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A Further Note on the Graph of Monogenic Semigroups

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Abstract

In [15], it has been recently defined a new graph $\Gamma(\mathscr{S}_M)$ on monogenic semigroups $\mathscr{S}_M$ (with zero) having elements $\{0, x, x^2, x^3, \cdots, x^n\}$. The vertices are the non-zero elements $x, x^2, x^3, \cdots, x^n$ and, for $1 \le i, j \le n$, any two distinct vertices x^i and x^j are adjacent if $x^i x^j = 0$ in $\mathscr{S}_M$. As a continuing study of [3] and [15], in this paper it will be investigated some special parameters (such as covering number, accessible number, independence number), first and second multiplicative Zagreb indices, and Narumi-Katayama index. Furthermore, it will be presented Laplacian eigenvalue and Laplacian characteristic polynomial for $\Gamma(\mathscr{S}_M)$.

Keywords: *Graph, Laplacian Eigenvalue, Laplacian Polynomial, Monogenic Semigroups, Narumi-Katayama Index.*
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1. Introduction and Preliminaries

The history of studying zero-divisor graphs has began over commutative rings by the paper [8], and then it followed over commutative and noncommutative rings by some of the joint paper written by Anderson (cf. [4, 5, 6]) and some other authors (see, for instance, [1]). After that DeMeyer et al. and some other authors studied this special graphs over commutative and noncommutative semigroups ([16, 17, 39]). Since then there are very huge number of studies have been added in the literature about zero-divisor graphs. In a recent study [15], the graph $\Gamma(\mathscr{S}_M)$ is defined by replacing the adjacent rule of vertices and not destroying the main idea. In detail, the authors first considered a finite multiplicative monogenic semigroup with zero as the set

$$\mathscr{S}_M = \{0, x, x^2, x^3, \cdots, x^n\} \tag{1.1}$$

and then, by considering the definition given in [17], it has been obtained an undirected (zero-divisor) graph $\Gamma(\mathscr{S}_M)$ associated to $\mathscr{S}_M$ as in the following. The vertices of the graph are labeled by the nonzero zero-divisors (in other words, all nonzero element) of $\mathscr{S}_M$, and any two distinct vertices x^i and x^j, where $(1 \le i, j \le n)$ are connected by an edge in case $x^i x^j = 0$ with the rule $x^i x^j = x^{i+j} = 0$ if and only if $i + j \ge n+1$. The fundamental spectral properties such as the diameter, girth, maximum and minimum degree, chromatic number, clique number, degree sequence, irregularity index and dominating number for this new graph are presented in [15]. Furthermore, in [3], it has been studied first and second Zagreb indices, Randić index, geometric-arithmetic index and atom-bond connectivity index, Wiener index, Harary index, first and second Zagreb eccentricity indices, eccentric connectivity index and the degree distance to indicate the importance of the graph $\Gamma(\mathscr{S}_M)$.
As a further study, in this paper, it will be investigated covering number, accessible number, independence number, first and second multiplicative Zagreb indices, and finally Narumi-Katayama index over the graph $\Gamma(\mathscr{S}_M)$. In addition, it will be investigated Laplacian eigenvalue and Laplacian characteristic polynomial for $\Gamma(\mathscr{S}_M)$. It is obvious that the reason for studying this subject is to give a great opportunity to a deep characterization over the algebraic structure that studied on it.
Throughout this paper $G = (V, E)$ will always denote a simple connected graph such that $V = V(G)$ represents the set of vertices and $E = E(G)$ represents the set of edges. In addition, for a real number r, we will denote by $\lfloor r \rfloor$ the greatest integer $\le r$, and by $\lceil r \rceil$ the least integer $\ge r$ in our results.

2. Some special numbers over $\Gamma(\mathscr{S}_M)$

In this section we will mainly deal with the special parameters, namely covering, independence and accessible numbers over the graph $\Gamma(\mathscr{S}_M)$ associated with $\mathscr{S}_M$ as given in (1.1). At this point we remind that such these properties can be obtained by calculating the distance between any two vertices or the total number of whole vertices in any simple graph G. So this idea will be applied in the proofs of results.

