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Generalized Statistical Convergence via Modulus Function in Octonion Valued *b*-Metric Spaces

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Abstract: The key ideas of summability theory have been the subject of extensive investigation in recent years in a variety of metric space extensions. Octonion-valued metric spaces are based on modifying the triangle inequality of a semi-metric space by multiplying one side of the inequality by a scalar b. This new generalisation of metric spaces is very interesting since octonions are not even a ring since they do not have the associative property of multiplication and the spaces do not satisfy the standard triangle inequality. We are prompted by this to study the notions of strong $\mathcal{J}$-Cesàro summability, $\mathcal{J}$-statistical convergence, $\mathcal{J}$-lacunary statistical convergence, and similar notions that respect the modulus function in octonion valued b-metric spaces, an extension of metric spaces. We also examine the connections among these ideas.

Keywords: Ideal convergence, Statistical convergence, Prime numbers, Octonion valued b-metric space, Modulus function

1. Introduction

With important contributions to computer science, topology, functional analysis, Fourier analysis, applied mathematics, mathematical modeling, and measure theory, sequence convergence and summability theory study has long been an important area in pure mathematics. The concept of statistical convergence of sequences has gained importance in recent years. Zygmund first referred to statistical convergence as "almost convergence" in the first edition of his well-known 1935 monograph (see [54]). Steinhaus later introduced the concept [44]. This development sparked a number of statistical convergence research. Fast introduced the idea of statistical convergence and its characteristics for the first time in 1951 [13]. It was later represented by Schoenberg [44], who described some basic properties of statistical convergence and studied statistical convergence as a summability technique. Fridy [18] and Šalát [38] also examined some intriguing aspects of statistical convergence (also see [22,23,29,30,42,49]). Introduced by Nuray and Ruckle in [34], who referred to it as generalized statistical convergence as a generalization of statistical convergence, and independently by Kostyrko, Salát, and Wilczyński in [27], the idea of ideal convergence (or $\mathcal{J}$-convergence) of real sequences offers a unifying perspective on various types of convergence related to statistical convergence. In his doctoral thesis, Gürdal [20] presented the ideal Cauchy sequence and its characteristics. $\mathcal{J}$-convergence, $\mathcal{J}$-Cauhy sequence, and related subjects have been the subject of several publications over the past 20 years; for further information, check [31,32,37,40,41,43,50,51,52], etc. Recently, Das et al. [9] introduced new concepts such as $\mathcal{J}$-statistical convergence and $\mathcal{J}$-lacunary statistical convergence within the framework of ideals. In mathematics, metric spaces are crucial, particularly in topology and analysis. The concept of a modulus function was introduced by Nakano [33] in 1953. In 1973, Ruckle [36] and in 1993, Maddox [28] defined certain sequence spaces using the modulus function. Connor [8] investigated the relationships between statistical convergence and strongly Cesaro summability with respect to the modulus function. For more details, refer to [2]. In the following years, the modulus function has been explored by numerous researchers, among which an important study was conducted by Gürdal and Özgür in 2015. In [21], the concepts of $\mathcal{J}^f$-statistical convergence and $\mathcal{J}^f$-Cauchy, which respect the modulus function, were introduced and the relationships between these concepts were examined in normed spaces. Fréchet [17] initially proposed the idea of metric space in 1906. Since then, the generalization of metric spaces has attracted the attention of several scholars, who have written numerous articles on the topic [10, 19, 25, 53]. The idea of octane-valued b-metric space, which was put up by Çetin et al. in 2025 [5] as a generalization of octane metrics-a logical extension of complex and quaternion-valued metrics as well as the traditional concept of metric-is one outcome of their investigations. Examining ideal convergence and its characteristics in octane valued b-metric spaces is the goal of this research.

Soon after quaternions were discovered in 1843, John T. Graves created octonions. Arthur Cayley later expanded and improved this idea on his own. A systematic extension in hypercomplex number theory controlled by the Cayley-Dickson structure is demonstrated by the evolution from real numbers to complex numbers, then to quaternions, and eventually to octonions. Because of their unique mathematical characteristics, octonions stand out in this sequence. Octonions are neither commutative nor associative, in contrast to real and complex numbers, which are commutative, and quaternions, which are non-commutative but nonetheless associative. In addition to their theoretical significance, octonions' special non-associative properties are useful in applications that need to manage multidimensional data relationships. Octonions have been employed in physics to create duality-invariant field equations for dyons, according to Kansu et al. [24]. These equations effectively represent electricmagnetic dualities, much like Maxwell's equations do. Eight-dimensional octonions' multicomponent character unifies the intricate interactions between electric and magnetic components. Octonions have emerged as a practical tool for processing and expressing high-dimensional data in the field of machine learning. Deep octonion networks (DONs), which incorporate multidimensional characteristics into various layers of neural networks by taking use of the compact structure of octonions, were first presented by Wu et al. [47]. Octonions facilitate effective data representation and processing in this context; activities like picture classification exhibit enhanced performance and convergence. Accordingly, octonions' associative and non-commutative characteristics have enabled creative applications in contemporary theoretical physics, artificial intelligence, and control systems where multidimensionality and flexible data representation are essential, despite initially posing difficulties for conventional algebraic applications. Acoording to [4,6,11,12,35,48], octonions, their subalgebraic structure, and multidisciplinary applications are covered in detail.

In this study, we extend certain fundamental ideas, such ideal convergence, statistical convergence and convergence, to octonion valued b-metric spaces, which were initially created by Çetin et al. [5], Savaş et al. [39] and Kişi et al. [26]. We present the essential ideas associated with this special mathematical structure, such as $\mathcal{I}$-statistical convergence, $\mathcal{I}$-lacunary statistical convergence, by developing a partial order relation on octonions via modulus function. We may investigate these ideas' characteristics and the relationships between them since they are generalized in the context of octonion valued b-metric spaces. Furthermore, we investigate the effects of the non-associative structure of octonions on the behavior of strong $\mathcal{I}$-Cesàro summability, $\mathcal{I}$-statistical convergence, $\mathcal{I}$-lacunary statistical convergence, and similar notions that respect the modulus function. This unique feature highlights the new opportunities and challenges that octonions bring in mathematical research in addition to providing more insight into convergence theory.

2. Material and Method

In the follow-up we will to examine on O, Octonions, a non-associative generalization of the division algebra of quaternions.

Now, we will begin by extending the basis elements of quaternions, represented as $\{1, i, j, k\}$, by incorporating an additional basis element ℓ. This extension enables us to construct the eightdimensional octonion division algebra in detail, as described in [16], including its diagrammatic representation and algebraic operations.

Thus, from [5, Diagram 1], each element $\mathfrak{x} \in O$ can be expressed in the form:

$$\mathfrak{x} = \mathrm{o}_0 + \mathrm{o}_1 e_1 + \mathrm{o}_2 e_2 + \mathrm{o}_3 e_3 + \mathrm{o}_4 e_4 + \mathrm{o}_5 e_5 + \mathrm{o}_6 e_6 + \mathrm{o}_7 e_7, \mathrm{o}_n \in \mathbb{R},$$

where $n = 0,1,2,3,4,5,6,7$. The basis elements of O are $1, e_1, e_2, e_3, e_4, e_5, e_6, e_7$. The detailed multiplication of these basis elements is shown in the table below.

$\cdot$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

The conjugate element $\bar{\mathfrak{x}}$ is given by

$$\bar{\mathfrak{x}} = \mathrm{o}_0 - \mathrm{o}_1 e_1 - \mathrm{o}_2 e_2 - \mathrm{o}_3 e_3 - \mathrm{o}_4 e_4 - \mathrm{o}_5 e_5 - \mathrm{o}_6 e_6 - \mathrm{o}_7 e_7.$$

The norm of an arbitrary octonion is calculated as

$$\|\mathfrak{x}\| = \sqrt{\mathfrak{x} \cdot \bar{\mathfrak{x}}} = \sqrt{\mathrm{o}_0^2 + \mathrm{o}_1^2 + \mathrm{o}_2^2 + \mathrm{o}_3^2 + \mathrm{o}_4^2 + \mathrm{o}_5^2 + \mathrm{o}_6^2 + \mathrm{o}_7^2}.$$

Additionally, the inverse of an arbitrary octonion $\mathfrak{x}$ is given in the form

$$\mathfrak{x}^{-1} = \frac{\bar{\mathfrak{x}}}{\|\mathfrak{x}\|^2}$$

Any quaternion's imaginary part can be represented as a vector in three-dimensional Euclidean space, analogous to a movement vector, with its real part indicating the time of this movement. Similarly, octonions can be redefined in a seven-dimensional Euclidean space as a pair consisting of a scalar and a vector, allowing for a different perspective. While quaternions differ from real and complex numbers in their non-commutative multiplication, octonions, as a more complex structure, lose the associative property from the group axioms in multiplication. Consequently, division algebra over octonions becomes non-associative, adding to its intriguing properties.

We can represent octonions as an ordered set of eight real numbers $(\mathrm{o}_0, \mathrm{o}_1, \mathrm{o}_2, \mathrm{o}_3, \mathrm{o}_4, \mathrm{o}_5, \mathrm{o}_6, \mathrm{o}_7)$ with coordinate-wise addition and multiplication defined by a specific table. Here, the first component, o_0, is called the real part, while the remaining seven-tuple $(\mathrm{o}_1, \mathrm{o}_2, \mathrm{o}_3, \mathrm{o}_4, \mathrm{o}_5, \mathrm{o}_6, \mathrm{o}_7)$ constitutes the imaginary part.

