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Research Article

Modulative categories and fixed point theorems on modulative metric spaces

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Abstract

In terms of enriched category theory, we obtain a categorical explanation of the concept of modular metric spaces, similar to the case of Lawvere metric spaces. To this end, we introduce a generalization of categories that we call modulative categories, and as an extension of this, we define a new concept of modulative metric spaces along with several other notions. We obtain some results on modulative metric spaces and modulative categories, and illustrate them with examples. Finally, we develop a theory of convergence in modulative metric spaces and generalize some fixed-point theorems to demonstrate the applicability of the results obtained.

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1. Introduction

The concept of distance between two things can be thought of in its simplest form as the length of the gap between them. From this perspective, if length is a first-level concept, one might say that distance is a second-level concept. However, in modern mathematics, things have somewhat reversed, and metric spaces, which are the most prominent tools for mathematically modeling the concept of distance, are structurally much simpler, more general, and more fundamental than normed spaces, which are prominent for modeling the concept of length.

The axioms of metric spaces originate from Fréchet's philosophical views on what properties a universally acceptable notion of the shortest distance should have. The accuracy of these views is evident from the fact that metric spaces appear abundantly in countless mathematical and applied fields today. Nevertheless, various alternative concepts have been proposed through different variations of the metrics. Indeed, the concept of a metric has undoubtedly more than a hundred first-level variations, such as quasi-metric, topology, bipolar metric, $b-$metric, $G-$metric, complex-valued metric, premetric, uniformity,

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extended metric, cone metric, partial metric, $\star-$metric, dislocated metric, proximity, probabilistic metric, fuzzy metric, random metric and so on. Additionally, there are even more variations that combine two or more of these under one framework, such as orthogonal neutrosophic $2-$metric [14], C^*-algebra valued bipolar $b-$metric [18], bipolar parametric v-metric [23], bicomplex-valued controlled metric [24], C^*-algebra-valued partial modular metric [28], and rectangular soft metric [32]. Despite such diversity, nearly all of these concepts have found a place, to some extent, in applications.

While the axiomatic framework for metric spaces, inherited from its period of conceptual definition, works well in applications, it is not particularly satisfying from a categorical theoretical standpoint, both in terms of its failure to satisfy certain desirable properties, such as categorical completeness, and due to its weaker categorical connections with some other known categories. Lawvere's well-known approach has succeeded in placing the concept of a metric space on an enriched categorical theoretical foundation, by generalizing it to some extent [20]. Similar approaches have been explored, for instance, also for probabilistic metric spaces [11] and partial metric spaces [12].

One of the variations that generalizes metric spaces is modular metric spaces. They provide a framework in which the distance between two points depends on a positive real parameter t. This parameter, though not necessarily, can be associated with physical quantities such as time or speed. When t is interpreted as time, modular metric spaces can be used to model systems where points move closer to each other over time, or at least do not move farther apart, with this approaching occurring in an orderly manner. If we denote the distance between points x and y at time $t > 0$ by $d_t(x, y)$, this order is maintained by a generalized triangle inequality of the form $d_{s+t}(x, z) \leq d_s(x, y) + d_t(y, z)$. Given that $(0, \infty)$ and $\mathbb{R}$ are homeomorphic, for a parameter that can take all real values, this can also be interpreted with the help of logarithms, putting $\ell_s := d_{e^s}$ for a given modular metric d, as $\ell_{\ln(e^s+e^t)}(x, z) \leq \ell_s(x, y) + \ell_t(y, z)$ or in other suitable forms. Here, if one considers the semigroup $(\mathbb{R}, *)$ given with the operation $a * b = \ln(e^a + e^b)$, it is possible to revise the structure to explain different physical systems where time, or whatever the parameter is interpreted as, progresses in a discrete, cyclic or quite sophisticated manner by replacing the semigroup with an arbitrary semigroup $(S, *)$. This is one of the approaches we will explore in this study.

Modular metric spaces have gained significant attention in the literature and have been extensively studied, particularly in fixed-point theory. A substantial body of work has been conducted on fixed-point theorems related to both modular metric spaces and their generalizations or special subclasses [2, 13, 16, 17, 22, 29, 33, 34]. The results obtained have found various applications in areas such as integral equations [4, 7, 15, 27, 35], dynamical programming [6, 9, 26, 31], linear equation systems [30], fuzzy fractional differential equations [3], and homotopy theory [8].

The primary objective of our study is to explain modular metric spaces in a more general framework using enriched category theory. We will refer to these more general structures as modulative Lawvere metric spaces. In doing so, we introduce the new concept of a modulative category, which we expect to have potentially versatile implications. In our setting, each modulative metric space will be based on a semigroup, and modular metric spaces will be thought as modulative metric spaces based on the semigroup $(\mathbb{R}^+, +)$. To demonstrate the applicability of modulative metric spaces, which are being introduced for the first time within this study, we also develop a theory of convergence on them, and apply the resulting findings to fixed-point theory by generalizing some known results.

2. Preliminaries

Aiming to fix the notation and eliminate ambiguity related to terms that are occasionally used with different meanings in the literature, here we introduce some fundamental concepts, notation, and terminology that will be necessary for the remainder of this work. The content of this section and other details on the topics can be found in [1], [21] and [25].

A category $\mathcal{C}$ consists of a class denoted by $|\mathcal{C}|$ (the class of objects), a set $\mathrm{Hom}(A, B)$ for each pair (A, B) of objects (the set of morphisms from A to B), an operation (composition)

$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \longrightarrow \mathrm{Hom}(A, C), \quad (f, g) \mapsto g \circ f$$

for each triple (A, B, C) of objects, a morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$ for each object A, such that the equalities $(h \circ g) \circ f = h \circ (g \circ f)$, $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_A \circ f = f$ hold, whenever they are defined.

The situation $f \in \mathrm{Hom}(A, B)$ is usually denoted briefly as $f : A \longrightarrow B$, and a diagram consisting of multiple objects and morphisms (also called arrows) is said to **commute** if different paths from one object to another, following the direction of the arrows, are equivalent, that is, the compositions of morphisms along all possible different paths yield the same result.

By definition, the collection of objects in a category does not have to be a set; it can also be a proper class. If all the objects of a category form a set, then it is called a small category. Additionally, in the literature, some definitions sometimes allow hom-sets (more accurately hom-classes) to be proper classes. Then, categories in which all hom-sets are sets are referred to as locally small. In this study, it is assumed that the hom-sets in all categories are sets, meaning that, according to the other terminology, all categories discussed here are assumed to be locally small. Moreover, some definitions impose the condition that hom-sets must be pairwise disjoint. In the categories used in this work, it is not required for hom-sets to be pairwise disjoint; however, the composition operations defined for given triples of objects are assumed to yield the same results for different triples, when a pair of morphisms that lie in suitable intersections is given. More precisely, if $f \in \mathrm{Hom}(A, B) \cap \mathrm{Hom}(D, E)$ and $g \in \mathrm{Hom}(B, C) \cap \mathrm{Hom}(E, F)$, then the compositions

$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \longrightarrow \mathrm{Hom}(A, C), \quad (f, g) \mapsto g \circ f$$
$$\mathrm{Hom}(D, E) \times \mathrm{Hom}(E, F) \longrightarrow \mathrm{Hom}(D, F), \quad (f, g) \mapsto g \circ f$$

are both defined, and take the same value $g \circ f \in \mathrm{Hom}(A, C) \cap \mathrm{Hom}(D, F)$.

A functor $\mathscr{F}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a rule that transforms the objects and morphisms in $\mathcal{C}$ into the objects and morphisms in $\mathcal{D}$, in a way that for each morphism $f : A \longrightarrow B$ in $\mathcal{C}$, one has $\mathscr{F}(f) : \mathscr{F}(A) \longrightarrow \mathscr{F}(B)$ in $\mathcal{D}$, and the equalities $\mathscr{F}(\mathrm{id}_A) = \mathrm{id}_{\mathscr{F}(A)}$ and $\mathscr{F}(g \circ f) = \mathscr{F}(g) \circ \mathscr{F}(f)$ are always satisfied. In this case the notation $\mathscr{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is used.

For two given categories $\mathcal{C}$ and $\mathcal{D}$, the product category $\mathcal{C} \times \mathcal{D}$ has the pairs (A, X) as objects, where $A \in |\mathcal{C}|$ and $X \in |\mathcal{D}|$. Morphisms from an object (A, X) to an object (B, Y) are the pairs (f, j), where $f : A \longrightarrow B$ in $\mathcal{C}$ and $j : X \longrightarrow Y$ in $\mathcal{D}$. In this case, the compositions in $\mathcal{C} \times \mathcal{D}$ are defined by the rule $(g, k) \circ (f, j) := (g \circ f, k \circ j)$, and then $\mathrm{id}_{(A,X)}$ is equal to $(\mathrm{id}_A, \mathrm{id}_X)$ for each $(A, X) \in |\mathcal{C} \times \mathcal{D}|$.

A functor $\mathscr{B} : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$ from a product category $\mathcal{C} \times \mathcal{D}$ to a category $\mathcal{E}$ is called a binary functor, or shortly, a bifunctor.

