE modeling were followed respectively.

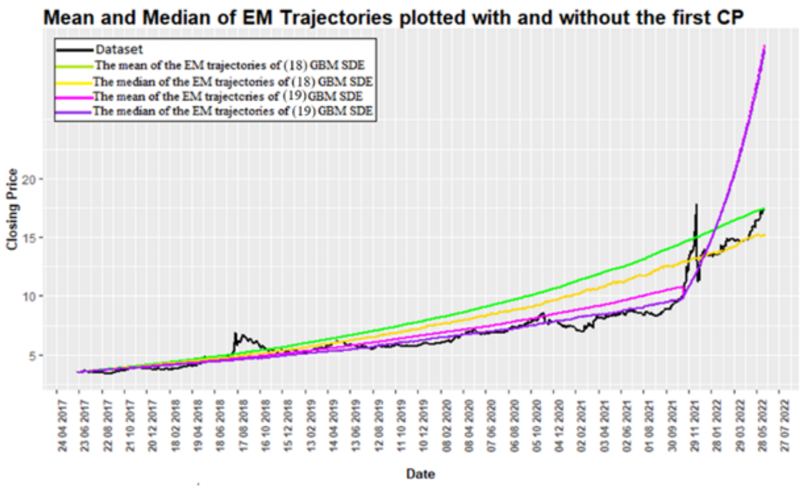

Figure 3.8: Mean and Median of EM Trajectories of (3.2) GBM SDE

The MLE of the parameter obtained for GBM SDE, taking into account the $\hat{\tau}_2$ change point, was given in Table 3.2.

$\tilde{\theta}_1$		$\tilde{\theta}_2$	
$t \in [\hat{\tau}_1, \hat{\tau}_2)$	$t \in [\hat{\tau}_2, T]$	$t \in [\hat{\tau}_1, \hat{\tau}_2)$	$t \in [\hat{\tau}_2, T]$
1.0453	0.2537	0.3727	0.0949

Table 3.2: Estimations of the MLE of the parameter obtained for GBM SDE considering the change point $\hat{\tau}_2$

Accordingly, when the initial value at the change point $\hat{\tau}_2$ is considered as the first value at $\hat{\tau}_2$; in other words, since the moment $\hat{\tau}_2$ corresponds to the 1656th data, the obtained GBM SDE equation is obtained as

$$dX_t = \begin{cases} 1.0453X_t dt + 0.3727X_t dW_t, & X_0 = x_0, \quad t \in [0, \hat{\tau}_1) \\ 1.0453X_t dt + 0.3727X_t dW_t, & X_0{}^* = x_{1611}, t \in [\hat{\tau}_1, \hat{\tau}_2) \\ 0.2537X_t dt + 0.0949X_t dW_t, & X_0{}^{**} = x_{1657}, t \in [\hat{\tau}_2, T] \end{cases} \tag{3.3}$$

where $0 = t_0 = 19.06.2017$ and $T = 19.06.2022$. Here, the number of data in the interval [$\hat{\tau}_1, \hat{\tau}_2$] is $N_3 = 43$ and the number of data in the interval $[\hat{\tau}_2, T]$ is $N_4 = 171\,(N = N_1 + N_3 + N_4)$; using N_3 steps for the interval $[\hat{\tau}_1, \hat{\tau}_2)$ and N_4 steps for the interval $[\hat{\tau}_2, T]$, the 1000 trajectories EM schema of the stochastic process, which is the solution of (3.3) GBM SDE for the interval $[\hat{\tau}_1, T]$, was obtained as in Figure 3.9 and Figure 3.10, respectively. For each interval, the mean and median of the 1000 trajectories specified here were taken by performing similar operations as in finding the optimum solutions of (3.2), and these two solutions are presented as optimum solutions for (3.3) GBM SDE for the interval $[\hat{\tau}_1, T]$. The results were given in Figure 3.11 for the interval [$\hat{\tau}_1, \hat{\tau}_2$) and in Figure 3.12 for the interval [$\hat{\tau}_2, T$]. In each figure, in the regions indicated above, black lines represent the dataset, red lines represent the mean of (3.3) GBM SDE's EM trajectories and blue lines represent the median of (3.3) GBM SDE's EM trajectories.