Thus, as noted above, any quaternion can be written in the form $(\mathrm{o}_0, \vec{u})$, where $\vec{u} = (\mathrm{o}_1, \mathrm{o}_2, \mathrm{o}_3, \mathrm{o}_4, \mathrm{o}_5, \mathrm{o}_6, \mathrm{o}_7)$ and o_0 represents the real part. From here, the following properties can be easily observed:

$$\begin{aligned} \mathfrak{x} &:= (\mathrm{o}_0, \vec{u}), \vec{u} \in \mathbb{R}^7;\ \mathrm{o}_0 \in \mathbb{R} \\ &= \left(\mathrm{o}_0, (\mathrm{o}_1, \mathrm{o}_2, \mathrm{o}_3, \mathrm{o}_4, \mathrm{o}_5, \mathrm{o}_6, \mathrm{o}_7)\right);\ \mathrm{o}_0, \mathrm{o}_1, \mathrm{o}_2, \mathrm{o}_3, \mathrm{o}_4, \mathrm{o}_5, \mathrm{o}_6, \mathrm{o}_7 \in \mathbb{R} \\ &= \mathrm{o}_0 + \mathrm{o}_1 e_1 + \mathrm{o}_2 e_2 + \mathrm{o}_3 e_3 + \mathrm{o}_4 e_4 + \mathrm{o}_5 e_5 + \mathrm{o}_6 e_6 + \mathrm{o}_7 e_7. \end{aligned}$$

Now, let us define a partial ordering relation $\preceq$ on the non-associative and non-commutative octonion algebra O as follows.

$\mathfrak{x} \preceq \mathfrak{x}'$ if and only if $\mathrm{Re}(\mathfrak{x}) \leq \mathrm{Re}(\mathfrak{x}'), \mathrm{Im}_e(\mathfrak{x}) \leq \mathrm{Im}_e(\mathfrak{x}'), \mathfrak{x}, \mathfrak{x}' \in \mathbb{H}; e = e_1, e_2, e_3, e_4, e_5, e_6, e_7$, where $\mathrm{Im}_{e_n} = o_n; n = 1,2,3,4,5,6,7$. To confirm that it is $\mathfrak{x} \preceq \mathfrak{x}'$, satisfying any one of the 256 conditions derived from the sum of all possible combinations of 8, from 0 to 8 in respectively, will suffice. Obtained from the 0 -combinations of 8, meaning none of its components are equal;this 1 case constitute

(1) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,2,3,4,5,6,7$.

Obtained from the 2-combinations of 8, meaning only one component is equal; these 8 cases constitute

(2) $\mathrm{Re}(\mathfrak{x}) = \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,2,3,4,5,6,7$.
(3) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 2,3,4,5,6,7; \mathrm{Im}_{e_1}(\mathfrak{x}) = \mathrm{Im}_{e_1}(\mathfrak{x}')$.
(4) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,3,4,5,6,7; \mathrm{Im}_{e_2}(\mathfrak{x}) = \mathrm{Im}_{e_2}(\mathfrak{x}')$.
(5) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,2,4,5,6,7; \mathrm{Im}_{e_3}(\mathfrak{x}) = \mathrm{Im}_{e_3}(\mathfrak{x}')$.
(6) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,2,3,5,6,7; \mathrm{Im}_{e_4}(\mathfrak{x}) = \mathrm{Im}_{e_4}(\mathfrak{x}')$.
(7) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,2,3,4,6,7; \mathrm{Im}_{e_5}(\mathfrak{x}) = \mathrm{Im}_{e_5}(\mathfrak{x}')$.
(8) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,2,3,4,5,7; \mathrm{Im}_{e_6}(\mathfrak{x}) = \mathrm{Im}_{e_6}(\mathfrak{x}')$.
(9) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,2,3,4,5,6; \mathrm{Im}_{e_7}(\mathfrak{x}) = \mathrm{Im}_{e_7}(\mathfrak{x}')$.

Obtained from the 2-combinations of 8, meaning only two components are equal; these 27 cases constitute
(10) $\mathrm{Re}(\mathfrak{x}) = \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 2,3,4,5,6,7; \mathrm{Im}_{e_1}(\mathfrak{x}) = \mathrm{Im}_{e_1}(\mathfrak{x}')$.
(11) $\mathrm{Re}(\mathfrak{x}) = \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,3,4,5,6,7; \mathrm{Im}_{e_2}(\mathfrak{x}) = \mathrm{Im}_{e_2}(\mathfrak{x}')$.
(12) $\mathrm{Re}(\mathfrak{x}) = \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,2,4,5,6,7; \mathrm{Im}_{e_3}(\mathfrak{x}) = \mathrm{Im}_{e_3}(\mathfrak{x}')$.
(13) $\mathrm{Re}(\mathfrak{x}) = \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,2,3,5,6,7; \mathrm{Im}_{e_4}(\mathfrak{x}) = \mathrm{Im}_{e_4}(\mathfrak{x}')$.
(14) $\mathrm{Re}(\mathfrak{x}) = \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,2,3,4,6,7; \mathrm{Im}_{e_5}(\mathfrak{x}) = \mathrm{Im}_{e_5}(\mathfrak{x}')$.
(15) $\mathrm{Re}(\mathfrak{x}) = \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,2,3,4,5,7; \mathrm{Im}_{e_6}(\mathfrak{x}) = \mathrm{Im}_{e_6}(\mathfrak{x}')$.
(16) $\mathrm{Re}(\mathfrak{x}) = \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,2,3,4,5,6; \mathrm{Im}_{e_7}(\mathfrak{x}) = \mathrm{Im}_{e_7}(\mathfrak{x}')$.
(17) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 3,4,5,6,7; \mathrm{Im}_{e_1}(\mathfrak{x}) = \mathrm{Im}_{e_1}(\mathfrak{x}'); \mathrm{Im}_{e_2}(\mathfrak{x}) = \mathrm{Im}_{e_2}(\mathfrak{x}')$.
(18) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 2,4,5,6,7; \mathrm{Im}_{e_1}(\mathfrak{x}) = \mathrm{Im}_{e_1}(\mathfrak{x}'); \mathrm{Im}_{e_3}(\mathfrak{x}) = \mathrm{Im}_{e_3}(\mathfrak{x}')$.
(19) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 2,3,5,6,7; \mathrm{Im}_{e_1}(\mathfrak{x}) = \mathrm{Im}_{e_1}(\mathfrak{x}'); \mathrm{Im}_{e_4}(\mathfrak{x}) = \mathrm{Im}_{e_4}(\mathfrak{x}')$.
(20) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 2,3,4,6,7; \mathrm{Im}_{e_1}(\mathfrak{x}) = \mathrm{Im}_{e_1}(\mathfrak{x}'); \mathrm{Im}_{e_5}(\mathfrak{x}) = \mathrm{Im}_{e_5}(\mathfrak{x}')$.
(21) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 2,3,4,5,7; \mathrm{Im}_{e_1}(\mathfrak{x}) = \mathrm{Im}_{e_1}(\mathfrak{x}'); \mathrm{Im}_{e_6}(\mathfrak{x}) = \mathrm{Im}_{e_6}(\mathfrak{x}')$.
(22) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 2,3,4,5,6; \mathrm{Im}_{e_1}(\mathfrak{x}) = \mathrm{Im}_{e_1}(\mathfrak{x}'); \mathrm{Im}_{e_7}(\mathfrak{x}) = \mathrm{Im}_{e_7}(\mathfrak{x}')$.
(23) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 1,4,5,6,7; \mathrm{Im}_{e_2}(\mathfrak{x}) = \mathrm{Im}_{e_2}(\mathfrak{x}'); \mathrm{Im}_{e_3}(\mathfrak{x}) = \mathrm{Im}_{e_3}(\mathfrak{x}')$.
(24) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 1,3,5,6,7; \mathrm{Im}_{e_2}(\mathfrak{x}) = \mathrm{Im}_{e_2}(\mathfrak{x}'); \mathrm{Im}_{e_4}(\mathfrak{x}) = \mathrm{Im}_{e_4}(\mathfrak{x}')$.
(25) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 1,3,4,6,7; \mathrm{Im}_{e_2}(\mathfrak{x}) = \mathrm{Im}_{e_2}(\mathfrak{x}'); \mathrm{Im}_{e_5}(\mathfrak{x}) = \mathrm{Im}_{e_5}(\mathfrak{x}')$.
(26) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 1,3,4,5,7; \mathrm{Im}_{e_2}(\mathfrak{x}) = \mathrm{Im}_{e_2}(\mathfrak{x}'); \mathrm{Im}_{e_6}(\mathfrak{x}) = \mathrm{Im}_{e_6}(\mathfrak{x}')$.
(27) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 1,3,4,5,6; \mathrm{Im}_{e_2}(\mathfrak{x}) = \mathrm{Im}_{e_2}(\mathfrak{x}'); \mathrm{Im}_{e_7}(\mathfrak{x}) = \mathrm{Im}_{e_7}(\mathfrak{x}')$.
(28) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 1,2,5,6,7; \mathrm{Im}_{e_3}(\mathfrak{x}) = \mathrm{Im}_{e_3}(\mathfrak{x}'); \mathrm{Im}_{e_4}(\mathfrak{x}) = \mathrm{Im}_{e_4}(\mathfrak{x}')$.