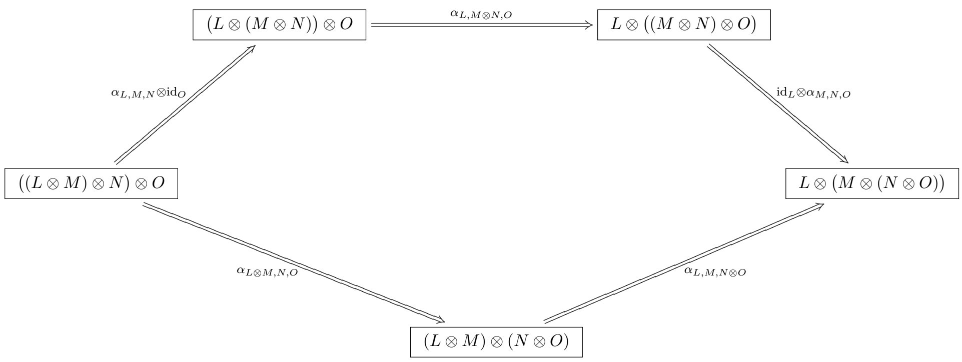
A monoidal category $\mathcal{M}$ is a category given together with a bifunctor $\otimes$ in the form

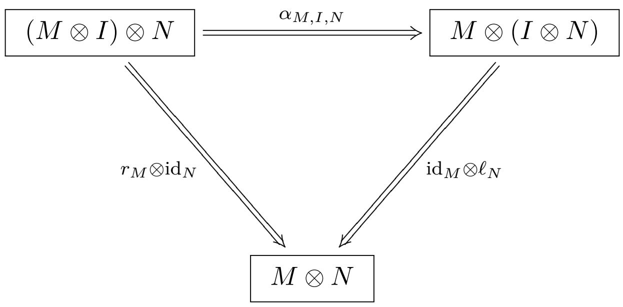
$$\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M},$$

which is called the tensor product, since it behaves like an operation and one uses the notation $M \otimes N := \otimes(M, N)$ for all objects M and N in $\mathcal{M}$, also with a distinguished object I, which is called the unit object, and three families of isomorphisms

$$\alpha_{LMN} : (L \otimes M) \otimes N \longrightarrow L \otimes (M \otimes N)$$
$$r_M : M \otimes I \longrightarrow M$$
$$\ell_M : I \otimes M \longrightarrow M$$

for all objects L, M and N in $\mathcal{M}$, called the natural isomorphisms of the monoidal category, such that the following diagrams commute, for all objects L, M, N and O in $\mathcal{M}$.



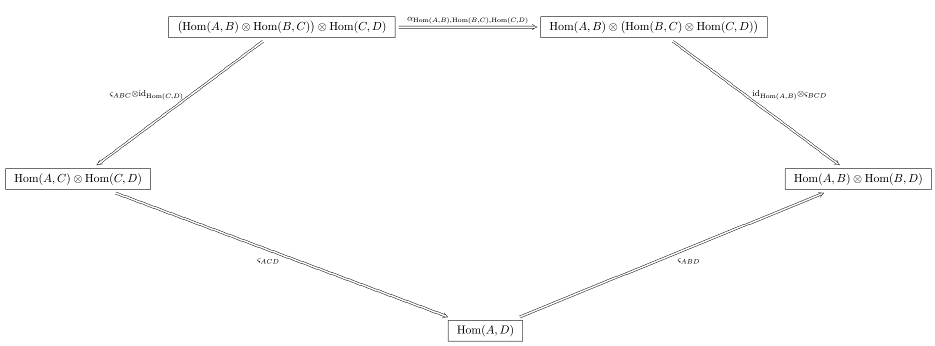


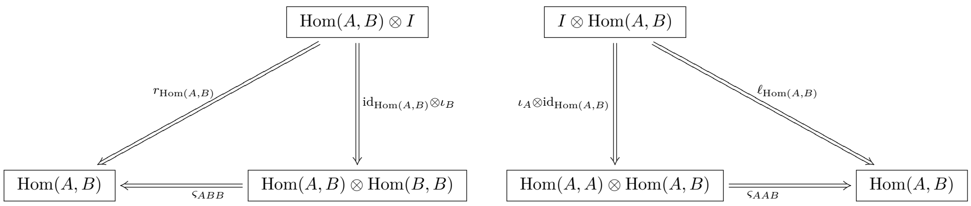
The first diagram is called the pentagon diagram, while the second one is called the triangle diagram. Their commutativities are often referred to as coherence conditions for the monoidal category, and the isomorphisms a_{LMN}, r_M and ℓ_M are called associators, right unitors and left unitors, respectively. In some cases, we also write a monoidal category as $(\mathcal{M}, \otimes, I)$ to clearly indicate the tensor product and the unit object.

A category $\mathcal{C}$ is said to be enriched over a monoidal category $(\mathcal{M}, \otimes, I)$, given with the natural isomorphisms α_{LMN}, r_M, ℓ_M for all objects L, M and N in $\mathcal{M}$, if the hom-sets $\mathrm{Hom}(A, B)$ in $\mathcal{C}$ are also objects in $\mathcal{M}$ and there are families of morphisms

$$\iota_A : I \longrightarrow \mathrm{Hom}(A, A)$$
$$\varsigma_{ABC} : \mathrm{Hom}(A, B) \otimes \mathrm{Hom}(B, C) \longrightarrow \mathrm{Hom}(A, C)$$

in $\mathcal{M}$, for all objects A, B and C in $\mathcal{C}$, such that the diagrams





in $\mathcal{M}$ commute. ι_A and ς_{ABC} are respectively called the identity arrows and the composition arrows.

A semigroup $(S, *)$ is an algebraic structure consisting of a set S and a binary operation $*$ on this set that satisfies the associativity property, and a monoid $(M, *, e)$ is a semigroup $(M, *)$ having an identity element e.

On the other hand, a lattice $(L, \lesssim)$ is a structure consisting of a set L and a partial order relation $\lesssim$ on L such that every two-element subset $\{a, b\}$ of L has an infimum and a supremum. These two values are denoted by $a \wedge b$ and $a \vee b$, respectively. More strongly, if every subset $A = \{a_i : i \in I\}$ of L has an infimum and a supremum, then L is called a complete lattice, and these values are respectively denoted by $\bigwedge_{i\in I} a_i$ and $\bigvee_{i\in I} a_i$, or alternatively by $\bigwedge A$ and $\bigvee A$. In particular, the elements $\top$ and $\bot$ are defined as $\top := \bigwedge \varnothing$ and $\bot := \bigvee \varnothing$. In this case, $\top$ is necessarily the maximum element in L, while $\bot$ is the minimum.

If $(Q, \rightarrow)$ is a complete lattice and $(Q, \otimes)$ is a semigroup, such that

$$q \otimes \bigvee_{i\in I} q_i = \bigvee_{i\in I} (q \otimes q_i) \quad \text{and} \quad \left(\bigvee_{i\in I} q_i\right) \otimes q = \bigvee_{i\in I} (q_i \otimes q)$$

for all $q \in Q$ and $\{q_i : i \in I\} \subseteq Q$, then the triple $(Q, \rightarrow, \otimes)$ is called a quantale. In particular, if $(Q, \otimes, \iota)$ is a monoid, then the quantale $(Q, \rightarrow, \otimes)$ is said to be unital with the identity element ι.

Each unital quantale $(Q, \rightarrow, \otimes)$ can also be regarded as a monoidal category. Thus, it makes sense to speak of enriching a category over a unital quantale. In this regard, the objects of the category are supposed to be elements of Q. For each $p, q \in Q$, $\mathrm{Hom}(p, q) = \varnothing$ if $p \not\rightarrow q$ (note that, $\rightarrow$ is, by definition, a partial order on Q), and $\mathrm{Hom}(p, q)$ is a singleton if $p \rightarrow q$. In the case $p \rightarrow q$, we denote the only element of the singleton $\mathrm{Hom}(p, q)$ by a_{pq}. The existence of identity morphisms $id_p = a_{pp}$ are guaranteed by the reflexivity of $\rightarrow$. If $a_{pq} : p \longrightarrow q$ and $a_{qr} : q \longrightarrow r$, then $a_{qr} \circ a_{pq}$ is defined as a_{pr}, where the transitivity of $\rightarrow$ guarantees the existence of a_{pr}. The operation $\otimes$ becomes the tensor product of the resulting monoidal category, and the identity element ι becomes the unit object. Then associators $\alpha_{pqr} : (p \otimes q) \otimes r \longrightarrow p \otimes (q \otimes r)$ are the morphisms $\mathrm{id}_{p\otimes q\otimes r}$ by the associativity of Q, while the right unitors $r_p : p \otimes \iota \longrightarrow p$ and the left $\ell_p : \iota \otimes p \longrightarrow p$ are both equal to id_p, as $p \otimes \iota = \iota \otimes p = p$.

Remark 2.1. Although it may seem unconventional to denote a partial order relation by the symbol $\rightarrow$, this convention is preferred for aligning the notation of the partial order relation with the arrows representing morphisms. The reason for this choice is the desire to more concisely emphasize the striking results for enriched categories over a unital quantale, viewed as a monoidal category. Indeed, when the two special quantales introduced below are considered, this convention will be reversed: by using the symbols of partial order relation instead of the arrows in the category. Thus the existence of morphisms between the objects in the enriched category will be interpreted as an ordering information.

We are particularly interested in two unital quantales. The first one is the quantale $\mathbf{2} := (\{0, 1\}, \leq, \cdot)$, where the usual order and multiplication on the two-element number set $\{0, 1\}$ are considered. The other important unital quantale is $[\infty, 0] := ([0, \infty], \geq, +)$, which is also known as the Lawvere quantale. The notation $[\infty, 0]$ is a playful inversion to emphasize that the order is reversed here; and hence in the lattice structure of the Lawvere quantale, the minimum element is ∞ and the maximum is 0.

It is known that a preordered set (a set endowed with a reflexive and transitive relation) can be represented as a small category enriched over the quantale **2**, and conversely each such category uniquely corresponds to a preordered set. More precisely, if $\mathcal{C}$ is a small category enriched over **2**, then for $A, B \in |\mathcal{C}|$, it is written $A \lesssim B$ if and only if $\mathrm{Hom}(A, B) = 1$, so that $A \not\lesssim B$ if and only if $\mathrm{Hom}(A, B) = 0$.