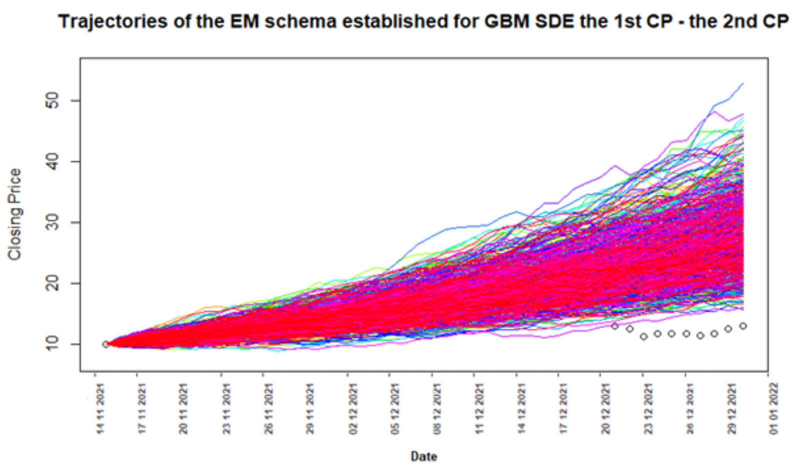

Figure 3.9: Trajectories of the EM schema of (3.3) GBM SDE for $t \in [\hat{\tau}_1, \hat{\tau}_2)$ using N_3 steps

When these solutions and for the interval [$0, \hat{\tau}_1$] earlier obtained solution are combined, the optimum solution of (3.3) GBM SDE is achieved, which is shown in Figure 3.13; in which the black line represents the dataset, the green line the mean of the EM trajectories of (3.1) GBM SDE, the yellow line the median of the EM trajectories of the (3.1) GBM SDE, the pink line the mean of the EM trajectories of (3.2)

GBM SDE, the purple line the median of the EM trajectories of the (3.2) GBM SDE and the red line the mean of the EM trajectories of (3.3) GBM SDE, the blue line the median of the EM trajectories of the (3.3) GBM SDE. Considering Figure 3.13, it can be said that the optimum solutions proposed for (3.3) GBM SDE are closer to the dataset than the optimum solutions proposed for (3.2) GBM SDE. Thus, GBM SDE EM approaches that are relatively more suitable than others are obtained with double CP for the current dataset. Apart from visually, the closeness of the solutions to the dataset were given in Table 3.3 by calculating the RMSE and MAPE values and the model that best fitted the dataset has been selected accordingly.

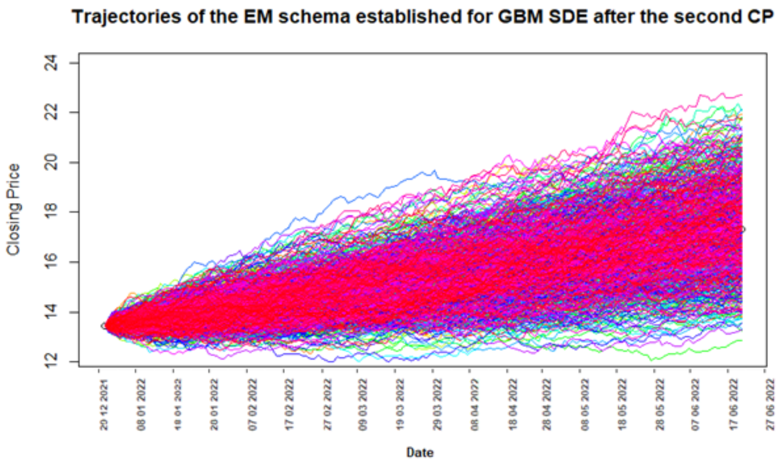

Figure 3.10: Trajectories of the EM schema of (3.3) GBM SDE for $t \in [\hat{\tau}_2, T]$ using N_4 steps