(29) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 1,2,4,6,7; \mathrm{Im}_{e_3}(\mathfrak{x}) = \mathrm{Im}_{e_3}(\mathfrak{x}'); \mathrm{Im}_{e_5}(\mathfrak{x}) = \mathrm{Im}_{e_5}(\mathfrak{x}')$.
(30) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 1,2,4,5,7; \mathrm{Im}_{e_3}(\mathfrak{x}) = \mathrm{Im}_{e_3}(\mathfrak{x}'); \mathrm{Im}_{e_6}(\mathfrak{x}) = \mathrm{Im}_{e_6}(\mathfrak{x}')$.
(31) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 1,2,4,5,6; \mathrm{Im}_{e_3}(\mathfrak{x}) = \mathrm{Im}_{e_3}(\mathfrak{x}'); \mathrm{Im}_{e_7}(\mathfrak{x}) = \mathrm{Im}_{e_7}(\mathfrak{x}')$.
(32) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 1,2,3,6,7; \mathrm{Im}_{e_4}(\mathfrak{x}) = \mathrm{Im}_{e_4}(\mathfrak{x}'); \mathrm{Im}_{e_5}(\mathfrak{x}) = \mathrm{Im}_{e_5}(\mathfrak{x}')$.
(33) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 1,2,3,5,7; \mathrm{Im}_{e_4}(\mathfrak{x}) = \mathrm{Im}_{e_4}(\mathfrak{x}'); \mathrm{Im}_{e_6}(\mathfrak{x}) = \mathrm{Im}_{e_6}(\mathfrak{x}')$.
(34) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 1,2,3,5,6; \mathrm{Im}_{e_4}(\mathfrak{x}) = \mathrm{Im}_{e_4}(\mathfrak{x}'); \mathrm{Im}_{e_7}(\mathfrak{x}) = \mathrm{Im}_{e_7}(\mathfrak{x}')$.
(35) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 1,2,3,4,7; \mathrm{Im}_{e_5}(\mathfrak{x}) = \mathrm{Im}_{e_5}(\mathfrak{x}'); \mathrm{Im}_{e_6}(\mathfrak{x}) = \mathrm{Im}_{e_6}(\mathfrak{x}')$.
(36) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) < \mathrm{Im}_{e_n}(\mathfrak{x}'), n = 1,2,3,4,6; \mathrm{Im}_{e_5}(\mathfrak{x}) = \mathrm{Im}_{e_5}(\mathfrak{x}'); \mathrm{Im}_{e_7}(\mathfrak{x}) = \mathrm{Im}_{e_7}(\mathfrak{x}')$.

Following a similar approach, we can easily list the 56 cases where exactly 3 components are equal (derived from the 3-combinations of 8), 70 cases with 4 equal components, 56 cases with 5 equal components, and 27 cases with 6 equal components. However, to avoid making the article overly tedious, we will not elaborate in detail on the remaining 211 intermediate cases. For simplicity, let us focus only on the 8 cases with exactly 7 equal components, corresponding to the 7-combinations of 8 where just one component differs.
(248) $\mathrm{Re}(\mathfrak{x}) < \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) = \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,2,3,4,5,6,7$.
(249) $\mathrm{Re}(\mathfrak{x}) = \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) = \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 2,3,4,5,6,7; \mathrm{Im}_{e_1}(\mathfrak{x}) < \mathrm{Im}_{e_1}(\mathfrak{x}')$.
(250) $\mathrm{Re}(\mathfrak{x}) = \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) = \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,3,4,5,6,7; \mathrm{Im}_{e_2}(\mathfrak{x}) < \mathrm{Im}_{e_2}(\mathfrak{x}')$.
(251) $\mathrm{Re}(\mathfrak{x}) = \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) = \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,2,4,5,6,7; \mathrm{Im}_{e_3}(\mathfrak{x}) < \mathrm{Im}_{e_3}(\mathfrak{x}')$.
(252) $\mathrm{Re}(\mathfrak{x}) = \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) = \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,2,3,5,6,7; \mathrm{Im}_{e_4}(\mathfrak{x}) < \mathrm{Im}_{e_4}(\mathfrak{x}')$.
(253) $\mathrm{Re}(\mathfrak{x}) = \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) = \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,2,3,4,6,7; \mathrm{Im}_{e_5}(\mathfrak{x}) < \mathrm{Im}_{e_5}(\mathfrak{x}')$.
(254) $\mathrm{Re}(\mathfrak{x}) = \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) = \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,2,3,4,5,7; \mathrm{Im}_{e_6}(\mathfrak{x}) < \mathrm{Im}_{e_6}(\mathfrak{x}')$.
(255) $\mathrm{Re}(\mathfrak{x}) = \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) = \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,2,3,4,5,6; \mathrm{Im}_{e_7}(\mathfrak{x}) < \mathrm{Im}_{e_7}(\mathfrak{x}')$.

Finally, let us consider the case derived from the 8-combinations of 8, where all corresponding components are equal, which indicates that the two octonions are identical.
(256) $\mathrm{Re}(\mathfrak{x}) = \mathrm{Re}(\mathfrak{x}'); \mathrm{Im}_{e_n}(\mathfrak{x}) = \mathrm{Im}_{e_n}(\mathfrak{x}')$, where $n = 1,2,3,4,5,6,7$.

Specifically, if $\|\mathfrak{x}\| \neq \|\mathfrak{x}'\|$ and any condition between (1) and (256) is satisfied, $\mathfrak{x} \preceq \mathfrak{x}'$ will be writen. If only condition (256) is satisfied, we will denote this by $\mathfrak{x} \prec \mathfrak{x}'$. We will briefly denote this situation as

$$\mathfrak{x} \preceq \mathfrak{x}' \Longrightarrow \|\mathfrak{x}\| \leq \|\mathfrak{x}'\|. \tag{2.1}$$

A careful examination of the 256 conditions above reveals that we can introduce octonion-valued metric spaces, which generalize the complex metric spaces defined by Azam et al. [3], by taking the codomain as the field of complex numbers.

These are then generalized to quaternion-valued metric spaces, as defined by Ahmed et al. [1], taking the codomain as the skew field of quaternions, which serve as a non-commutative extension of these metric spaces to Clifford algebra analysis.

Following, we will define octonion-valued metric spaces, an interesting generalization of metric spaces that are neither commutative nor associative.

Definition 1. ([7]) Given a nonempty set $\digamma$. If the transformation $\Pi_O: \digamma \times \digamma \to \mathrm{O}$ on this set satisfies following conditions,
(1) $0_{\mathrm{O}} \preceq \Pi_{\mathrm{O}}(s,t)$ for all $s,t \in \digamma$ and $\Pi_{\mathrm{O}}(s,t) = 0_{\mathrm{O}}$ if and only if $s = t$,
(2) $\Pi_{\mathrm{O}}(s,t) = \Pi_{\mathrm{O}}(t,s)$ for all $s,t \in \digamma$,
(3) $\Pi_{\mathrm{O}}(s,t) \preceq \Pi_{\mathrm{O}}(s,v) + \Pi_{\mathrm{O}}(v,t)$ for all $s,t,v \in \digamma$.

Then Π_{O} is called be an octonion valued metric on $\digamma$, and the pair $(\digamma, \Pi_{\mathrm{O}})$ is called be an octonion valued metric space.

Example 1. Let $\Pi_{\mathrm{O}}: \mathrm{O} \times \mathrm{O} \to \mathrm{O}$ be an octanion valued function defined as $\Pi_{\mathrm{O}}(\mathrm{o}, \mathrm{o}') = |\mathrm{o}_0 - \mathrm{o}_0'| + |\mathrm{o}_1 - \mathrm{o}_1'|e_1 + |\mathrm{o}_2 - \mathrm{o}_2'|e_2 + |\mathrm{o}_2 - \mathrm{o}_2'|e_2 + |\mathrm{o}_3 - \mathrm{o}_3'|e_3 + |\mathrm{o}_4 - \mathrm{o}_4'|e_4 + |\mathrm{o}_5 - \mathrm{o}_5'|e_5 + |\mathrm{o}_6 - \mathrm{o}_6'|e_6 + |\mathrm{o}_7 - \mathrm{o}_7'|e_7$, where $\mathrm{o}, \mathrm{o}' \in \mathbb{O}$ with

$$\mathfrak{x} = \mathrm{o}_0 + \mathrm{o}_1 e_1 + \mathrm{o}_2 e_2 + \mathrm{o}_3 e_3 + \mathrm{o}_4 e_4 + \mathrm{o}_5 e_5 + \mathrm{o}_6 e_6 + \mathrm{o}_7 e_7,$$

$$\mathfrak{x}' = \mathrm{o}_0' + \mathrm{o}_1' e_1 + \mathrm{o}_2' e_2 + \mathrm{o}_3' e_3 + \mathrm{o}_4' e_4 + \mathrm{o}_5' e_5 + \mathrm{o}_6' e_6 + \mathrm{o}_7' e_7;$$
$$\mathrm{o}_i, \mathrm{o}_i' \in \mathbb{R}; i = 0,1,2,3,4,5,6,7.$$

Then $(\mathrm{O}, \Pi_{\mathrm{O}})$ defines an octanion valued metric space.