Consider the identity arrows $\iota_A : 1 \leq \mathrm{Hom}(A, A)$ (note that I was replaced with the identity element 1 of **2** and the arrow was replaced with the symbol $\leq$, as described in Remark 2.1). Since **2** has no element greater than 1, the existence of ι_A forces $\mathrm{Hom}(A, A)$ to equal 1, from which the reflexivity $A \lesssim A$ is obtained. Similarly, keeping in mind that the tensor product of **2** is the multiplication, the existence of the composition arrows ς_{ABC} : $\mathrm{Hom}(A, B) \cdot \mathrm{Hom}(B, C) \leq \mathrm{Hom}(A, C)$ suggest that $\mathrm{Hom}(A, B) = 1$ and $\mathrm{Hom}(B, C) = 1$ together force $\mathrm{Hom}(A, C) = 1$, that is $A \lesssim B$ and $B \lesssim C$ implies $A \lesssim C$, and this is the transitivity.

A Lawvere metric space is an extended pseudo-quasi-metric space. In other words, (X, L) is a Lawvere metric space, if $L : X \times X \longrightarrow [0, \infty]$ is a function satisfying

(M1) $L(x, x) = 0$
(M2) $L(x, z) \leq L(x, y) + L(y, z)$

for all $x, y, z \in X$. The adjective extended can be dropped if infinity values are not allowed, and also the prefixes pseudo and quasi can be dropped, respectively, if

(M0) If $L(x, y) = 0$, then $x = y$

and

(M3) $L(x, y) = L(y, x)$

for all $x, y \in X$.

Small categories enriched over the Lawvere quantale $[\infty, 0]$ are equivalent to Lawvere metric spaces. Taking into account that in $[\infty, 0]$, the unit is 0, the tensor product is $+$, and the arrow notation is $\geq$, it is possible to recover the axiom (M1) from the identity arrows $\iota_A : 0 \geq \mathrm{Hom}(A, A)$, and (M2) from the composition arows $\varsigma_{ABC} : \mathrm{Hom}(A, B) + \mathrm{Hom}(B, C) \geq \mathrm{Hom}(A, C)$.

As a final matter, modular metric spaces represent a generalized form of metric spaces, related to the concept of a modular subspace in a vector space, which is explained in terms of real functionals known as (metric) modulars [5]. A modular metric space is defined as a pair (X, ω), where $\omega = (\omega_\lambda)_{\lambda \in \mathbb{R}^+}$ is a family of functions $\omega_\lambda : X \times X \to \mathbb{R}_0^+ = [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$:

- If $\omega_\lambda(x, y) = 0$ for all $\lambda \in \mathbb{R}^+$, then $x = y$.
- $\omega_\lambda(x, x) = 0$ for all $\lambda \in \mathbb{R}^+$.
- $\omega_{\lambda+\mu}(x, z) \leq \omega_\lambda(x, y) + \omega_\mu(y, z)$ for all $\lambda, \mu \in \mathbb{R}^+$.
- $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda \in \mathbb{R}^+$.

It is known that a modular metric $\omega = (\omega_\lambda)_{\lambda \in \mathbb{R}^+}$ is always nonincreasing with respect to λ, that is $\lambda \leq \mu$ implies $\omega_\lambda(x, y) \geq \omega_\mu(x, y)$.

3. Modulative categories

Similar to how a more general form of metric spaces can be obtained from categories enriched over the Lawvere quantale, we aim to achieve a categorical equivalence for modular metric spaces. One evident candidate for such an endeavor could be to take another quantale or make alternations based on composition, similar to the case of approach spaces. However, beyond these, another alternative approach is to manipulate the underlying concept of category, which would fundamentally change things. When examining modular metric spaces, the solution we found for adapting the approach from metric spaces was to label the morphisms in categories by positive real numbers and redefine composition accordingly, without altering the notion of an object. This approach led us to the new concept of a modulative category.

Although modular metric spaces feature a family of distance functions indexed by positive real numbers and satisfy a sort of generalized triangle inequality in a manner compatible with addition, it can actually be observed that this construction is structurally quite suitable for transferring beyond the additive semigroup of positive real numbers to any semigroup. Such a general framework is expected to yield more interesting and much broader results in a generalized categorical sense. Therefore, we situate this new concept on an arbitrary semigroup and we adopt the term modulative instead of modular for this more general case.

Intuitively, one can think of a modulative category as a category where a kind of semigroup grading is applied to its hom-sets, with the caveat that the graded parts need not be disjoint. This means that, they are not direct summands in general, and due to grading incompatibility, some compositions may not be defined even if the domain and codomain appear compatible. For a clearer understanding of this structure, we focus on the following formal definition and subsequent examples.

Definition 3.1. A **modulative category** $\mathscr{C}$ over a semigroup $(S,*)$ consists of a class $|\mathscr{C}|$ of objects and a set $\mathrm{Hom}_s(A,B)$ for each $s \in S$ and $A,B \in |\mathscr{C}|$, whose elements are called the morphisms, or more precisely $s-$**morphisms**. There are a family of operations (compositions)

$$\mathrm{Hom}_s(A,B) \times \mathrm{Hom}_t(B,C) \longrightarrow \mathrm{Hom}_{s*t}(A,C), \quad (f,g) \mapsto g \circ f$$

such that $(h \circ g) \circ f = h \circ (g \circ f)$ whenever they are defined, and it is required that the given family of composition operations satisfy the property called the **compositional consistency**: if there are morphisms g and f that lie in the intersection of different hom-sets, then for all operations where the expression $g \circ f$ is defined, it must yield the same result; more precisely, if $f \in \mathrm{Hom}_r(A,B) \cap \mathrm{Hom}_s(X,Y)$ and $g \in \mathrm{Hom}_t(B,C) \cap \mathrm{Hom}_u(Y,Z)$, then the pair (f,g) is mapped into both $\mathrm{Hom}_{r*t}(A,C)$ and $\mathrm{Hom}_{s*u}(X,Z)$, and it should take the same value $g \circ f \in \mathrm{Hom}_{r*t}(A,C) \cap \mathrm{Hom}_{s*u}(X,Z)$ in both cases. A modulative category $\mathscr{C}$ is called **small**, if its class of objects is a set.

The presence of identities is not required in a modulative category. If identity morphisms exist in the hom-sets for each element of the semigroup, we will say that this modulative category is unital. The formal definitions of unitality and some of its weaker forms are provided below.

Definition 3.2. A modulative category $\mathscr{C}$ is called **locally weakly unital** at an object A, if for an $s \in S$, there is a morphism $\mathrm{id}_A \in \mathrm{Hom}_s(A,A)$ such that $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_A \circ g = g$ for all morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow A$. More strongly, $\mathscr{C}$ is called **locally unital** at A, if $\mathrm{id}_A \in \mathrm{Hom}_s(A,A)$ for all $s \in S$. A modulative category that is locally weakly unital at every object is called **weakly unital**, and in particular, if it is locally unital at every object, then it is called **unital**.

Let $\mathscr{C}$ be a unital modulative category over a monoid $(M,*,e)$. In this case, due to the fact that $e * e = e$, the composition operations between the hom-sets corresponding to the identity element of the monoid, are of the form $\mathrm{Hom}_e(A,B) \times \mathrm{Hom}_e(B,C) \to \mathrm{Hom}_e(A,C)$. Therefore, if one only considers the $e-$morphisms and disregards the others, then a category in the usual sense is obtained. This clear relationship between categories and modulative categories is stated below without the need for a proof.

Proposition 3.3. *Each unital modulative category $\mathscr{C}$ over a monoid $(M,*,e)$, gives a category, whose objects are the objects of $\mathscr{C}$, and whose morphisms are the $e-$morphisms* *in $\mathscr{C}$. Conversely, each category $\mathcal{C}$ can be regarded as a modulative category over the trivial group $G = \{e\}$.*

We now present a series of potentially interesting examples to concretize the concept of a modulative category.

Example 3.4. If the objects of $\mathscr{C}$ are groups, the $\overline{0}-$morphisms are group homomorphisms, and the $\overline{1}-$morphisms are antihomomorphisms (i.e., functions $f : G \longrightarrow H$ that satisfy $f(ab) = f(b)f(a)$ for all $a, b \in G$), then it is easy to check that $\mathscr{C}$ is a modulative category over the group $(\mathbb{Z}_2, +)$. This modulative category is weakly unital, but not unital, since the identity homomorphisms are $\overline{0}-$morphisms, but they are not $\overline{1}-$morphisms for nonabelian groups.

Example 3.5. This time, consider the monoid $(\mathbb{Z}_2, \cdot, \overline{1})$ and choose the objects as topological spaces, with the $\overline{1}-$morphisms being continuous functions and the $\overline{0}-$morphisms being sequentially continuous functions. This gives a unital modulative category.

Example 3.6. It is possible to extend the previous example as follows: Let Ω be any set of cardinals. By assuming the Axiom of Choice, we ensure that every pair of cardinals in Ω is comparable, meaning that Ω is totally ordered. Now, consider ∞ as a formal element not belonging to the set Ω and assumed to be greater than all elements in Ω. Take the set $\Omega_\infty := \Omega \cup \{\infty\}$. Consider the monoid $(\Omega_\infty, \wedge, \infty)$, where $\wedge$ denotes the operation taking infimum (hence, in fact minimum). Let the objects of $\mathscr{C}$ be topological spaces, the $\infty-$morphisms be continuous functions, and for each cardinal α in Ω, the $\alpha-$morphisms be functions that preserve the limits of all nets whose directed set has cardinality at most α. In this case, $\mathscr{C}$ forms a unital modulative category.

Example 3.7. If the objects of $\mathscr{C}$ are sets, and for each positive integer n, the relation $\beta \subseteq A \times B$ is defined as an $n-$morphism from A to B, provided that $1 \leq \#\{b \in B : (a, b) \in \beta\} \leq n$ for all $a \in A$, where $\#$ denotes the cardinality of the set it precedes (and in this case, $1-$morphisms correspond to functions), then with composition of relations, $\mathscr{C}$ becomes a unital modulative category over the monoid $(\mathbb{Z}^+, \cdot, 1)$.