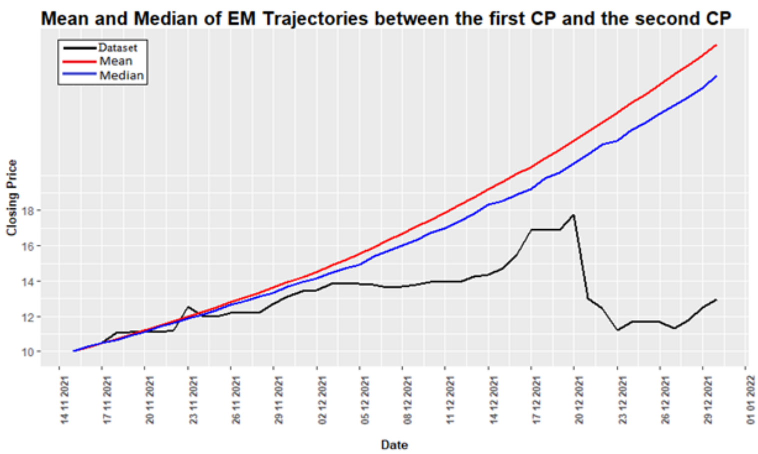

Figure 3.11: Mean and Median of EM Trajectories of (3.3) GBM SDE for $t \in [\hat{\tau}_1, \hat{\tau}_2)$

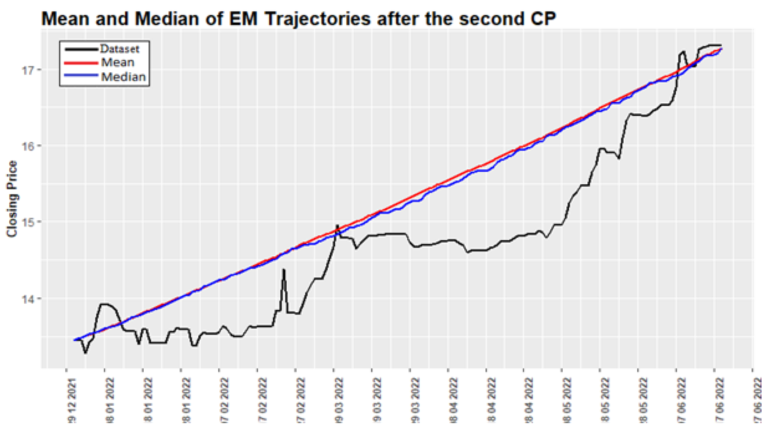

Figure 3.12: Mean and Median of EM Trajectories of (3.3) GBM SDE for $t \in [\hat{\tau}_2, T]$

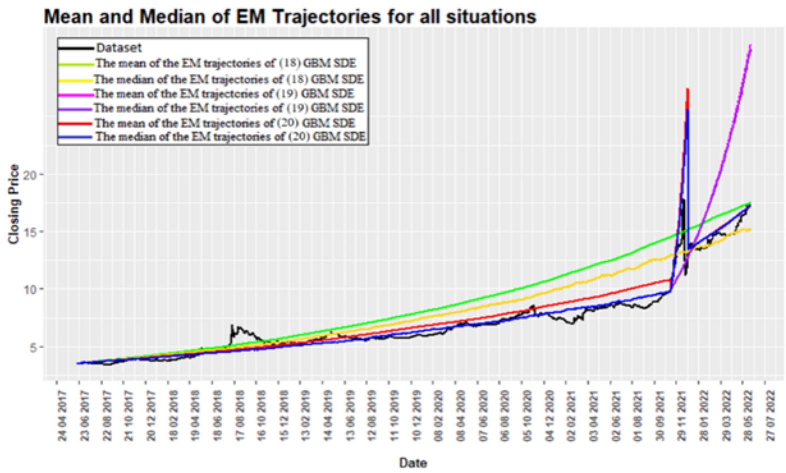

Figure 3.13: Mean and Median of EM Trajectories of (3.3) GBM SDE

	RMSE		*MAPE*	
Model of GBM SDE	Mean	Median	Mean	Median
(3.1) GBM SDE	2.265709	1.823897	23.42425	15.79356
(3.2) GBM SDE	1.162308	1.560190	8.941137	7.771112
(3.3) GBM SDE	1.598107	**1.018635**	7.066688	**6.129356**