Below, we provide an example of an octonion-valued metric that does not have a known numerical set as its domain.

Example 2. Let $X = \{a, b, c\}$ be an arbitrary set with three elements. Define the distances between the elements of the set by

$$\Pi_{\mathrm{O}}(a,b) = \Pi_{\mathrm{O}}(b,a) = 3 + 4e_1 - 6e_2 + 4e_3 + 3e_4 + 3e_5 - 2e_6 + e_7$$
$$\Pi_{\mathrm{O}}(b,c) = \Pi_{\mathrm{O}}(c,b) = 1 + 2e_1 + 3e_3 - 5e_4 - 3e_6 + 4e_7$$
$$\Pi_{\mathrm{O}}(a,c) = \Pi_{\mathrm{O}}(c,a) = 2 + 3e_1 + e_2 + e_3 - 2e_4 + 2e_5 - e_6 + 5e_7$$
$$\Pi_{\mathrm{O}}(a,a) = \Pi_{\mathrm{O}}(b,b) = \Pi_{\mathrm{O}}(c,c) = 0 + 0e_1 + 0e_2 + 0e_3 + 0e_4 + 0e_5 + 0e_6 + 0e_7.$$

Since they are $\|\Pi_{\mathrm{O}}(a,b)\| = 10, \|\Pi_{\mathrm{O}}(a,c)\| = 7, \|\Pi_{\mathrm{O}}(c,b)\| = 8, \|\Pi_{\mathrm{O}}(a,b) + \Pi_{\mathrm{O}}(a,c)\| = \sqrt{195}$, $\|\Pi_{\mathrm{O}}(a,b) + \Pi_{\mathrm{O}}(b,c)\| = \sqrt{200}$ and $\|\Pi_{\mathrm{O}}(c,b) + \Pi_{\mathrm{O}}(a,c)\| = \sqrt{169} = 13$, it can be seen through straightforward calculations that the conditions given in Definition 1 above are satisfied.

Definition 2. ([5]) Given a nonempty set $\digamma$. If the transformation $\Pi_{\mathrm{O}}: \digamma \times \digamma \to \mathrm{O}$ on this set satisfies following conditions,

(1) $0_{\mathrm{O}} \preceq \Pi_{\mathrm{O}}(s,t)$ for all $s,t \in \digamma$ and $\Pi_{\mathrm{O}}(s,t) = 0_{\mathrm{O}}$ if and only if $s = t$,
(2) $\Pi_{\mathrm{O}}(s,t) = \Pi_{\mathrm{O}}(t,s)$ for all $s,t \in \digamma$,
(3) $\Pi_{\mathrm{O}}(s,t) \preceq b \cdot (\Pi_{\mathrm{O}}(s,v) + \Pi_{\mathrm{O}}(v,t))$ for all $s,t,v \in \digamma$, $1 \leq b \in \mathbb{R}$.
Then Π_{O} is called be an octonion valued b-metric on $\digamma$, and the pair $(\digamma, \Pi_{\mathrm{O}})$ is called be an octanion valued b-metric space.

Example 3. Example 1 and Example 2 are instances of octonion-valued 1-metric spaces for the real scalar $b = 1$.

Remark 1. It should be explicitly noted that, as seen from Definition 1 and Definition 2, every octonion-valued metric space is an octonion-valued b-metric space in the special case where $b = 1$.

The converse of the remark we provided above is not true, except for the special case of $b = 1$. The next example we will present is an octonion-valued b-metric space for $b = 2$, yet it is not an octonion-valued metric space.

Example 4. Let $\Pi_{\mathrm{O}}^{b}: \mathrm{O} \times \mathrm{O} \to \mathrm{O}$ be an octonion valued function defined as $\Pi_{\mathrm{O}}^{b}(\mathrm{o},\mathrm{o}') = |\mathrm{o}_0 - \mathrm{o}_0'|^2 + |\mathrm{o}_1 - \mathrm{o}_1'|^2 e_1 + |\mathrm{o}_2 - \mathrm{o}_2'|^2 e_2 + |\mathrm{o}_3 - \mathrm{o}_3'|^2 e_3 + |\mathrm{o}_4 - \mathrm{o}_4'|^2 e_4 + |\mathrm{o}_5 - \mathrm{o}_5'|^2 e_5 + |\mathrm{o}_6 - \mathrm{o}_6'|^2 e_6 + |\mathrm{o}_7 - \mathrm{o}_7'|^2 e_7$, where $\mathrm{o},\mathrm{o}' \in \mathrm{O}$ with

$$\mathrm{o} = \mathrm{o}_0 + \mathrm{o}_1 e_1 + \mathrm{o}_2 e_2 + \mathrm{o}_3 e_3 + \mathrm{o}_4 e_4 + \mathrm{o}_5 e_5 + \mathrm{o}_6 e_6 + \mathrm{o}_7 e_7,$$
$$\mathrm{o}' = \mathrm{o}_0' + \mathrm{o}_1' e_1 + \mathrm{o}_2' e_2 + \mathrm{o}_3' e_3 + \mathrm{o}_4' e_4 + \mathrm{o}_5' e_5 + \mathrm{o}_6' e_6 + \mathrm{o}_7' e_7;$$
$$\mathrm{o}_i, \mathrm{o}_i' \in \mathbb{R}; i = 0,1,2,3,4,5,6,7$$

Then $(\mathrm{O}, \Pi_{\mathrm{O}})$ defines an octonion valued b-metric space. Indeed, note that if we take

$$\mathrm{o} = 3 + 3e_1 + 3e_2 + 3e_3 + 3e_4 + 3e_5 + 3e_6 + 3e_7$$
$$\mathrm{o}' = 2 + 2e_1 + 2e_2 + 2e_3 + 2e_4 + 2e_5 + 2e_6 + 2e_7$$
$$\mathrm{o}'' = 1 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7,$$

although they are comparable under the partial ordering relation defined on octonions in [7],

$$\Pi_{\mathrm{O}}^{b}(\mathrm{o},\mathrm{o}'') = 4 + 4e_1 + 4e_2 + 4e_3 + 4e_4 + 4e_5 + 4e_6 + 4e_7$$
$$\Pi_{\mathrm{O}}^{b}(\mathrm{o},\mathrm{o}') = 1 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7$$
$$\Pi_{\mathrm{O}}^{b}(\mathrm{o}',\mathrm{o}'') = 1 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7$$
$$\Pi_{\mathrm{O}}^{b}(\mathrm{o},\mathrm{o}') + \Pi_{\mathrm{O}}^{b}(\mathrm{o}',\mathrm{o}'') = 2 + 2e_1 + 2e_2 + 2e_3 + 2e_4 + 2e_5 + 2e_6 + 2e_7,$$

which would violate the third property of the axioms for being an octonion-valued metric space as stated in Definition 1, making it not an octonion-valued metric space. However, if we take $b = 2$, in this case, the partial ordering $\preceq$ satisfies the axioms in Definition 2.

As can be seen from the definitions and example above, the definition we provided is a natural generalization of the classical b-metric definition, as well as complex and quaternion-valued b-metrics. To express the connections between them, let us present the following propositions.

Proposition 1. Every quaternion-valued b-metric space can be embedded into an octonion-valued b-metric space.

Proposition 2. Every complex-valued b-metric space can be embedded into a quaternion-valued bmetric space and an octonion-valued b-metric space.

Proposition 3. Every b-metric space can be embedded into a complex-valued b-metric space, a quaternion-valued b-metric space and an octonion-valued b-metric space.

This is accomplished by generalizing scalar fields from conventional metric spaces to complexvalued metric spaces. The integral domain is further generalized to quaternion-valued metric spaces, and non-associative, higher-dimensional extensions result in octonion-valued metric spaces. The categories of classical, complex-valued, quaternion-valued, and octonion-valued b-metric spaces are introduced by relaxing the triangle inequality for $b \geq 1$. While forgetful functors cause reverse transitions, inclusion functors help with these transitions. Here, instead of concentrating on the algebraic and categorical features of octonion valued b-metric spaces, we concentrate on their calculus aspects.

Thus, we can now proceed to define some fundamental concepts related to the definition above (see [5], [39] and [26]).

Definition 3. Any point $s \in \digamma$ is called be an interior point of set $A \subset \digamma$ whenever there exists $0_{\mathbb{O}} \prec r \in \mathbb{O}$ such that

$$B(\mathfrak{f}, r) = \{t \in \digamma : \Pi_{\mathbb{O}}(\mathfrak{f}, t) \prec r\} \subset A.$$

Definition 4. Any point $\mathfrak{f} \in \digamma$ is called be a limit point of $A \subset \digamma$ whenever for every $0_{\mathbb{O}} \prec r \in \mathbb{O}$

$$B(\mathfrak{f}, r) \cap (A - \{\mathfrak{f}\}) \neq \emptyset.$$

Definition 5. Set O is said to be an open set whenever each element of O is an interior point of O. Subset $C \subset \digamma$ is called a closed set whenever each limit point of C belongs to C. The family

$$F = \{B(\mathfrak{f}, r) : \mathfrak{f} \in \digamma, 0_{\mathbb{O}} \prec r\}$$

is a subbase for Hausdorff topology τ on $\digamma$.