Example 3.8. Consider, as objects of $\mathscr{C}$, pairs (X, μ), where X is a set and $\mu : X \longrightarrow [0, 1]$ is a function, or in other words, μ is a fuzzy subset of X. For each $a \in [0, 1]$, we regard a function $f : X \longrightarrow Y$ as an $a-$morphism $f : (X, \mu) \longrightarrow (Y, \nu)$ in $\mathscr{C}$, if $\nu(f(x)) \geq a\mu(x)$ for all $x \in X$. In this case, $\mathscr{C}$ is a unital modulative category over the monoid $([0, 1], \cdot, 1)$.

Example 3.9. Each semigroup $(S, *)$ can be regarded as a modulative category over itself. To do this, we assume the existence of only one object, say O, and we view each $s \in S$ as an $s-$morphism $s : O \longrightarrow O$. If S is a monoid, then this modulative category becomes weakly unital.

Example 3.10. A one-object unital modulative category over a semigroup $(S, *)$, as the case of the categories, corresponds to a monoid. Moreover, if hom-sets are disjoint for different elements of S, then this monoid is $S-$graded.

Example 3.11. Let objects in $\mathscr{C}$ be all complex Hilbert spaces, and define the set of $\alpha-$objects, for each $\alpha \in [1, \infty)$, as bounded linear operators with operator norm at most α. Then $1-$morphisms are nonexpansive (also known as short) linear maps, and the composition of an $\alpha-$morphism and a $\beta-$morphism is an $\alpha\beta-$morphism, since operator norm always satisfies the inequality $\|ST\| \leq \|S\|\|T\|$. This gives a unital modulative category over the monoid $([1, \infty), \cdot, 1)$.

Example 3.12. Take the objects in $\mathscr{C}$ as all bipolar sets [10], and let $\overline{0}-$morphisms be covariant mappings, while $\overline{1}-$morphisms are contravariant mappings. This gives a weakly unital modulative category $\mathscr{C}$ over the group $(\mathbb{Z}_2, +)$, and a similar approach can be applied to the case of bipolar metric spaces through nonexpansive covariant and contravariant mappings. Alternatively, if one regards a covariant mapping with a Lipschitz constant α as an α-morphism, and a contravariant mapping with a Lipschitz constant α as a $-\alpha$-morphism, then this gives a weakly unital modulative category over the monoid $(\mathbb{R}, \cdot, 1)$.

Definition 3.13. An $s-$morphism $f : A \longrightarrow B$ in a modulative category which is locally weakly unital at A and B and over a semigroup $(S, *)$, is called an $s-$**isomorphism**, where $s \in S$, if there exists a $t-$morphism $g : B \longrightarrow A$ for some $t \in S$, such that $g \circ f = \mathrm{id}_A$, $f \circ g = \mathrm{id}_B$, and in this case g is denoted by f^{-1}.

By definition of a modulative category, when a morphism f is given, one might consider the risk that f could have two different inverses, one being an $s-$morphism and the other a $t-$morphism for some s and t in the underlyig semigroup. Therefore, it is important to know that the notation f^{-1} as given above is well-defined. The following proposition aims to clarify that no well-definedness issue exists.

Proposition 3.14. *Let $f : A \longrightarrow B$ be an $r-$morphism in a weakly unital modulative category $\mathscr{C}$ on a semigroup $(S, *)$, where $r \in S$. If $g \circ f = \mathrm{id}_A$, $f \circ g = \mathrm{id}_B$ for an $s-$morphism $g : B \longrightarrow A$ for $s \in S$ and $h \circ f = \mathrm{id}_A$, $f \circ h = \mathrm{id}_B$ for a $t-$morphism $h : B \longrightarrow A$ for $t \in S$, then $g = h$.*

Proof. Observe that the equalities

$$g = g \circ \mathrm{id}_B = g \circ (f \circ h) = (g \circ f) \circ h = \mathrm{id}_A \circ h = h$$

hold independently of the elements of the semigroup, by the compositional consistency. □

The following definition of an opposite modulative category is quite standard, but it will be used later to explain symmetry in modulative metric spaces.

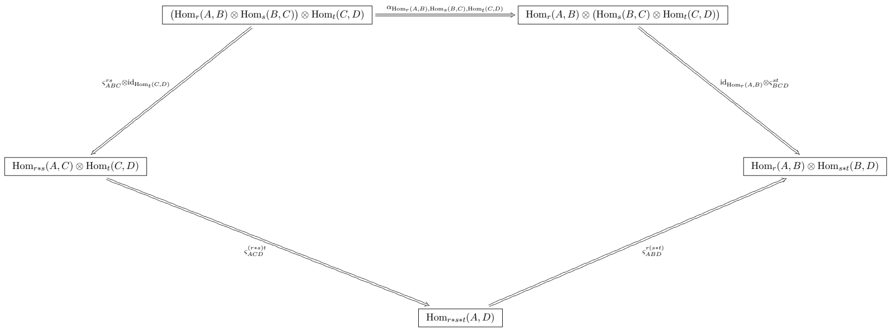
Definition 3.15. Let $\mathscr{C}$ be a modulative category on a semigroup $(S, *)$. The operation $\hat{*} : S \times S \to S$, $s \,\hat{*}\, t := t * s$ gives another semigroup $(S, \hat{*})$. In this case, the **opposite** of $\mathscr{C}$ is defined to be the modulative category $\mathscr{C}^{\mathrm{op}}$ over the semigroup $(S, \hat{*})$, having same objects as $\mathscr{C}$, but each $s-$morphism from an object A to an object B in $\mathscr{C}$ is viewed in $\mathscr{C}^{\mathrm{op}}$ in opposite direction, as an $s-$morphism from B to A.

In order to explain modular metric spaces in categorical terms, we now adapt the concept of enriched categories, which we briefly discussed in Section 2, to modulative categories.

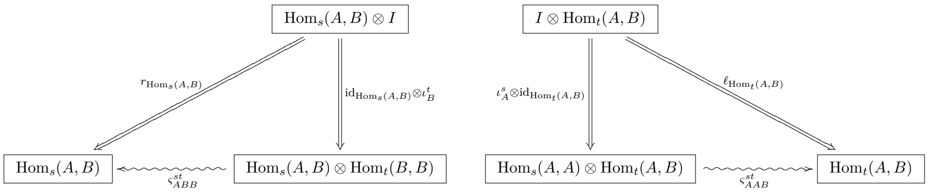
Definition 3.16. A modulative category $\mathscr{C}$ over a semigroup $(S, *)$ is said to be enriched over a monoidal category $(\mathcal{M}, \otimes, I)$ with the natural isomorphisms α_{LMN}, r_M and ℓ_M, if the hom-sets $\mathrm{Hom}_s(A, B)$ in $\mathscr{C}$ are also objects in $\mathcal{M}$ for all $s \in S$, and there are families of morphisms (respectively called the identity arrows and the composition arrows)

$$\iota^s_A : I \longrightarrow \mathrm{Hom}_s(A, A)$$
$$\varsigma^{st}_{ABC} : \mathrm{Hom}_s(A, B) \otimes \mathrm{Hom}_t(B, C) \longrightarrow \mathrm{Hom}_{s*t}(A, C)$$

in the monoidal category $\mathcal{M}$, for all given objects A, B and C in the category $\mathscr{C}$ and for all elements $s, t \in S$, such that the diagram



and the possible diagrams



commute in $\mathcal{M}$, whenever the wavy arrows are defined.

Given an $s-$morphism $f : A \longrightarrow B$ and a $t-$morphism $g : B \longrightarrow B$ in the diagram on the left. Then the composition $g \circ f : A \longrightarrow B$ is an $(s * t)-$morphism. If it is also an $s-$morphism, then the wavy arrow exists; otherwise, the diagram on the left does not impose any condition. Similarly, in the diagram on the right, the necessary and sufficient condition for the existence of the wavy arrow for given $f \in \mathrm{Hom}_s(A, A)$ and $g \in \mathrm{Hom}_t(A, B)$ is that $g \circ f \in \mathrm{Hom}_t(A, B) \cap \mathrm{Hom}_{s*t}(A, B)$. Noting that hom-sets with different indices are not necessarily disjoint and modulative categories possess the property of compositional consistency, these two diagrams make sense.

Small categories enriched over the quantale $\mathbf{2} = (\{0, 1\}, \leq, \cdot)$ correspond to preordered sets. Now, we explore which related concept small modulative categories enriched over $\mathbf{2}$ correspond to.

Suppose that $\mathscr{C}$ is a $\mathbf{2}-$enriched small modulative category over a semigroup $(S, *)$. Then $\mathrm{Hom}_s(A, B) \in \{0, 1\}$ for all $A, B \in |\mathscr{C}|$ and $s \in S$. In the case that $\mathrm{Hom}_s(A, B) = 1$, we use the notation $A \,\hat{\mathrm{s}}\, B$. Otherwise, if $\mathrm{Hom}_s(A, B) = 0$, we write $A \not{\hat{\mathrm{s}}}\, B$.

If we recall, as previously explained, that in the quantale $\mathbf{2}$, the arrows are denoted by $\leq$, the tensor product is multiplication, and the unit is 1, presence of identity arrows ι^s_A : $1 \leq \mathrm{Hom}_s(A, A)$ requires $\mathrm{Hom}_s(A, A) = 1$, meaning that $A \,\hat{\mathrm{s}}\, A$. Similarly, the composition arrows $\varsigma^{st}_{ABC} : \mathrm{Hom}_s(A, B) \cdot \mathrm{Hom}_t(B, C) \leq \mathrm{Hom}_{s*t}(A, C)$ implies that if both $\mathrm{Hom}_s(A, B)$ and $\mathrm{Hom}_t(B, C)$ are 1, that is, if $A \,\hat{\mathrm{s}}\, B$ and $B \,\hat{\mathrm{t}}\, C$ for $s, t \in S$, then $\mathrm{Hom}_{s*t}(A, C)$ must be 1, leading to the conclusion $A \,\widehat{\mathrm{s} * \mathrm{t}}\, C$.