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For a (simple) graph G, two vertices are said to be a *cover* for each other in G if they are incident. On the other hand, a *vertex cover* in G is a set of vertices that covers all edges of G. Depending on this, the *covering number* (cf. [10, 22]) of G, denoted by $\tau(G)$, is the number which obtained by the minimum cardinality of a vertex cover in G. Thus, the first result of this paper is the following.

Theorem 2.1. *For any $\mathscr{S}_M$ as in (1.1), we certainly have $\tau(\Gamma(\mathscr{S}_M)) = \lfloor \frac{n}{2} \rfloor$.*

Proof. We need to define a cover set C for the graph $\Gamma(\mathscr{S}_M)$. To do that, let us start by adding the vertices in C that having highest number of neighborhoods. This gives an opportunity to cover highest number of edges with the less number of vertices. Now, we will investigate the proof in the meaning of even and odd cases of n. For simplicity, let us call C_1 and C_2 to the cover sets depends on the even or odd cases over n, respectively.
The set C_1: For the vertex x^n, since $x^n x^{i_1} = 0$, it must be in the set C_1 (where $1 \leq i_1 \leq n-1$). Secondly, the vertex x^{n-1} must also be in C_1 since x^{n-1} is adjacent with all vertices holds the condition $x^n x^{i_2} = 0$ where $2 \leq i_2 \leq n-2$. By keeping this idea, we finally see that the vertex $x^{\frac{n}{2}+1}$ must be in C_1 which covers $x^{\frac{n}{2}} x^{\frac{n}{2}+1} \in E(\Gamma(\mathscr{S}_M))$. As a result of this, the vertex cover set C_1 (for n is even) of $\Gamma(\mathscr{S}_M)$ is given by $C_1 = \left\{x^{\frac{n}{2}+1}, x^{\frac{n}{2}+2}, \cdots, x^n\right\}$ that implies the covering number is equal to $\frac{n}{2}$.
The set C_2: As a similar approximation in the above case, the vertex x^n must be in C_2 since x^n is adjacent with all the vertices of the form $x^n x^{i_1} = 0$, where $1 \leq i_1 \leq n-1$. Secondly the vertex x^{n-1} must also be in C_2 since x^{n-1} is adjacent with all vertices of the form $x^n x^{i_2} = 0$, where $2 \leq i_2 \leq n-2$. By iterating this idea, we finally see that the vertex $x^{\frac{n+1}{2}+1}$ must be in C_2 which covers the edges $x^{\frac{n+1}{2}+1} x^{\frac{n+1}{2}}$ and $x^{\frac{n+1}{2}+1} x^{\frac{n-1}{2}}$. Thus, the vertex cover set C_2 (for n is odd) of $\Gamma(\mathscr{S}_M)$ is given by $C_2 = \left\{x^{\frac{n+1}{2}+1}, x^{\frac{n+1}{2}+2}, \cdots, x^n\right\}$ such that the covering number is equal to $\frac{n-1}{2}$.
Considering the sets C_1 and C_2 (for both even and odd cases of n), one can obtain the covering number as the number $\lfloor \frac{n}{2} \rfloor$, as required. □

The following two theorems will be about the accessible and independence numbers for $\Gamma(\mathscr{S}_M)$. A subset A of $V(G)$ is called *accessible* if and only if each $v \in \{V(G) - A\}$ is adjacent to $N[A]$. The minimal number of vertices over all accessible sets of G is called *accessible number* (cf. [20]) and denoted by $\eta(G)$.

Theorem 2.2. *For any $\mathscr{S}_M$ as in (1.1), we always have $\eta\,(\Gamma(\mathscr{S}_M)) = 1$.*

Proof. It is easy to see that for the graph $\Gamma(\mathscr{S}_M)$, if A is chosen as the set $A = \{x^n\}$, then $N[A] = \{x^1, x^2, \cdots, x^{n-1}\}$. Therefore, we get each vertex v in the set $\{V\,(\Gamma(\mathscr{S}_M)) - A\}$ is adjacent to $N[A]$. So $\eta\,(\Gamma(\mathscr{S}_M)) = 1$, as required. □

For a simple graph G, the non-empty subset $J \subseteq V(G)$ is called *independent* if there are no edges among the vertices in J. If an independent set that is not a subset of another independent set, then it is called the *maximal*. In fact such sets are also named as *dominating sets*. The cardinality of a largest independent set in G is called the *independence number* of G and denoted by $ind(G)$.
The following lemma plays a central role in the proof next theorem.