Definition 6. Let $\mathfrak{f} \in \digamma$ and $\mathfrak{f}_\alpha$ be a sequence in the set $\digamma$. If for each $o \in \mathbb{O}$ with $0_{\mathbb{O}} \prec o$ there is $\alpha_0 \in \mathbb{N}$ such that for all $\alpha > \alpha_0, \Pi_{\mathbb{O}}(\mathfrak{f}_\alpha, \mathfrak{f}) \prec o$, then $(\mathfrak{f}_\alpha)$ is called convergence sequence. Then, in this case $(\mathfrak{f}_\alpha)$ sequence converges to the limit point $\mathfrak{f}$; as notation, $\mathfrak{f}_\alpha \to \mathfrak{f}$ as $\alpha \to \infty$ or $\lim_{\alpha} \mathfrak{f}_\alpha = \mathfrak{f}$.

Definition 7. If there exists $\alpha_0 \in \mathbb{N}$ such that for all $\alpha > \alpha_0, \Pi_{\mathbb{O}}(\mathfrak{f}_{\alpha+m}, \mathfrak{f}_\alpha) \prec o$, then $(\mathfrak{f}_\alpha)$ is said to be a Cauchy sequence in the octonion valued b-metric space $(\digamma, \Pi_{\mathbb{O}})$. If every

Cauchy sequence is convergent in $(\digamma, \Pi_O)$, then $(\digamma, \Pi_O)$ is said to be a complete octonion valued b-metric space.

Definition 8. If there exists $\alpha_0 \in \mathbb{N}$ such that for all $\alpha > \alpha_0, \Pi_O(\mathfrak{f}_{\alpha+m}, \mathfrak{f}_\alpha) \prec$ o, then $(\mathfrak{f}_\alpha)$ is said to be a Cauchy sequence in the octanion-valued b-metric space $(\digamma, \Pi_O)$. If every Cauchy sequence is convergent in $(\digamma, \Pi_O)$, then $(\digamma, \Pi_O)$ is said to be a complete octonion valued b-metric space.

Definition 9. A sequence $(\mathfrak{f}_\alpha)$ in an octonion valued b-metric space $(\digamma, \Pi_O)$ is said to converge statistically to a point $\mathfrak{f} \in \digamma$ (denoted as $\mathfrak{f}_\alpha \overset{\text{stg}}{\to} \mathfrak{f}$), if as for all $0_O \prec \mathfrak{x}$, we have

$$\lim_{N\to\infty} \frac{1}{N} |\{\alpha \leq N: \Pi_O(\mathfrak{f}_\alpha, \mathfrak{f}) \nprec \mathfrak{x}\}| = 0.$$

Definition 10. A sequence $(\mathfrak{f}_\alpha)$ in an octonion valued b-metric space $(\digamma, \Pi_O)$ is said to be statistical Cauchy sequence, if as for all $0_O \prec \mathfrak{x}$, we have $l \in \mathbb{N}^+$depending on the norm of $\mathfrak{x} \in O$

$$\lim_{N} \frac{1}{N} |\{\alpha \leq N: \Pi_O(\mathfrak{f}_\alpha, \mathfrak{f}_l) \nprec \mathfrak{x}\}| = 0$$

Theorem 1. Every statistically convergent sequence in an octonion valued b-metric space is a statistical Cauchy sequence.

Definition 11. If every statistically Cauchy sequence is statistically convergent in $(\digamma, \Pi_O)$, then $(\digamma, \Pi_O)$ is said to be a statistically complete octonion valued b-metric space.

Definition 12. A sequence $(\mathfrak{f}_\alpha)$ in an octonion valued b-metric space $(\digamma, \Pi_O)$ ideally converges to a point $\mathfrak{f} \in \digamma$ (represented as $\mathfrak{f}_\alpha \overset{\mathcal{J}_{(\digamma,\Pi_O)}}{\to} \mathfrak{f}$), if, for all $0_O \prec \varrho$ with $0_O \prec \varrho$,

$$\{\alpha \in \mathbb{N}: \Pi_O(\mathfrak{f}_\alpha, \mathfrak{f}) \nprec \varrho\} \in \mathcal{J}.$$

Definition 13. A sequence $(\mathfrak{f}_\alpha)$ in an octonion-valued b-metric space $(\digamma, \Pi_O)$ is said to be an ideal Cauchy sequence if, for every $\varrho \in O$ with $0_O \prec \varrho$, there exists an $l \in \mathbb{N}^+$depending on the norm of $\varrho \in O$ such that

$$\{\alpha \in \mathbb{N}: \Pi_O(\mathfrak{f}_\alpha, \mathfrak{f}_l) \nprec \varrho\} \in \mathcal{J}.$$

3. Results

In this section, by using modulus functions, we present some definitions for octonion-valued b-metric spaces as given in [5], [39] and [26], focusing on concepts such as $\mathcal{J}$-statistically convergence, $\mathcal{J}$-lacunary statistically convergence, strong $\mathcal{J}$-Cesàro summability, and strong $\mathcal{J}$-lacunary summability in octonion-valued b-metric space. Additionally, various properties associated with these notions are explored.

A modulus function is a function $f: [0, \infty) \to [0, \infty)$ such that (i) $f(\mathfrak{a}) = 0$ iff $\mathfrak{a} = 0$, (ii) $f(\mathfrak{a} + \mathfrak{b}) \leq f(\mathfrak{a}) + f(\mathfrak{b})$ for all $\mathfrak{a}, \mathfrak{b} \geq 0$, (iii) f is increasing, (iv) f is continuous from right at 0. The modulus function may be bounded or unbounded. For example, if we take $f(\mathfrak{a}) = \mathfrak{a}/(\mathfrak{a}+1)$, then $f(\mathfrak{a})$ is bounded. If $f(\mathfrak{a}) = \mathfrak{a}^{\varsigma}, \varsigma \in (0,1)$, then the modulus $f(\mathfrak{a})$ is unbounded. Unless otherwise stated throughout the paper, consider f as an unbounded modulus function.

We can now present some definitions, among which we will give the inclusion relations.

Definition 14. Let $(\digamma, \Pi_{\mathrm{O}})$ represent an octonion-valued b-metric space, where $\omega \in \digamma$ is a point, and $(\omega_\alpha) \subseteq \digamma$ is a sequence. A sequence (ω_α) is considered $\mathcal{J}$-statistically convergent to ω if, for every $\varrho, \tau \in \mathrm{O}$ with $0_{\mathrm{O}} \prec \varrho, \tau$ such that

$$\left\{\gamma \in \mathbb{N}: \frac{f(|\{\alpha \leq \gamma: \Pi_{\mathrm{O}}(\omega_\alpha, \omega) \nprec \varrho\}|)}{f(\gamma)} \nprec \tau\right\} \in \mathcal{J}.$$

This type of convergence is denoted by $\mathrm{F}(\mathcal{J}, f)^{(\digamma, \Pi_{\mathrm{O}})} - \lim_{\alpha \to \infty} \omega_\alpha = \omega$ or $(\omega_\alpha) \overset{\mathrm{F}(\mathcal{J}, f)^{(\digamma, \Pi_{\mathrm{O}})}}{\rightarrow} \omega$. The set of all $\mathcal{J}$-statistically convergent sequences in octonion-valued b-metric space is represented as $\mathrm{F}(\mathcal{J}, f)^{(\digamma, \Pi_{\circ})}$.

Definition 15. A sequence (ω_α) is known as $\mathcal{J}$-lacunary statistically convergent to $\omega \in \digamma$ if, for every $\varrho, \tau \in \mathrm{O}$ with $0_{\mathrm{O}} \prec \varrho, \tau$ such that

$$\left\{\mathfrak{r} \in \mathbb{N}: \frac{f(|\{\alpha \in I_{\mathfrak{r}}: \Pi_{\mathrm{O}}(\omega_\alpha, \omega) \nprec \varrho\}|)}{f(h_{\mathfrak{r}})} \nprec \tau\right\} \in \mathcal{J}.$$

It is indicated by $\mathrm{F}_\theta(\mathcal{J}, f)^{(\digamma, \Pi_{\mathrm{O}})} - \lim_{\alpha \to \infty} \omega_\alpha = \omega$ or $(\omega_\alpha) \overset{\mathrm{F}_\theta(\mathcal{J}, f)^{(\digamma, \Pi_{\mathrm{O}})}}{\rightarrow} \omega$. The set of all $\mathcal{J}$-lacunary statistically convergent sequences in octonion-valued b-metric space is symbolized as $\mathrm{F}_\theta(\mathcal{J}, f)^{(\digamma, \Pi_{\mathrm{O}})}$.

Definition 16. A sequence (ω_α) is defined as strongly $\mathcal{J}$-Cesàro summable to $\omega \in \digamma$ if, for any $\varrho \in \mathrm{O}$ with $0_{\mathrm{O}} \prec \varrho$,

$$\left\{t \in \mathbb{N}: \sum_{\alpha=1}^{t} f\big(\Pi_{\mathrm{O}}(\omega_\alpha, \omega)\big) \nprec \varrho\right\} \in \mathcal{J}.$$

This is denoted by $C_1^{(\digamma, \Pi_{\mathrm{O}})}[\mathcal{J}, f] - \lim_{\alpha \to \infty} \omega_\alpha = \omega$ or $(\omega_\alpha) \; C_1^{(\digamma, \Pi_{\mathrm{O}})} \; [\mathcal{J}, f] \omega$.