Conversely, it is easily seen that a similar family of relations given on a set and satisfying the conditions

$$x \,\hat{\mathrm{s}}\, x$$

$$x \,\hat{\mathrm{s}}\, y \text{ and } y \,\hat{\mathrm{t}}\, z \implies x \,\widehat{\mathrm{s} * \mathrm{t}}\, z$$

corresponds to a modulative category over $(S, *)$, assuming that each hom-set contains exactly one morphism.

We now introduce some formal definitions related to this type of family of relations.

Definition 3.17. Let X be a set, $(S,*)$ be a semigroup and $\frown = \{\hat{s} : s \in S\}$ be a family of relations on X. Consider the properties

- $\forall s \in S,\ x\,\hat{s}\,x$ (reflexivity)
- $\forall s \in S,\ (x\,\hat{s}\,y \implies y\,\hat{s}\,x)$ (symmetry)
- $(\forall s \in S\ (x\,\hat{s}\,y\ \wedge\ y\,\hat{s}\,x)) \implies x = y$ (modulative antisymmetry)
- $\forall s,t \in S\ ((x\,\hat{s}\,y\ \wedge\ y\,\hat{t}\,z) \implies x\,\widehat{s*t}\,z)$ (modulative transitivity)

given for all $x,y,z \in X$. If the family $\frown$ is reflexive and modulative transitive, then we call it a **modulative preorder**. If it is also modulative antisymmetric, then we call it a **modulative order**. On the other hand a symmetric modulative preorder is called a **modulative equivalence**.

In this context, **2**−enriched small modulative categories correspond exactly to modulative preordered sets.

Modulative orders and modulative equivalences are potentially interesting in their own right and are likely to carry various insights related to the semigroup structure. Moreover, modulative equivalence classes such as $[x]_s = \{y \in X : x\,\hat{s}\,y\}$ can be defined for each $s \in S$, which will cover X but are not necessarily pairwise disjoint. However, since this is not the focus of this study, we do not explore any details concerning modulative relations and leave them to the interest of other researchers.

Now, we will investigate what kind of structure results from a small modulative category enriched over the quantale $[\infty,0] := ([0,\infty],\geq,+)$. As is well known, in the similar situation for classical categories, Lawvere metric spaces were obtained.

Let $\mathscr{C}$ be a $[\infty,0]$−enriched small modulative category over a semigroup $(S,*)$. Keep in mind that in the quantale $[\infty,0]$, the arrows are denoted by $\geq$, the tensor product is addition, and the unit is 0. In this case, the identity arrows $\iota_A^s : 0 \geq \mathrm{Hom}_s(A,A)$, force $\mathrm{Hom}_s(A,A) = 0$ for every $s \in S$. On the other hand, the existence of the composition arrows $\varsigma_{ABC}^{st} : \mathrm{Hom}_s(A,B) + \mathrm{Hom}_t(B,C) \geq \mathrm{Hom}_{s*t}(A,C)$ implies the inequality $\mathrm{Hom}_{s*t}(A,C) \leq \mathrm{Hom}_s(A,B) + \mathrm{Hom}_t(B,C)$, which can be interpreted as a weaker form of the triangle inequality, which we refer to as the **modulative triangle inequality**.

Conversely, given a semigroup $(S,*)$ and a set X, if for each $s \in S$ there exists a function $d_s : X \to [0,\infty]$, such that the properties

$$d_s(x,x) = 0,$$
$$d_{s*t}(x,z) \leq d_s(x,y) + d_t(y,z),$$

are satisfied for all $s,t \in S$, then we can construct a small modulative category enriched over the Lawvere quantale by defining the hom-sets as pairwise disjoint singletons. We will refer to such a metric-space-like structure as a **modulative Lawvere metric space**.

Based on all these observations and by also taking some inspiration from the definition of a modular metric, we can now introduce the concept of a modulative metric space.

Definition 3.18. Let X be a set, $(S,*)$ be a semigroup and $w = (w_s)_{s\in S}$ be a family of functions $w_s : X \times X \to \mathbb{R}_0^+$. Then the quadruple $(X,w,S,*)$ is called a **modulative metric space**, if the following hold for all $x,y,z \in X$.

(MM0) If $w_s(x,y) = 0$ for all $s \in S$, then $x = y$.
(MM1) $w_s(x,x) = 0$ for all $s \in S$.
(MM2) $w_{s*t}(x,z) \leq w_s(x,y) + w_t(y,z)$ for all $s,t \in S$
(MM3) $w_s(x,y) = w_s(y,x)$ for all $s \in S$.

Similar to the case of the metric spaces, the adjective extended and the prefixes pseudo and quasi are implemented to the word metric, respectively when infinity values are allowed, when (MM0) is dropped, and when (MM3) is dropped. Hence, a modulative Lawvere metric space is an modulative extended pseudo-quasi-metric space, and it satisfies (MM3), if for the $[\infty, 0]-$enriched category $\mathscr{C}$, corresponding to this Lawvere metric space, the equality $\mathscr{C} = \mathscr{C}^{\mathrm{op}}$ is satisfied.

Note also that, a modular metric space in the classical sense, is nothing other than a modulative metric space over the semigroup $(\mathbb{R}^+, +)$. This makes modulative metric spaces, a generalization of modular metric spaces.

4. Modulative metric spaces

In this section, we briefly discuss some of the intrinsic properties of modulative metric spaces, which were formally introduced in Definition 3.18 and whose connection to modulative categories was explained in the previous section.

First, let's present some examples of modulative metric spaces.

Example 4.1. Consider the semigroup $(\mathbb{Z}^+, +)$, and the family of functions $w_m : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}_0^+$,

$$w_m(x, y) = \frac{(x-y)^2}{m}$$

be given. In this case (MM0), (MM1) and (MM3) are readily satisfied. Note that none of the functions w_m itself satisfies the triangle inequality, and thus they are not metrics on $\mathbb{R}$. However they define a modulative metric. In fact, if we say $a := x - y$ and $b := y - z$, then

$$\begin{aligned} \frac{a^2}{m} + \frac{b^2}{n} - \frac{(a+b)^2}{m+n} &= \frac{a^2 n(m+n) + b^2 m(m+n) - a^2 mn - b^2 mn - 2abmn}{mn(m+n)} \\ &= \frac{a^2n^2 + b^2m^2 - 2abmn}{mn(m+n)} = \frac{(an-bm)^2}{mn(m+n)} \geq 0 \end{aligned}$$

gives $\frac{(x-z)^2}{m+n} \leq \frac{(x-y)^2}{m} + \frac{(y-z)^2}{n}$, that is $w_{m+n}(x, z) \leq w_m(x, y) + w_n(y, z)$, and thus (MM2) holds.

Example 4.2. For any given set X and a semigroup $(\mathbb{R}^+, *)$ on $\mathbb{R}^+$ such that $r * s \leq r + s$ for all $r, s \in \mathbb{R}$, the family of functions

$$w_r(x, y) = \begin{cases} r, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

gives a modulative metric space $(X, w, \mathbb{R}^+, *)$.

Example 4.3. It is easy to verify that $(\mathbb{R}, w, (0, 1), \cdot)$ is a modulative metric space, where the family w is defined by

$$w_\lambda(x, y) = \lambda|x - y|$$

for all $x, y \in X$ and $\lambda \in (0, 1)$.

Example 4.4. Let (X, d) be an arbitrary metric space, and define

$$w_r(x, y) = \max\{d(x, y) - r, 0\}$$

for each real number $r > 0$ and for all $x \in X$. Then $(X, w, \mathbb{R}^+, +)$ becomes a modulative metric space.

Now, we present a series of brief results that examine the interaction of a modulative metric space $(X, w, S, *)$ with certain properties of the underlying semigroup $(S, *)$. Given such a modulative (Lawvere) metric space, one can define the partial order

$$w_s \leq w_t \iff w_s(x, y) \leq w_t(x, y) \text{ for all } x, y \in X$$

on the family w.

Proposition 4.5. *If $(X, w, S, *)$ is a modulative metric space (or more generally, a modulative Lawvere metric space), then for every $s, t \in S$, it holds that $w_{s*t} \leq w_s$ and $w_{s*t} \leq w_t$.*

Proof. By Definition 3.18 (MM1)–(MM2), $w_{s*t}(x, y) \leq w_s(x, y) + w_t(y, y) = w_s(x, y)$ and $w_{s*t}(x, y) \leq w_s(x, x) + w_t(x, y) = w_t(x, y)$ is immediate. □

Proposition 4.6. *If $(X, w, S, *)$ is a modulative Lawvere metric space and $(S, *)$ has a right or left identity e, then $w_e \geq w_s$ for every $s \in S$.*

Proof. We assume that e is a right identity. Then $w_s = w_{s*e} \leq w_e$ is obtained by Proposition 4.5. The similar argument also applies for a left identity. □

Proposition 4.7. *If $(X, w, S, *)$ is a modulative metric space and $(S, *)$ has a right or left identity e, then w_e is a metric on X.*

Proof. (MM1) and (MM3) are equivalent to the corresponding metric axioms. The modulative triangle inequality (MM2) reduces to the classical one, since $w_e(x, z) = w_{e*e}(x, z) \leq w_e(x, y) + w_e(y, z)$. If $w_e(x, y) = 0$, then by Proposition 4.6, $w_s(x, y) \leq w_e(x, y) = 0$ for all $s \in S$, gives $x = y$ by (MM0). □

Proposition 4.8. *Given a modulative Lawvere metric space $(X, w, M, *)$ over a monoid $(M, *, e)$. If $m \in M$ has a right or left inverse, then $w_m = w_e$.*

Proof. We prove for the right inverses; the arguments are similar for the left inverses. If $\widetilde{m}$ is the a right inverse of m, then we have $w_e = w_{m*\widetilde{m}} \leq w_m$ by Proposition 4.5. On the other hand, $w_e \geq w_m$ by Proposition 4.6. From these, the equality follows. □

Since every element in a group has an inverse, this proposition suggests that studying modulative metric spaces over groups is not particularly meaningful. If $(G, *)$ is a group, then existence of inverses implies $w_g = w_h$ for every $g, h \in G$, meaning that instead of studying the modulative metric space $(X, w, G, *)$, it suffices to study the metric space (X, w_g) for any $g \in G$. Therefore, our special interest in modulative metrics focuses on semigroups that are not groups.