Lemma 2.3 ([22])**.** *A set J is independent if and only if its complement is a vertex cover. So the sum of $ind(G)$ and the size of a minimum vertex cover $\eta(G)$ is the number of vertices in the graph.*

Adapting this lemma into our graph, we obtain the following theorem.

Theorem 2.4. *For any $\mathscr{S}_M$ as in (1.1), we have $ind(\Gamma(\mathscr{S}_M)) = \lfloor \frac{n}{2} \rfloor$.*

Proof. As we proved in Theorem 2.1, the minimum vertex cover has been obtained as the set $C = \left\{x^{\lfloor \frac{n}{2} \rfloor+1}, x^{\lfloor \frac{n}{2} \rfloor+2}, \cdots, x^n\right\}$. Thus, by Lemma 2.3, the set $\left\{x^1, x^2, \cdots, x^{\lfloor \frac{n}{2} \rfloor}\right\}$ is actually independent which has the largest cardinality. So the independence number of $\Gamma(\mathscr{S}_M)$ is $\lfloor \frac{n}{2} \rfloor$, in other words, $ind(\Gamma(\mathscr{S}_M)) = \lfloor \frac{n}{2} \rfloor$. □

By considering maximum or minumum degrees, another result can be stated as follows.

Example 2.5. *Let us consider the graph $\Gamma(\mathscr{S}_{M_6})$ in Figure 2.1 with the vertex set $V(\Gamma(\mathscr{S}_{M_6})) = \{x, x^2, x^3, x^4, x^5, x^6\}$ (see [15] for the details of this graph).*

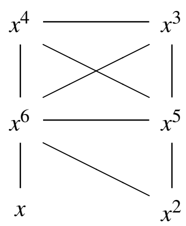

Figure 2.1: The graph of $\Gamma(\mathscr{S}_{M_6})$

Hence, by Theorems 2.1, 2.2 and 2.4, we certainly have $\tau(\Gamma(\mathscr{S}_{M_6})) = 3$, $\eta(\Gamma(\mathscr{S}_{M_6})) = 1$ and $ind(\Gamma(\mathscr{S}_{M_6})) = 3$.

3. Multiplicative Zagreb Indices

Topological indices play a significant role in chemistry, pharmacology, etc. (see, for instance, [18, 25, 26, 37]). Many of the topological indices of current interest in mathematical chemistry are defined in terms of vertex degrees of the molecular graph. For example, the first and second Zagreb indices are defined as

$$M_1(G)=\sum_{v_i\in V(G)} d_G(v_i)^2 \quad \text{and} \quad M_2(G)=\sum_{v_iv_j\in E(G)} d_G(v_i)d_G(v_j),$$

respectively (cf. [28, 29]). The Zagreb indices and their applications have been used to study molecular complexity, chirality, ZE-isomerism, heterosystems, etc. We encourage the reader to consult [9, 24, 31, 32, 35, 40] for historical background, computational techniques, and mathematical properties of Zagreb indices.
Following an earlier idea of Narumi and Katayama [34] (one may also cite the work [27]) who put forward what nowadays is referred to as the Narumi-Katayama index

$$NK(G)=\prod_{i=1}^{n} d_G(v_i) \tag{3.1}$$

of a simple graph G. There also introduced the multiplicative versions of the Zagreb indices ([23]). Actually, the first and second multiplicative Zagreb indices [21, 36, 37] are defined, respectively, as follows:

$$\Pi_1(G)=\prod_{v_i\in V(G)} (d_G(v_i))^2 \quad \text{and} \quad \Pi_2(G)=\prod_{i=1}^{n} d_G(v_i)^{d_G(v_i)}. \tag{3.2}$$

These above facts clearly give us that $\Pi_1=(NK(G))^2$. Therefore, as another main result of this paper, we have the following theorem.