Definition 17. A sequence (ω_α) is considered as strongly $\mathcal{J}$-lacunary convergent to $\omega \in \digamma$ if, for all $\varrho \in \mathrm{O}$ with $0_{\mathrm{O}} \prec \varrho$,

$$\left\{\mathfrak{r} \in \mathbb{N}: \frac{1}{f(\mathfrak{h}_{\mathfrak{r}})} \sum_{\alpha \in I_{\mathfrak{r}}} f\big(\Pi_{\mathrm{O}}(\omega_\alpha, \omega)\big) \nprec \varrho\right\} \in \mathcal{J}.$$

and is indicated by $N_\theta^{(\digamma, \Pi_{\mathrm{O}})}[\mathcal{J}, f] - \lim_{\alpha \to \infty} \omega_\alpha = \omega$ or $(\omega_\alpha)^{N_\theta^{(\digamma, \Pi_{\mathrm{O}})} \; [\mathcal{J}, f]} \omega$.

Definition 18. A sequence (ω_α) is called to be $[V,\lambda]^{(\digamma,\Pi_O)}(\mathcal{J})$-summable to $\omega \in \digamma$, if for any every $\varrho \in O$ with $0_O \prec \varrho$,

$$\{\gamma \in \mathbb{N}: \Pi_O\big(t_\gamma(\omega_\alpha),\omega\big) \nprec \varrho\} \in \mathcal{J},$$

where

$$t_\gamma(\omega_\alpha) := \frac{1}{f(\lambda_\gamma)} \sum_{\alpha \in I_\gamma} f\big(\Pi_O(\omega_\alpha,\omega)\big), I_\gamma = [\gamma - \lambda_\gamma + 1, \gamma].$$

In this context, we write $[V,\lambda]^{(\digamma,\Pi_O)}(\mathcal{J},f) - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ or $(\omega_\alpha) \xrightarrow{[V,\lambda]^{(\digamma,\Pi_O)}(\mathcal{J},f)} \omega$. The set of all $[V,\lambda](\mathcal{J},f)$-summable sequences in octonion valued b-metric space is symbolized as $[V,\lambda]^{(\digamma,\Pi_O)}(\mathcal{J},f)$.

Definition 19. A sequence (ω_α) is known as $\mathcal{J}$ - λ, f-statistically convergent or $\mathcal{J}$ - $\digamma_{\lambda,f}$-convergent to τ, if, for every $\varrho,\tau \in O$ with $0_O \prec \varrho,\tau$,

$$\left\{\gamma \in \mathbb{N}: \frac{f\big(|\{\alpha \in I_\gamma: \Pi_O(\omega_\alpha,\omega) \nprec \varrho\}|\big)}{f(\lambda_\gamma)} \nprec \tau\right\} \in \mathcal{J}.$$

In this case we write $\mathcal{J}^{(\digamma,\Pi_O)} - \digamma_{\lambda,f} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ or $(\omega_\alpha) \xrightarrow{\mathcal{J}^{(\digamma,\Pi_O)} - \digamma_{\lambda,f}} \omega$. The collection of all $\mathcal{J}$ - λ, f-statistically convergent sequences in octonion valued b-metric space is symbolized as $\mathcal{J}^{(\digamma,\Pi_O)} - \digamma_{\lambda,f}$.

Theorem 2. Let $\theta = (k_r)$ be a lacunary sequence. Then, the following statements hold:

(i) (a) If $N_\theta^{(\digamma,\Pi_O)}[\mathcal{J},f] - \lim_{\alpha\to\infty} \omega_\alpha = \omega$, then $\mathrm{F}_\theta(\mathcal{J},f)^{(\digamma,\Pi_O)} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$, and (b) $N_\theta^{(\digamma,\Pi_O)}[\mathcal{J},f]$ is a proper subset of $\mathrm{F}_\theta(\mathcal{J},f)^{(\digamma,\Pi_O)}$.

(ii) If $(\omega_\alpha) \in l_\infty(\digamma)$, the space of all bounded sequences of $(\digamma,\Pi_O)$ and $\mathrm{F}_\theta(\mathcal{J},f)^{(\digamma,\Pi_O)} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$, then $N_\theta^{(\digamma,\Pi_O)}[\mathcal{J},f] - \lim_{\alpha\to\infty} \omega_\alpha = \omega$.

(iii) $\mathrm{F}_\theta(\mathcal{J},f)^{(\digamma,\Pi_O)} \cap l_\infty = N_\theta^{(\digamma,\Pi_O)}[\mathcal{J},f] \cap l_\infty$.

Proof. (i) (a) If $\varrho \in O$ with $0_O \prec \varrho$ and $N_\theta^{(\digamma,\Pi_O)}[\mathcal{J},f] - \lim_{\alpha\to\infty} \omega_\alpha = \omega$, then we can write

$$\sum_{\alpha \in I_r} f\big(\Pi_O(\omega_\alpha,\omega)\big) \geq \sum_{\alpha \in I_r \& f(\Pi_O(\omega_\alpha,\omega)) \nprec \varrho} f\big(\Pi_O(\omega_\alpha,\omega)\big) \geq \varrho \cdot f(|\{\alpha \in I_r: \Pi_O(\omega_\alpha,\omega) \nprec \varrho\}|),$$

and so

$$\frac{1}{\varrho f(\mathfrak{h}_r)} \sum_{\alpha \in I_r} f\big(\Pi_O(\omega_\alpha,\omega)\big) \geq \frac{1}{f(\mathfrak{h}_r)} f(|\{\alpha \in I_r: \Pi_O(\omega_\alpha,\omega) \nprec \varrho\}|).$$

For any $\tau \in O$ with $0_O \prec \tau$, we obtain

$$\left\{\mathfrak{r} \in \mathbb{N}: \frac{f(|\{\alpha \in I_{\mathfrak{r}}: \Pi_{O}(\omega_{\alpha}, \omega) \nprec \varrho\}|)}{f(\mathfrak{h}_{\mathfrak{r}})} \nprec \tau\right\} \subseteq \left\{\mathfrak{r} \in \mathbb{N}: \frac{\sum_{\alpha \in I_r} f\big(\Pi_{O}(\omega_{\alpha}, \omega)\big)}{f(\mathfrak{h}_{\mathfrak{r}})} \nprec \varrho\tau\right\} \in \mathcal{I}.$$

Hence, $\mathrm{F}_{\theta}(\mathcal{I}, f)^{(\digamma,\Pi_{O})} - \lim_{\alpha\to\infty} \omega_{\alpha} = \omega$.

(b) To show that the inclusion $N_{\theta}^{(\digamma,\Pi_{O})}[\mathcal{I}, f] \subseteq \mathrm{F}_{\theta}(\mathcal{I}, f)^{(\digamma,\Pi_{O})}$ is strict, consider a given θ. Define (ω_{α}) as follows: for the first $\sqrt{\mathfrak{h}_{\mathfrak{r}}}$ integers in $I_{\mathfrak{r}}$, let (ω_{α}) be $1,2, \dots, \sqrt{\mathfrak{h}_{\mathfrak{r}}}$; for all other values, let $\omega_{\alpha} = 0$. For any $\varrho \in O$ with $0_{O} \prec \varrho$, one writes

$$\frac{(|\{\alpha \in I_{\mathfrak{r}}: \Pi_{O}(\omega_{\alpha}, \omega) \nprec \varrho\}|)}{f(h_{\mathfrak{r}})} \le \frac{f\big(\sqrt{h_{\mathfrak{r}}}\big)}{f(h_{\mathfrak{r}})},$$

and for any $\tau \in O$ with $0_{O} \prec \tau$,

$$\left\{\mathfrak{r} \in \mathbb{N}: \frac{(|\{\alpha \in I_{\mathfrak{r}}: \Pi_{O}(\omega_{\alpha}, \omega) \nprec \varrho\}|)}{f(h_{\mathfrak{r}})} \nprec \tau\right\} \subseteq \left\{\mathfrak{r} \in \mathbb{N}: \frac{f\big(\sqrt{h_{\mathfrak{r}}}\big)}{f(h_{\mathfrak{r}})} \ge \tau\right\}$$

Since the set on the right-hand side is finite and thus belongs to $\mathcal{I}$, it follows that $\mathrm{F}_{\theta}(\mathcal{I}, f)^{(\digamma,\Pi_{O})} - \lim_{\alpha\to\infty} \omega_{\alpha} = \omega$. On the other hand,

$$\frac{1}{f(\mathfrak{h}_{\mathfrak{r}})} \sum_{\alpha \in I_{\mathfrak{r}}} f\big(\Pi_{O}(\omega_{\alpha}, 0)\big) = \frac{1}{f(\mathfrak{h}_{\mathfrak{r}})} \frac{f\big(\sqrt{\mathfrak{h}_{\mathfrak{r}}}\big)\Big(f\big(\sqrt{\mathfrak{h}_{\mathfrak{r}}} + 1\big)\Big)}{2} \to \frac{1}{2} \neq 0,$$

Thus, we have

$$\left\{\mathfrak{r} \in \mathbb{N}: \frac{1}{f(\mathfrak{h}_{\mathfrak{r}})} \sum_{\alpha \in I_{\mathfrak{r}}} f\big(\Pi_{O}(\omega_{\alpha}, 0)\big) \nprec \frac{1}{4}\right\} = \left\{\mathfrak{r} \in \mathbb{N}: \frac{f\big(\sqrt{\mathfrak{h}_{\mathfrak{r}}}\big)\Big(f\big(\sqrt{\mathfrak{h}_{\mathfrak{r}}} + 1\big)\Big)}{2f(\mathfrak{h}_{\mathfrak{r}})} \ge \frac{1}{2}\right\} = \{n, n+1, \dots\}$$

for some $n \in \mathbb{N}$ that belongs to $\mathcal{F}(\mathcal{I}, f)$, given that $\mathcal{I}$ is admissible. Hence, $N_{\theta}^{(\digamma,\Pi_{O})}[\mathcal{I}, f] - \lim_{\alpha\to\infty} \omega_{\alpha} \neq 0$.