Definition 4.9. Let $(X, w, S, *)$ be a modulative metric space, (x_n) a sequence in X, and $x \in X$. The subset

$$\mathfrak{CD}(x_n, x) = \{s \in S : \lim_{n\to\infty} w_s(x_n, x) = 0\}$$

of S is called the **convergence domain** of (x_n) to x. If $\mathfrak{CD}(x_n, x) = S$, in other words, if $w_s(x_n, x) \to 0$ in $\mathbb{R}$ for every $s \in S$, then the sequence (x_n) is said to **converge** to x, and this is denoted by $(x_n) \to x$.

According to this definition, in a modulative metric space $(X, w, S, *)$, the necessary and sufficient condition for a sequence (x_n) to converge to a point $x \in X$ is that for every given $\varepsilon > 0$ and every $s \in S$, there exists a natural number $n_{0(s)}$ such that $n \geq n_{0(s)}$ implies $w_s(x_n, x) < \varepsilon$.

The stronger notion of convergence, obtained by replacing the values $n_{0(s)}$ with a single universal n_0 independent of the elements $s \in S$, is referred to as **synchroconvergence**. Accordingly, a sequence (x_n) synchroconverges to x, if for every $\varepsilon > 0$, there exists an n_0 such that $n \geq n_0$ implies $w_s(x_n, x) < \varepsilon$ for every $s \in S$.

We present an example of a sequence that is convergent but not synchroconvergent.

Example 4.10. Consider the set $\mathbb{R}^+$ of positive real numbers and the operation $p*q = \frac{pq}{p+q}$ given on $\mathbb{R}^+$. It is easy to check that $(\mathbb{R}^+, *)$ is a semigroup, and then $([0, 1], w, \mathbb{R}^+, *)$ is a modulative metric space, where $w_q(x, y) = |qx - qy|$ for all $x, y \in [0, 1]$ and $q \in \mathbb{R}^+$. Clearly $(\frac{1}{n}) \to 0$ by $\lim w_q(\frac{1}{n}, 0) = \lim |\frac{q}{n}| = 0$ as $n \to \infty$. However for $\varepsilon = 1$, to guarantee that $|\frac{q}{n}| < \varepsilon$, one must have $n > q$. Hence a universal $q-$independent n_0 must be greater than any $q \in \mathbb{R}^+$, which is impossible. Thus the convergent sequence $(\frac{1}{n})$ is not synchroconvergent.

Recall that for a semigroup $(S,*)$ and two given subsets $R,T \subseteq S$, the set $R*T$ is given by $R*T = \{r*t : r \in R,\, t \in T\}$. If a modulative metric space $(X,w,S,*)$ satisfies the condition

$$\text{(QMM0)}\ \left(\forall s \in S*S,\ w_s(x,y)=0\right) \implies x=y$$

or equivalently

$$\text{(QMM0)}\ \left(\forall s,t \in S,\ w_{s*t}(x,y)=0\right) \implies x=y$$

which is stronger than (MM0) since $S*S \subseteq S$, then this space is said to be **quadratically separated**.

As a particular case, if the semigroup $(S,*)$ has the property $S*S=S$, then (MM0) and (QMM0) are equivalent and then each modulative metric space over $(S,*)$ is quadratically separated.

We call a modulative metric space $(X,w,S,*)$ **idempotent**, if $S*S=S$. Under this definition, idempotence implies quadratically separatedness.

Since modular metric spaces are exactly the modulative metric spaces over the monoid $(\mathbb{R}^+,+,0)$, and $\mathbb{R}^+ + \mathbb{R}^+ = \mathbb{R}^+$, each modular metric space is idempotent. Moreover, if a semigroup $(S,*)$ has a right or left identity, then every modulative metric space over $(S,*)$ is idempotent.

Proposition 4.11. *Every convergent sequence in a quadratically separated modulative metric space has a unique limit.*

Proof. Let (x_n) be a sequence on a quadratically separated modulative metric space $(X,w,S,*)$ such that $(x_n) \to x_1 \in X$ and $(x_n) \to x_2 \in X$. Then by (MM2) and (MM3),

$$w_{s*t}(x_1,x_2) = \lim_{n\to\infty} w_{s*t}(x_1,x_2) \leq \lim_{n\to\infty} \left(w_s(x_1,x_n) + w_t(x_n,x_2)\right) = 0$$

on $\mathbb{R}$, for all $s,t \in S$ and $n \in \mathbb{Z}^+$. This means that $w_{s*t}(x_1,x_2)=0$ for all $s,t \in S$, and (QMM0) ensures $x_1 = x_2$. □

We provide a counterexample to show that in a modulative metric space that is not quadratically separated, a convergent sequence may have more than one limit.

Example 4.12. Inspired by the topological space known as the line with two origins, we construct the following structure: Consider the set $\mathbb{R}\setminus\{0\}$, and add two new elements, $\acute{0}$ and $\grave{0}$, to form the set $X := (\mathbb{R}\setminus\{0\}) \cup \{\grave{0},\acute{0}\}$. Define the function $w_1 : X\times X \to \mathbb{R}_0^+$ for every $a,b \in \mathbb{R}\setminus\{0\}$ as follows:

$$w_1(a,b)=|a-b| \qquad w_1(\grave{0},\acute{0}) = w_1(\acute{0},\grave{0}) = 1$$
$$w_1(\grave{0},\grave{0}) = w_1(\acute{0},\acute{0}) = 0 \qquad w_1(a,\acute{0}) = w_1(\acute{0},a) = w_1(a,\grave{0}) = w_1(\grave{0},a) = |a|$$

For other $k \in \mathbb{Z}^+$, where $k \geq 2$, define the functions $w_k : X\times X \to \mathbb{R}_0^+$ for every $a,b \in \mathbb{R}\setminus\{0\}$ as follows:

$$w_k(a,b)=|a-b| \qquad w_k(\grave{0},\acute{0}) = w_k(\acute{0},\grave{0}) = 0$$
$$w_k(\grave{0},\grave{0}) = w_k(\acute{0},\acute{0}) = 0 \qquad w_k(a,\acute{0}) = w_k(\acute{0},a) = w_k(a,\grave{0}) = w_k(\grave{0},a) = |a|$$

In this case, it can be observed that $(X,w,\mathbb{Z}^+,+)$ is a modulative metric space. In particular, note that the elements $\acute{0}$ and $\grave{0}$ satisfy the (MM0) axiom due to w_1. However, (QMM0) is not satisfied in this space because, although $\grave{0} \neq \acute{0}$, we have $m+n \geq 2$ for any $m,n \in \mathbb{Z}^+$, and hence $w_{m+n}(\grave{0},\acute{0}) = 0$. Therefore, $(X,w,\mathbb{Z}^+,+)$ is not quadratically separated. Now, consider the sequence $(x_n) = \left(\frac{1}{n}\right)$ on X. Since $\lim w_k(x_n,\acute{0}) = \lim w_k(x_n,\grave{0}) = \lim \frac{1}{n} = 0$ in $\mathbb{R}$ for every $k \in \mathbb{Z}^+$, the sequence (x_n) converges to both $\acute{0}$ and $\grave{0}$.

In Proposition 4.11, we stated that the uniqueness of limits is guaranteed in quadratically separated spaces, and in the example above, we have illustrated that the lack of this property can lead to multiple limits. However quadratic separation is not a necessary condition for the uniqueness of limits. Below, we present a minimalist example demonstrating that some modulative metric spaces that are not quadratically separated can still possess the property that every convergent sequence has a unique limit.

Example 4.13. Consider the set $X = \{a_1, a_2\}$. Then defining

$$\begin{array}{llll} \omega_{\bar{0}}(a_1,a_1)=0 & \omega_{\bar{0}}(a_1,a_2)=0 & \omega_{\bar{0}}(a_2,a_1)=0 & \omega_{\bar{0}}(a_2,a_2)=0 \\ \omega_{\bar{2}}(a_1,a_1)=0 & \omega_{\bar{2}}(a_1,a_2)=1 & \omega_{\bar{2}}(a_2,a_1)=1 & \omega_{\bar{2}}(a_2,a_2)=0 \end{array}$$

gives a modulative metric space $(X, w, 2\mathbb{Z}_4, \cdot)$. It does not satisfy (QMM0), since $\bar{0}\cdot\bar{0} = \bar{0}\cdot\bar{2} = \bar{2}\cdot\bar{0} = \bar{2}\cdot\bar{2} = \bar{0}$ and $\omega_{\bar{0}}(a_1, a_2) = 0$. Hence, the elements in $2\mathbb{Z}_4 \cdot 2\mathbb{Z}_4 = \{\bar{0}\}$ does not seperate a_1 and a_2, and thus $(X, w, 2\mathbb{Z}_4, \cdot)$ is not quadratically seperated. Nevertheless, each convergent sequence in this space has a unique limit, since in particular for $\bar{2} \in 2\mathbb{Z}_4$, $\lim w_{\bar{2}}(x_n, a_i) = 0$ implies that (x_n) must be eventually constant at a_i, for $i = 1, 2$.