Table 3.3: RMSE and MAPE criteria values of GBM SDE's obtained by considering the CP estimations

Looking at Table 3.3, when each model is evaluated within itself, the models presented by taking the median of the EM approximation solutions among the optimum solutions were chosen as the most appropriate solutions, since they gave smaller values according to both RMSE and MAPE criteria. Among these three models, the median of the EM trajectories of (3.3) GBM SDE with the smallest RMSE value of 1.018635 and the smallest MAPE value of 6.129356 was chosen as the most appropriate model presented for the given dataset. The graph of the proposed model is given in Figure 3.14; where the black line represents the dataset, and the blue line the proposed solution.

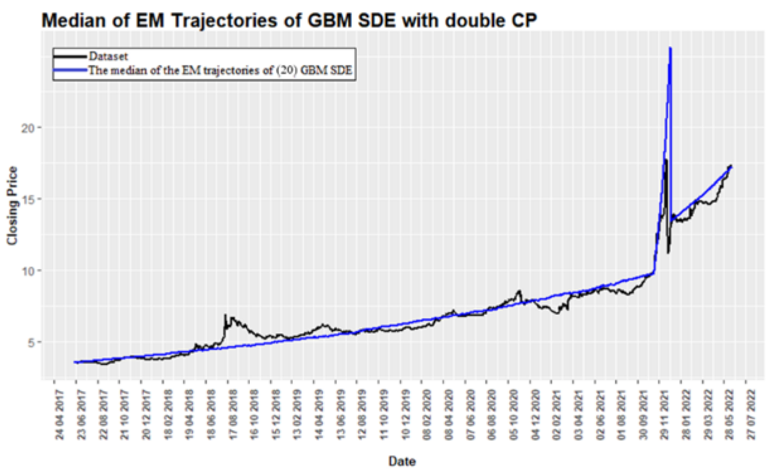

Figure 3.14: The proposed GBM SDE model for USD/TL data

4. Conclusion

In this study, the data showing the fluctuation of the TL against the USD were modeled by GBM SDE by considering the double CP for the first time. With the smallest RMSE and the smallest MAPE values, this solution of SDE has been acquired as the median of the EM trajectories of (3.3) GBM SDE for the given dataset.
Based on these results, when working with abruptly changing datasets, it can be said that considering the CP has a positive effect on the performance of the established model. In other words, the GBM SDE model built with double CP outperforms the GBM SDE model installed without considering the CP. In addition, there may be more than one CP in the dataset, and as the number of CP increases, more suitable models are obtained for the dataset.
Sudden changes in international capital flows can lead to sharp price movements in foreign exchange markets, causing various financial problems in emerging market economies such as Türkiye. For this reason, modeling such sudden changes in exchange rates may have

consequences that may have an impact on market risk management and investment decisions of international investors. The model recommended in this study suggests information that will assist policymakers in implementing effective policies in such areas.
In summary, the main contribution of this research can be outlined as follows:
The primary contribution of this study is the development of a novel modeling approach that integrates GBM with a double CP framework within the SDE setting, implemented using the YUIMA Project. This methodology enables the detection of structural breaks in volatile exchange rate regimes such as USD/TRY, providing a more flexible and accurate alternative to traditional constant-parameter models. The empirical results, based on real market data, demonstrate a significant improvement in model fit and forecasting performance. Furthermore, the use of validated open-source tools ensures transparency, reproducibility, and practical applicability, especially for high-volatility emerging market currencies like the Turkish Lira.

Article Information

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Author's Contributions: All authors read and approved the final manuscript. **S.O.C.** developed the main idea of the study, established the methodological framework, introduced the concept of change point estimation, coded the model, performed the analyses, and interpreted the results. **F.E.** contributed to the literature review, supported the selection of the dataset, and assisted in the interpretation of the findings. **N.I.** coded the model, prepared all figures, implemented technical components of the methodology, and provided support in resolving analytical issues during the study.

Artificial Intelligence Statement: Artificial intelligence was used in terms of language proficiency to enhance the quality of the sentences added after the revision.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the authors.

Plagiarism Statement: This article was scanned by the plagiarism program.

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