(ii) Suppose that $(\omega_{\alpha}) \in l_{\infty}(\digamma)$ and $\mathrm{F}_{\theta}(\mathcal{I}, f)^{(\digamma,\Pi_{\circ})} - \lim_{\alpha\to\infty} \omega_{\alpha} = \omega$. Then, there exists a $\mathfrak{D} > 0$ such that $\Pi_{O}(\omega_{\alpha}, \omega) \nprec \mathfrak{D}$. Given $\varrho \in O$ with $0_{O} \prec \varrho$, we have

$$\begin{aligned}
\frac{1}{f(\mathfrak{h}_{\mathfrak{r}})} \sum_{\alpha \in I_{\mathfrak{r}}} f\big(\Pi_{O}(\omega_{\alpha}, \omega)\big) &\ge \frac{1}{f(\mathfrak{h}_{\mathfrak{r}})} \sum_{\alpha \in I_{\mathfrak{r}} \& \Pi_{O}(\omega_{\alpha}, \omega) \nprec \frac{\varrho}{2}} f\big(\Pi_{O}(\omega_{\alpha}, \omega)\big) + \frac{1}{f(h_{\mathfrak{r}})} \sum_{\alpha \in I_{\mathfrak{r}} \& \Pi_{O}(\omega_{\alpha}, \omega) \prec \frac{\varrho}{2}} f\big(\Pi_{O}(\omega_{\alpha}, \omega)\big) \\
&\ge \frac{\mathcal{D}}{f(\mathfrak{h}_{\mathfrak{r}})} f\left(\left|\left\{\alpha \in I_{\mathfrak{r}}: \Pi_{O}(\omega_{\alpha}, \omega) \nprec \frac{\varrho}{2}\right\}\right|\right) + \frac{\varrho}{2}.
\end{aligned}$$

Therefore, we have

$$\left\{\mathfrak{r} \in \mathbb{N}: \frac{1}{f(\mathfrak{h}_{\mathfrak{r}})} \sum_{\alpha \in I_{\mathfrak{r}}} f\big(\Pi_{\mathbb{O}}(\omega_\alpha, \omega)\big) \not\prec \varrho\right\}$$
$$\subseteq \left\{\mathfrak{r} \in \mathbb{N}: \frac{1}{f(\mathfrak{h}_{\mathfrak{r}})} f\left(\left|\left\{\alpha \in I_{\mathfrak{r}}: \Pi_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \frac{\varrho}{2}\right\}\right|\right) \not\prec \frac{\varrho}{2\mathfrak{D}}\right\} \in \mathcal{J}.$$

As a result $N_\theta^{(\mathfrak{f},\Pi_{\mathbb{O}})}[\mathcal{J}, f] - \lim_{\alpha\to\infty} \omega_\alpha = \omega$.

Theorem 3. Let $\theta = \{k_{\mathfrak{r}}\}$ be a lacunary sequence. If $\liminf_{\mathfrak{r}} \frac{f(\mathfrak{h}_{\mathfrak{r}})}{f(k_{\mathfrak{r}})} > 1$, then

$$\mathrm{F}(\mathcal{J}, f)^{(\mathfrak{f},\Pi_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega \Rightarrow \mathrm{F}_\theta(\mathcal{J}, f)^{(\mathfrak{f},\Pi_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega.$$

If $\lim\inf\mathfrak{r}_{\mathfrak{r}} \frac{f(\mathfrak{h}_{\mathfrak{r}})}{f(k_{\mathfrak{r}})} = 1$, then there exists a bounded sequence (ω_α) which is $\mathcal{J}$-statistically convergent but not $\mathcal{J}$-lacunary statistically convergent in octonion valued b-metric space.

Proof. If $\lim_{\mathfrak{r}}\inf \frac{f(\mathfrak{h}_{\mathfrak{r}})}{f(k_{\mathfrak{r}})} > 1$, then there exists $\varsigma > 0$ such that $\frac{f(\mathfrak{h}_{\mathfrak{r}})}{f(k_{\mathfrak{r}})} \geq 1 + \varsigma$ for all $\mathfrak{r} \geq 1$. Since $h_{\mathfrak{r}} = k_{\mathfrak{r}} - k_{\mathfrak{r}-1}$, we have $\frac{f(k_{\mathfrak{r}})}{f(h_{\mathfrak{r}})} \leq \frac{1+\varsigma}{\varsigma}$ and $\frac{f(k_{\mathfrak{r}-1})}{f(\mathfrak{h}_{\mathfrak{r}})} \leq \frac{1}{\varsigma}$. If $\mathrm{F}(\mathcal{J}, f)^{(\mathfrak{f},\Pi_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$, then for all $\varrho \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \varrho$, we have

$$\frac{1}{f(k_{\mathfrak{r}})} f(|\{\alpha \leq k_{\mathfrak{r}}: \Pi_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}|) \geq \frac{1}{f(k_{\mathfrak{r}})} f(|\{\alpha \in I_{\mathfrak{r}}: \Pi_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}|)$$
$$\geq \frac{\varsigma}{1+\varsigma} \frac{1}{f(\mathfrak{h}_{\mathfrak{r}})} f(|\{\alpha \in I_{\mathfrak{r}}: \Pi_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}|);$$

Then, for every $\tau \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \tau$, we get

$$\left\{\mathfrak{r} \in \mathbb{N}: \frac{1}{f(\mathfrak{h}_{\mathfrak{r}})} f(|\{\alpha \in I_{\mathfrak{r}}: \Pi_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}|) \not\prec \tau\right\}$$
$$\subseteq \left\{k_{\mathfrak{r}} \in \mathbb{N}: \frac{1}{f(k_{\mathfrak{r}})} f(|\{\alpha \leq k_{\mathfrak{r}}: \Pi_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}|) \not\prec \frac{\tau\varsigma}{1+\varsigma}\right\} \in \mathcal{J}.$$

As a result, we obtain $\mathrm{F}_\theta(\mathcal{J}, f)^{(\mathfrak{f},\Pi_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$. Suppose that $\liminf_{\mathfrak{r}} \frac{f(\mathfrak{h}_{\mathfrak{r}})}{f(k_{\mathfrak{r}})} = 1$. Then, we can select a subsequence $\left(k_{\mathfrak{r}_{\mathfrak{v}}}\right)$ of θ such that

$$\frac{f\left(k_{\mathfrak{r}_{\mathfrak{v}}}\right)}{f\left(k_{\mathfrak{r}_{\mathfrak{v}}-1}\right)} < 1 + \frac{1}{\mathfrak{v}} \text{ and } \frac{f\left(k_{\mathfrak{v}_{\mathfrak{v}}-1}\right)}{f\left(k_{\mathfrak{r}_{\mathfrak{v}}-1}\right)} > \mathfrak{v}$$

where $\mathfrak{r}_{\mathfrak{v}} > \mathfrak{r}_{\mathfrak{v}-1} + 2$. Construct a sequence (ω_α) such that

$$\omega_\alpha = \begin{cases} 1, & \text{if } \alpha \in I_{\mathfrak{r}_v}, \\ 0, & \text{if not.} \end{cases}$$

Then, we have

$$\frac{1}{f\left(\mathfrak{h}_{\mathfrak{r}_{\mathfrak{v}}}\right)} \sum_{\alpha \in I_{\mathfrak{r}}} f\big(\Pi_{\mathbb{O}}(\omega_\alpha, \omega)\big) = 1 - \omega \text{ for } \mathfrak{v} = 1,2,.. \text{ and } \frac{1}{f(\mathfrak{h}_{\mathfrak{r}})} \sum_{\alpha \in I_{\mathfrak{r}}} f\big(\Pi_{\mathbb{O}}(\omega_\alpha, \omega)\big) = \omega \text{ for } \mathfrak{r} \neq \mathfrak{r}_{\mathfrak{v}}.$$

Given that (ω_α) does not belong to $N_\theta^{(\digamma,\Pi_O)}[\mathcal{I},f]$, and considering that (ω_α) is bounded, Theorem 2(iii) implies that $\mathrm{F}_\theta(\mathcal{I},f)^{(\digamma,\Pi_O)}-\lim_{\alpha\to\infty}\omega_\alpha\neq\omega$. Next, let $k_{\mathfrak{r}_v-1}\leq\gamma\leq k_{\mathfrak{r}_{v+1}-1}$. Then, from Theorem 2.1 in [14], we can write

$$\frac{1}{f(\gamma)}f(|\{\alpha\leq\gamma:\Pi_O(\omega_\alpha,\omega)\nprec\varrho\}|)\leq\frac{1}{f(\gamma)}\sum_{\alpha=1}^{\gamma}f\big(\Pi_O(\omega_\alpha,\omega)\big)\prec\frac{k_{\mathfrak{r}_\mathfrak{v}-1}+\mathfrak{h}_{\mathfrak{r}_\mathfrak{v}}}{k_{\mathfrak{r}_\mathfrak{v}-1}}\prec\frac{1}{\mathfrak{v}}+\frac{1}{\mathfrak{v}}=\frac{2}{\mathfrak{v}}.$$

Hence, $\mathrm{F}(\mathcal{I},f)^{(\digamma,\Pi_O)}-\lim_{\alpha\to\infty}\omega_\alpha=\omega$.