Recall that a semigroup S is called divisible, if for every $s \in S$ and $n \in \mathbb{Z}^+$, there exists an $r \in S$ (which is not needed to be unique) such that the equality $r^n = s$ holds [19]. We call a modulative metric space $(X, w, S, *)$ **divisible**, if $(S, *)$ is a divisible semigroup. We note that, each divisible modulative metric space $(X, w, S, *)$ readily satisfies the equality $S*S = S$, and thus it is idempotent. Hence, we obtain the following flowchart of properties.

Divisible $\Longrightarrow$ Idempotent $\Longrightarrow$ Quadratically separated $\Longrightarrow$ Unique limits

In particular, each modular metric space is divisible, since the semigroup $(R^+, +)$ is divisible, having $\frac{x}{n}$ as an nth "root" of x for all $x \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$.

Definition 4.14. A sequence (x_n) on a modulative metric space $(X, w, S, *)$ is called a **Cauchy sequence**, if $\lim\limits_{m,n\to\infty} w_s(x_n, x_m) = 0$ on $\mathbb{R}$ for all $s \in S$.

In the following, we present an expected relationship between convergent sequences and Cauchy sequences in modulative metric spaces, but under the assumption of idempotency. The need for this additional condition will be clarified later with an example.

Proposition 4.15. *Every convergent sequence on an idempotent modulative metric space is a Cauchy sequence.*

Proof. Let $(x_n) \to x$ in an idempotent modulative metric space $(X, w, S, *)$. Then for a given $s \in S$, we can pick $r, t \in S$ such that $s = r * t$. Then

$$w_s(x_n, x_m) = w_{r*t}(x_n, x_m) \leq w_r(x_n, x) + w_t(x, x_m).$$

Taking limits on both sides forces $\lim\limits_{m,n\to\infty} w_s(x_n, x_m) = 0$. □

Now, we present a counterexample, in which, a convergent sequences is not a Cauchy sequences, in a non-idempotent modulative metric space.

Example 4.16. Consider the semigroup $(2\mathbb{Z}_4, \cdot)$ and define

$$w_{\bar{0}}(x, y) = |x - y|, \quad w_{\bar{2}}(x, y) = \begin{cases} 0, & \text{if } x = y \\ |x - y|, & \text{if } x = 0 \text{ or } y = 0 \\ 1, & \text{otherwise.} \end{cases}$$

for all $x, y \in [0, 1]$. Then $([0, 1], w, 2\mathbb{Z}_4, \cdot)$ is a modulative metric space. However, it is not idempotent, since there is no $r, s \in 2\mathbb{Z}_4$ such that $\bar{2} = r \cdot s$.

Consider the sequence $(x_n) = (\frac{1}{n})$ on $[0, 1]$.

$$\lim_{n\to\infty} w_{\bar{0}}(\tfrac{1}{n}, 0) = \lim_{n\to\infty} |\tfrac{1}{n}| = 0 \ \text{ and } \ \lim_{n\to\infty} w_{\bar{2}}(\tfrac{1}{n}, 0) = \lim_{n\to\infty} |\tfrac{1}{n}| = 0$$

gives $(x_n) \to 0$. On the other hand

$$\lim_{m,n\to\infty} w_{\bar{0}}(\tfrac{1}{m}, \tfrac{1}{n}) = \lim_{m,n\to\infty} |\tfrac{1}{m} - \tfrac{1}{n}| = 0, \text{ but} \lim_{m,n\to\infty} w_{\bar{2}}(\tfrac{1}{m}, \tfrac{1}{n}) \neq 0,$$

since the double sequence $w_{\bar{2}}\left(\frac{1}{m}, \frac{1}{n}\right)$ on $\mathbb{R}$, has all its terms equal to 1, except for those where $m = n$. Hence, the convergent sequence (x_n) is not Cauchy.

However, note that even if the given modulative metric space is not idempotent, it may still have the property that all convergent sequences are Cauchy sequences. It is easy to see that the modulative metric space provided in Example 4.13 is an example of this situation.

Given Proposition 4.16 and Example 4.16, as well as Proposition 4.11 and Example 4.12 keeping in mind that every idempotent modulative metric space is also quadratically separated, it is easy to see that idempotency is a critical property in the context of modulative metric spaces. The fundamental reason for this is that when idempotency does not hold, the function w_s cannot be included on the left side of the modulative triangle inequality for any $s \in S \setminus (S * S) \neq \varnothing$, and therefore, it essentially remains uncontrolled by not being subject to any structural upper bound. For these reasons, non-idempotent modulative metric spaces tend to be quite pathological, and it would not be surprising if future studies, especially those involving convergence, focus primarily on idempotent modulative metric spaces.

We provide a definition of completeness for modulative metric spaces, but it should be noted that in the case where the space is not idempotent, this completeness does not equate being a Cauchy sequence with being convergent. In this respect, this concept of completeness will, of course, be more meaningful in idempotent modulative metric spaces. Nevertheless, the definition is given in the most general terms because the concept of completeness, even under weaker assumptions, will be effectively used in this study to obtain some more general results, particularly in the context of fixed-point theory.

Definition 4.17. A modulative metric space $(X, w, S, *)$ is said to be **complete**, if every Cauchy sequence converges.

Another important topic in modulative metric spaces is the concept of continuity and the question of how maps should be defined. In this context, it is possible to limit our interest solely to transformations between modulative metric spaces defined over the same semigroup $(S, *)$. Such a fixed-base approach, which is also the most prominent choice in other areas such as module theory, is evidently less problematic in many cases. On the other hand, it is also possible to propose transformations that offer an acceptable transition mechanism between modulative metric spaces based on different semigroups. Below, we present some possible notions of continuity for both approaches.

Definition 4.18. Let $(X, w, S, *)$ and $(Y, v, S, *)$ be modulative metric spaces over the same semigroup $(S, *)$. A function $f : X \to Y$ is said to be **continuous** at a point $a \in X$ if, for every $\varepsilon > 0$ and for every $s \in S$, there exists a $\delta_s > 0$ such that for all $x \in X$, $w_s(x, a) < \delta_s$ implies $v_s(f(x), f(a)) < \varepsilon$. If f is continuous at every point $a \in X$, then f is simply said to be a continuous function. In particular, if δ can be selected independently of s, then f is called **synchrocontinous**.

More generally, if $(X, w, S, *)$ and $(Y, v, T, \#)$ are two modulative metric spaces, then a pair (f, φ) of functions $f : X \longrightarrow Y$ and $\varphi : S \to T$ is said to be a **continuous bimapping**, if f is continuous. Moreover, if (f, φ) is a continuous bimapping and φ is a homomorphism, then (f, φ) is called a **modulative bimapping**. We also use the terms **synchrocontinous bimapping** and **synchromodulative bimapping**, when the choice of δ is independent of $s \in S$.

The following proposition and its corollary, as expected, demonstrate that continuous bimappings preserve convergence of sequences.

Proposition 4.19. *Let $(X, w, S, *)$ and $(Y, v, T, \#)$ be two modulative metric spaces and let $(f, \varphi) : (X, w, S, *) \longrightarrow (Y, v, T, \#)$ be a continuous bimapping. Then $\varphi[\mathfrak{CD}(a_n, a)] \subseteq \mathfrak{CD}(f(a_n), f(a))$.*

Proof. Let $s \in \mathfrak{CD}(a_n, a)$. Then $\lim w_s(a_n, a) = 0$ on $\mathbb{R}$, as $n \to \infty$. Given a number $\varepsilon > 0$. Since f is continuous at a, there exists a $\delta_s > 0$ such that $v_{\varphi(s)}(f(x), f(x)) < \varepsilon$, whenever $w_s(x, a) < \delta$. Since $\lim w_s(a_n, a) = 0$ as $n \to \infty$, for this $\delta > 0$, there exists an $n_{0(s)} \in \mathbb{N}$ such that $n \geq n_{0(s)}$ implies $w_s(a_n, a) < \delta$, and thus $v_{\varphi(s)}(f(a_n), f(a)) < \varepsilon$. Given that ε is chosen arbitrarily, we have $\lim v_{\varphi(s)}(f(a_n), f(a)) = 0$ as $n \to \infty$, which in turn means that $\varphi(s) \in \mathfrak{CD}(f(a_n), f(a))$. ☐

The inclusion $\varphi[\mathfrak{CD}(a_n, a)] \subseteq \mathfrak{CD}(f(a_n), f(a))$ proven in the above, implies that if $\varphi[\mathfrak{CD}(a_n, a)] = S$, then $\mathfrak{CD}(f(a_n), f(a))$ must also be equal to S. Thus, the following result becomes evident.

Corollary 4.20. *If $(X, w, S, *)$ and $(Y, v, T, \#)$ are two modulative metric spaces and the bimapping $(f, \varphi) : (X, w, S, *) \longrightarrow (Y, v, T, \#)$ is continuous, then $(a_n) \to a$ on X implies $(f(a_n)) \to f(a)$ on Y.*

5. Some results in fixed point theory

To illustrate the applicability of the concept of a modulative metric space, which was introduced with a categorical-theoretic approach but developed further through a metric convergence theory, we finally aim to lay the groundwork for potential future research by presenting a couple of contraction-type fixed-point theorems in this section. To this end, we first redefine the concept of contraction within the framework of modulative metric spaces.

Definition 5.1. Given a modulative metric space $(X, w, S, *)$ and a function $T : X \longrightarrow X$. Then T is called a **contraction**, if for each $s \in S$, there is a constant $\lambda_s \in (0, 1)$ such that

$$w_s(Tx, Ty) \leq \lambda_s w_s(x, y) \tag{5.1}$$

for all $x, y \in X$.

According to this definition, note that the coefficients λ_s depend on the element $s \in S$ and do not necessarily have an upper bound within the open interval $(0, 1)$. In parallel with our terminology for convergence and continuity, if independently of the elements of the set S, there exists a universal constant λ for (5.1), which is valid for every $s \in S$, then we can refer to such a contraction as a **synchrocontraction**.