Theorem 3.1. *Let $\mathscr{S}_M$ be a monogenic semigroup as given in (1.1). Then*

- *the Narumi-Katayama index of $\Gamma(\mathscr{S}_{M_n})$ is $(n-1)!\left\lfloor \frac{n}{2}\right\rfloor$,*
- *the first mutliplicative Zagreb index of $\Gamma(\mathscr{S}_{M_n})$ is $\left((n-1)!\left\lfloor \frac{n}{2}\right\rfloor\right)^2$,*
- *the second mutliplicative Zagreb index of $\Gamma(\mathscr{S}_{M_n})$ is $\left((n-1)!\left\lfloor \frac{n}{2}\right\rfloor\right)^{\left((n-1)!\left\lfloor \frac{n}{2}\right\rfloor\right)}$.*

Proof. By the Narumi-Katayama index as defined in (3.1), we get

$$NK(\Gamma(\mathscr{S}_{M_n}))=\prod_{i=1}^{n} d_{\Gamma(\mathscr{S}_{M_n})}(v_i)=1.2.3.\cdots.(n-1)\left\lfloor \frac{n}{2}\right\rfloor=(n-1)!\left\lfloor \frac{n}{2}\right\rfloor$$

that gives the first condition of the theorem. Secondly, by the indices given in (3.2), we have

$$\Pi_1(\Gamma(\mathscr{S}_{M_n}))=\prod_{i=1}^{n}\left(d_{\Gamma(\mathscr{S}_{M_n})}(v_i)\right)^2=\left((n-1)!\left\lfloor \frac{n}{2}\right\rfloor\right)^2$$

and

$$\Pi_2(\Gamma(\mathscr{S}_{M_n}))=\prod_{ij\in E(\Gamma(\mathscr{S}_{M_n}))}^{n} d_{\Gamma(\mathscr{S}_{M_n})}(v_i)d_{\Gamma(\mathscr{S}_{M_n})}(v_j)=\prod_{i=1}^{n} d_{\Gamma(\mathscr{S}_{M_n})}(v_i)^{d_{\Gamma(\mathscr{S}_{M_n})}(v_i)}=\left((n-1)!\left\lfloor \frac{n}{2}\right\rfloor\right)^{\left((n-1)!\left\lfloor \frac{n}{2}\right\rfloor\right)}.$$

Hence the result. □

4. Laplacian characteristic polynomial of $\Gamma(\mathscr{S}_M)$

The main goal of this section is to study on the Laplacian eigenvalue and Laplacian characteristic polynomial on $\Gamma(\mathscr{S}_M)$.
The Laplacian matrix is a discrete analog of the Laplacian operator in multivariable calculus and serves a similar purpose by measuring to what extent a graph differs at one vertex from its values at nearby vertices. The Laplacian matrix arises in the analysis of random walks and electrical networks on graphs ([19]), and in particular in the computation of resistance distances ([7]). Furthermore, recently, it has been added so many studies (see, for example, [2, 11, 12, 14, 30, 33]) in the literature about these important characterizations.
Let us again consider G with the vertex set $V=\{v_1,v_2,v_3,...,v_n\}$ and the edge set E of cardinality e. Assume that the vertices are ordered as $d_1\geq d_2\geq\cdots\geq d_n$, where d_i is the degree of each v_i for $i=1,2,...,n$. For each $v_i\in V$, the set of neighbors of v_i and the average of the degrees of the vertices adjacent to v_i are denoted by N_{v_i} and m_v, respectively. Furthermore, as usually, let $A(G)$ be the adjacency matrix and let $D(G)$ be the diagonal matrix of vertex degrees of G. It is quite well known that the Laplacian matrix of G is defined by $L(G)=D(G)-A(G)$. In here, $L(G)$ is a real symmetric matrix and so its eigenvalues are all non-negative real numbers. Moreover, since the sum of the rows is equal to 0, this implies that 0 is the smallest eigenvalue of $L(G)$. Additionally, the spectrum of G is $S(G)=(\lambda_1(G),\lambda_2(G),\cdots,\lambda_n(G))$, where $\lambda_1(G)\geq\lambda_2(G)\geq\cdots\geq\lambda_n(G)=0$ are the eigenvalues of $L(G)$ arranged in non-increasing order. Thus, it is clear that for a star graph of order n, the spectrum is $(n,\underbrace{1,1,\cdots,1}_{n-2},0)$.
The following lemma will be the keynote step in the proof next theorem which can be proved directly by induction steps.