It is typically expected that contractions should be continuous, which is indeed the case also in modulative metric spaces.

Proposition 5.2. *Contractions on modulative metric spaces are continuous. In particular, each synchrocontraction is synchrocontinuous.*

Proof. Let $T : (X, w, S, *) \to (X, w, S, *)$ be a contraction satisfying (5.1). Given an $\varepsilon > 0$, and choose $\delta_s := \frac{\varepsilon}{\lambda_s}$ for each $s \in S$. If $w_s(x, y) < \delta_s$, then for every $x, y \in X$, we have $w_s(Tx, Ty) \leq \lambda_s w_s(x, y) < \lambda_s \delta_s = \varepsilon$. Therefore, T is continuous at all points of X.

Moreover, if λ is given independently of the elements $s \in S$, then we can choose δ as $\frac{\varepsilon}{\lambda}$, which is also independent of s, and this proves that every synchrocontraction is synchrocontinuous. ☐

We are now in a position to present and prove a generalization of the Banach fixed-point principle within the framework of modulative metric spaces.

Theorem 5.3. *Let $(X, w, S, *)$ be a divisible complete modulative metric space. Then each contraction $T : X \longrightarrow X$ has a unique fixed point.*

Proof. Take an arbitrary point $x_0 \in X$, and define $x_{n+1} := Tx_n$ for all $n \in \mathbb{N}$, so that $T^n(x_0) = x_n$ for all $n \in \mathbb{Z}^+$. Then by (5.1),

$$w_s(x_{n+1}, x_n) = w_s(Tx_n, Tx_{n-1}) \leq \lambda_s w_s(x_n, x_{n-1}) \leq \cdots \leq \lambda_s^n w_s(x_1, x_0)$$

for all $s \in S$ and $n \in \mathbb{Z}^+$.

Let $m, n \in \mathbb{Z}^+$. We assume without loss of generality that $m \geq n$. For $m = n$, $w_s(x_m, x_n) = 0$, and for $m > n$, say $k := m - n$. Since the semigroup $(S, *)$ is divisible, there is an $r \in S$ such that $r^k = s$, and thus

$$\begin{aligned} w_s(x_{n+k}, x_n) &= w_{r^k}(x_{n+k}, x_n) \\ &\leq w_r(x_{n+k}, x_{n+k-1}) + w_r(x_{n+k-1}, x_{n+k-2}) + \cdots + w_r(x_{n+1}, x_n) \\ &\leq \lambda_r^{n+k-1} w_r(x_1, x_0) + \lambda_r^{n+k-2} w_r(x_1, x_0) + \cdots + \lambda_r^n w_r(x_1, x_0) \\ &= \left(\sum_{\alpha=0}^{n+k-1} \lambda_r^\alpha - \sum_{\alpha=0}^{n-1} \lambda_r^\alpha \right) w_r(x_1, x_0) \\ &= \left(\frac{1 - \lambda_r^{n+k}}{1 - \lambda_r} - \frac{1 - \lambda_r^n}{1 - \lambda_r} \right) w_r(x_1, x_0) \\ &= \frac{\lambda_r^n (1 - \lambda_r^k)}{1 - \lambda_r} w_r(x_1, x_0) \leq \frac{\lambda_r^n}{1 - \lambda_r} w_r(x_1, x_0). \end{aligned}$$

In this case, we have

$$w_s(x_m, x_n) = w_s(x_{n+(m-n)}, x_n) \leq \frac{\lambda_r^n}{1 - \lambda_r} w_r(x_1, x_0)$$

and taking limits in both sides gives

$$\lim_{m,n\to\infty} w_s(x_m, x_n) \leq \lim_{m,n\to\infty} \frac{\lambda_r^n}{1 - \lambda_r} w_r(x_1, x_0) = 0$$

Hence (x_n) is a Cauchy sequence and by the completeness of $(X, w, S, *)$, (x_n) converges to a point $a \in X$, and also does its subsequence (x_{n+1}). However, at the same time we have by Proposition 5.2 and Corollary 4.20 that $(x_{n+1}) = (Tx_n) \to Ta$, and this gives $Ta = a$ by Proposition 4.11. Therefore T possesses a fixed point.

Now, to show the uniqueness, we assume that a_1 and a_2 are two fixed points of T. Then, for all $s \in S$,

$$w_s(a_1, a_2) = w_s(Ta_1, Ta_2) \leq \lambda_s w_s(a_1, a_2),$$

and $\lambda_s \in (0, 1)$ gives $w_s(a_1, a_2) = 0$. Hence $a_1 = a_2$. □

As a final result in this section, we present the following fixed-point theorem for modulative metric spaces.

Theorem 5.4. *Let $(X, w, S, *)$ be a divisible complete modulative metric space. If*

$$w_s(Tx, Ty) \leq \rho \left[w_{s^2}(Tx, x) + w_{s^2}(Ty, y) \right] \tag{5.2}$$

for all $x, y \in X$ and for all $s \in S$, with $\rho \in \left[0, \frac{1}{2}\right)$, then $T : X \longrightarrow X$ has a unique fixed point in X. Additionally, for any $x \in X$, $(T^n x)$ converges to the fixed point.

Proof. We choose an arbitrary point $x_0 \in X$, and describe the iterative sequence (x_n) by $x_n = Tx_{n-1}$. Clearly $T^n x_0 = x_n$ for all $n \in \mathbb{Z}^+$. If $Tx_{n_0} = Tx_{n_0 - 1}$ for some $n_0 \in \mathbb{Z}^+$, then we obtain that $Tx_{n_0} = x_{n_0}$. In this case x_{n_0} will be a fixed point of T. Say $Tx_n \neq Tx_{n-1}$

for all $n \in \mathbb{Z}^+$. Then by (5.2),

$$\begin{aligned} w_s(x_{n+1}, x_n) &= w_s(Tx_n, Tx_{n-1}) \\ &\leq \rho\left[w_{s^2}(Tx_n, x_n) + w_{s^2}(Tx_{n-1}, x_{n-1})\right] \\ &\leq \rho\left[w_s(Tx_n, x_n) + w_s(x_n, x_n) + w_s(Tx_{n-1}, x_{n-1}) + w_s(x_{n-1}, x_{n-1})\right] \\ &= \rho\left[w_s(x_{n+1}, x_n) + w_s(x_n, x_{n-1})\right] \end{aligned}$$

for all $s \in S$ and $n \in \mathbb{Z}^+$, where $\rho \in \left[0, \frac{1}{2}\right)$. Hence, we obtain that

$$w_s(x_{n+1}, x_n) \leq \frac{\rho}{1-\rho} w_s(x_n, x_{n-1})$$

for all $s \in S$ and $n \in \mathbb{Z}^+$. Say $\gamma := \frac{\rho}{1-\rho}$, then $\gamma \in [0, 1)$. Hence,

$$\begin{aligned} w_s(x_{n+1}, x_n) &\leq \gamma w_s(x_n, x_{n-1}) \\ &\leq \gamma^2 w_s(x_{n-1}, x_{n-2}) \\ &\;\;\vdots \\ &\leq \gamma^n w_s(x_1, x_0) \end{aligned}$$

for all $s \in S$ and $n \in \mathbb{Z}^+$. Similar to the proof of Theorem 5.3, we can observe that (x_n) is a Cauchy sequence, so that by the completeness of $(X, w, S, *)$, (x_n) converges to a point $\xi \in X$, and ξ is a fixed point of T. In order to establish the uniqueness, we assume that ξ_1 and ξ_2 are two distinct fixed points of T. Then by (5.2),

$$w_s(\xi_1, \xi_2) = w_s(T\xi_1, T\xi_2) \leq \rho\left[w_{s^2}(T\xi_1, \xi_1) + w_{s^2}(T\xi_2, \xi_2)\right] = 0$$

for all $s \in S$, which gives $\xi_1 = \xi_2$. □

6. Conclusion

In this study, the new concepts of modulative categories and modulative metric spaces along with numerous auxiliary notions, have been presented for the first time. Various networks of relationships have been identified, and to demonstrate the adaptability of the concept of a modulative metric to potential applications, a couple of well-known fixed-point theorems have been generalized, thereby opening the doors to many new areas of research.

To keep the study concise and focused, many details have been left out of the main text, leading to the emergence of numerous research topics and questions. One such topic is the concept of modulative topology, which is expected to offer a new perspective and area of study. However, findings related to this concept, which exceed the scope of this work, are planned to be addressed in an individual follow-up study.

The fixed-point theorems presented here initiate the study of fixed-point theory in modulative metric spaces. Given the applicability of fixed-point theorems concerning modular metric spaces, it is expected that modulative metric spaces may find a potentially broader range of applications in various fields. On the other hand, it is known that modular metrics are closely related to functions called metric modular. Similarly, it can be anticipated that there may be analogous functions, which we might call metric modulations, corresponding to modulative metric spaces.

Other structures not detailed here but which we believe are worth investigating include modulative equivalence relations and modulative partially ordered sets. These structures may lead to interesting results and constructions. Additionally the modulative categories and, though its significance is uncertain, the potential concept of a modulative ultrametric space could possibly give rise to some definitions, generalizations or results on the graph theory, and especially on trees.

Finally, the concept of modulative categories, which forms the backbone of this study, has been discussed only as needed, without delving into some of the prominent concepts in category theory. Even fixed-base or variable-base covariant and contravariant functors for modulative categories have not been introduced yet. In this respect, a multitude of unknowns await researchers' attention in the field of modulative categories. Considering that modulative categories are based on semigroups and are thus related to the category of semigroups, another open question is whether a similar structure could be generalized to a broader family of categories, beyond semigroups.

Conflicts of Interest

The authors declare no conflicts of interest.

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