Lemma 4.1 ([12]). *Let $G=(V,E)$ be a graph with a vertex subset $V'=\{v_1,v_2,\cdots,v_k\}$. Also, let $E^+=E\cup E'$, where $E'\subseteq V'\times V'$. If $G'=(V',E')$ has eigenvalues $a_1\geq a_2\geq\cdots\geq a_k=0$ then the eigenvalues of $L(G^+)$, where $G^+=(V,E^+)$ are as follows: those eigenvalues of the graph $G=(V,E)$ which are equal to N (k-1 in number) are incremented by $a_i,1,2,\cdots,k-1$ and the remaining eigenvalues are same.*

To catch up the aim of this section, we also need the following result.

Theorem 4.2. *The Laplacian characteristic polynomial of chromatic* $\Gamma(\mathscr{S}_{M_n})$ *is*

$$\phi_n(x) = (x-n)(x-(n-1)),\cdots,(x-(\left\lceil\frac{n}{2}\right\rceil+1))(x-(\left\lceil\frac{n}{2}\right\rceil-1)),\cdots,(x-1)x.$$

Proof. The mathematical induction will be used for the proof.

Case i) n is odd:
Step 1: By taking into account the graph $\Gamma(\mathscr{S}_{M_3})$, the induction can be started by $n=3$. In here the Laplacian eigenvalues are $\{3,1,0\}$. Now if we apply Lemma 4.1 and the material before it (in another words, if we add one more vertex in the graph $\Gamma(\mathscr{S}_{M_3})$ and then adjoint it with all vertices of that graph, and also if we add another vertex with adjoint it only the last vertex which added in final), then we obtain the Laplacian eigenvalues are $\{5,4,2,1,0\}$ which belongs the graph $\Gamma(\mathscr{S}_{M_5})$.
Step 2: Let us assume that the spectrum of $\Gamma(\mathscr{S}_{M_n})$ (where n is odd) is

$$\{n,n-1,\cdots,\left\lceil\frac{n}{2}\right\rceil+1,\left\lceil\frac{n}{2}\right\rceil-1,\cdots,2,1,0\}.$$

Step 3: As a similar idea in Step 1 (i.e. adding a new vertex in the graph $\Gamma(\mathscr{S}_{M_n})$ and then joining it with all vertices of $\Gamma(\mathscr{S}_{M_n})$, and also adding another vertex with joining it only the last vertex which added in final), we obtain the graph $\Gamma(\mathscr{S}_{M_{n+2}})$ since $\Gamma(\mathscr{S}_{M_{n+2}}) = (\Gamma(\mathscr{S}_{M_n})\cup K_1)\vee K_1$. Thus the spectrum of it is

$$\{n+2,n+1,n,\cdots,\left\lceil\frac{n}{2}\right\rceil+2,\left\lceil\frac{n}{2}\right\rceil,\left\lceil\frac{n}{2}\right\rceil-1,\cdots,2,1,0\}.$$

That gives the odd case.
Case ii) n is even:
Step 1: Let us take $n=4$ and so start it with the graph $\Gamma(\mathscr{S}_{M_4})$. In fact the Laplacian eigenvalues of it is $\{4,3,1,0\}$. By Lemma 4.1 and the material before it (applying same steps as in Step 1 in n is odd case), we obtain the Laplacian eigenvalues $\{6,5,4,2,1,0\}$ which are belong to $\Gamma(\mathscr{S}_{M_6})$.
Step 2: Let us assume that the spectrum of $\Gamma(\mathscr{S}_{M_n})$ (where n is even) is

$$\{n,n-1,\cdots,\frac{n}{2}+1,\frac{n}{2}-1,\cdots,2,1,0\}.$$

Step 3: Iterating the progress applied in Step 1, we obtain the graph $\Gamma(\mathscr{S}_{M_{n+2}})$ since $\Gamma(\mathscr{S}_{M_{n+2}}) = (\Gamma(\mathscr{S}_{M_n})\cup K_1)\vee K_1$. In here, the spectrum is

$$\{n+2,n+1,n,\cdots,\frac{n}{2}+2,\frac{n}{2},\frac{n}{2}-1,\cdots,2,1,0\}.$$

Hence the result. □

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