Theorem 4. If $\theta=\{k_\mathfrak{r}\}$ be a lacunary sequence with limsup $\frac{f(k_\mathfrak{v}-k_{\mathfrak{v}-1})}{f(k_\mathfrak{v}-1)}=\mathfrak{B}_\mathfrak{v}<\infty(\mathfrak{v}=1,2,\dots,\mathfrak{r})$, then

$$\mathrm{F}_\theta(\mathcal{I},f)^{(\digamma,\Pi_O)}-\lim_{\alpha\to\infty}\omega_\alpha=\omega\Rightarrow\mathrm{F}(\mathcal{I},f)^{(\digamma,\Pi_O)}-\lim_{\alpha\to\infty}\omega_\alpha=\omega.$$

Proof. Assume that $\mathrm{F}_\theta(\mathcal{I},f)^{(\digamma,\Pi_O)}-\lim_{\alpha\to\infty}\omega_\alpha=\omega$, and for $\varrho,\tau\in O$ with $0_O\prec\varrho,\tau$, define the sets

$$\mathfrak{C}=\left\{\mathfrak{r}\in\mathbb{N}:\frac{1}{f(\mathfrak{h}_\mathfrak{r})}f(|\{\alpha\in I_\mathfrak{r}:\Pi_O(\omega_\alpha,\omega)\nprec\varrho\}|)<\tau\right\}$$

and

$$\mathfrak{T}=\left\{\gamma\in\mathbb{N}:\frac{1}{f(\gamma)}f(|\{\alpha\leq\gamma:\Pi_O(\omega_\alpha,\omega)\nprec\varrho\}|)<\tau_1\right\}.$$

Based on our assumption that $\mathfrak{C}\in\mathcal{F}(\mathcal{I})$, the filter associated with the ideal $\mathcal{I}$, it is also evident that

$$\mathfrak{K}_\mathfrak{v}=\frac{1}{f(\mathfrak{h}_\mathfrak{v})}f(|\{\alpha\in I_\mathfrak{v}:\Pi_O(\omega_\alpha,\omega)\nprec\varrho\}|)<\tau$$

for all $\mathfrak{v}\in\mathfrak{C}$. Let $\gamma\in\mathbb{N}$ be such that $k_{\mathfrak{r}-1}<\gamma\leq k_\mathfrak{r}$ for some $r\in\mathfrak{C}$. Now

$$
\begin{aligned}
&\frac{1}{f(\gamma)} f(|\{\alpha \leq \gamma : \Pi_O(\omega_\alpha, \omega) \nless \varrho\}|) \leq \frac{1}{f(k_r - 1)} f(|\{\alpha \leq k_r : \Pi_O(\omega_\alpha, \omega) \nless \varrho\}|) \\
&= \frac{1}{f(k_{r-1})} f(|\{\alpha \in I_1 : \Pi_O(\omega_\alpha, \omega) \nless \varrho\}|) + \frac{1}{f(k_{r-1})} f(|\{\alpha \in I_2 : \Pi_O(\omega_\alpha, \omega) \nless \varrho\}|) \\
&+ \cdots + \frac{1}{f(k_{r-1})} f(|\{\alpha \in I_r : \Pi_O(\omega_\alpha, \omega) \nless \varrho\}|) \\
&= \frac{f(k_1)}{f(k_{r-1})} \frac{1}{f(\mathfrak{h}_1)} f(|\{\alpha \in I_1 : \Pi_O(\omega_\alpha, \omega) \nless \varrho\}|) + \frac{f(k_2 - k_1)}{f(k_r - 1)} \frac{1}{f(\mathfrak{h}_2)} f(|\{\alpha \in I_2 : \Pi_O(\omega_\alpha, \omega) \nless \varrho\}|) \\
&+ \cdots + \frac{f(k_r - k_r - 1)}{f(k_r)} \frac{1}{f(\mathfrak{h}_r)} f(|\{\alpha \in I_r : \Pi_O(\omega_\alpha, \omega) \nless \varrho\}|) \\
&= \frac{f(k_1)}{f(k_{r-1})} \mathfrak{K}_1 + \frac{f(k_2 - k_1)}{f(k_r - 1)} \mathfrak{K}_2 + \cdots + \frac{f(k_r - k_{r-1})}{f(k_{r-1})} \mathfrak{K}_t \\
&\leq \sup_{\mathfrak{v} \in \mathbb{C}} \mathfrak{K}_{\mathfrak{v}} . \sup \frac{f\big(k_{\mathfrak{v}} - k_{\mathfrak{v}-1)}}{f(k_{\mathfrak{v}-1})} < \mathfrak{B}_{\mathfrak{v}} \tau.
\end{aligned}
$$

Given that $\tau_1 = \frac{\tau}{\mathfrak{B}_{\mathfrak{v}}}$ and considering that $\bigcup \ \{t : k_{r-1} < t \leq k_r, r \in \mathfrak{C}\} \subset \mathfrak{T}$ where $\mathfrak{C} \in \mathcal{F}(\mathcal{I})$ it follows from our assumption on θ that the set $\mathfrak{T}$ also belongs to $\mathcal{F}(\mathcal{I})$. Therefore, we have $\mathrm{F}(\mathcal{I}, f)^{(\mathrm{F}, \Pi_O)} - \lim_{\alpha \to \infty} \omega_\alpha = \omega$.

Theorem 5. Let $\mathcal{I}$ be an admissible ideal satisfying condition (AP), and let $\theta \in \mathcal{F}(\mathcal{I})$. If $\omega \in \mathrm{F}(\mathcal{I}, f)^{(\mathrm{F}, \Pi_O)} \cap \mathrm{F}_\theta(\mathcal{I}, f)^{(\mathrm{F}, \Pi_O)}$, then $\mathrm{F}(\mathcal{I}, f)^{(\mathrm{F}, \Pi_O)} - \lim_{\alpha \to \infty} \omega_\alpha = \mathrm{F}_\theta(\mathcal{I}, f)^{(\mathrm{F}, \Pi_O)} - \lim_{\alpha \to \infty} \omega_\alpha$.

Proof. Suppose that $\mathrm{F}(\mathcal{I}, f)^{(\mathrm{F}, \Pi_\circ)} - \lim_{\alpha \to \infty} \omega_\alpha = \omega_1$ and $\mathrm{F}_\theta(\mathcal{I}, f)^{(\mathrm{F}, \Pi_\circ)} - \lim_{\alpha \to \infty} \omega_\alpha = \omega_2$, and $\omega_1 \neq \omega_2$. Let $0_O < \varrho < \frac{1}{2} |\omega_1 - \omega_2|$, where $\varrho \in O$ with $0_O \prec \varrho$. As $\mathcal{I}$ meets the condition (AP), there exists an $\mathfrak{M} \in \mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N} \setminus \mathfrak{M} \in \mathcal{I}$) such that

$$
\lim_{r \to \infty} \frac{1}{f(t_r)} f(|\{\alpha \leq t_r : \Pi_O(\omega_\alpha, \omega_1) \nless \varrho\}|) = 0, \text{ where } \mathfrak{M} = \{t_1, t_2, \ldots\}
$$

Let

$$
\mathfrak{U} = \{\alpha \leq t_r : \Pi_O(\omega_\alpha, \omega_1) \nless \varrho\}, \mathfrak{T} = \{\alpha \leq t_r : \Pi_O(\omega_\alpha, \omega_2) \nless \varrho\}.
$$

The inequality $t_r = |\mathfrak{U} \cup \mathfrak{T}| \leq |\mathfrak{U}| + |\mathfrak{T}|$ leads to $1 \leq \frac{f(|\mathfrak{U}|)}{f(t_r)} + \frac{f(|\mathfrak{T}|)}{f(t_r)}$. Given that $\frac{f(|\mathfrak{T}|)}{f(t_r)} \leq 1$ and $\lim_{r \to \infty} \frac{f(|\mathfrak{L}|)}{f(t_r)} = 0$, it follows that $\lim_{r \to \infty} \frac{f(|\mathfrak{F}|)}{f(t_r)} = 1$. Let $\mathfrak{M}^* = \{\alpha_{j_1}, \alpha_{j_2}, \ldots\} = \mathfrak{M} \cap \theta \in \mathcal{F}(\mathcal{I})$. Then, the α_{j_s} th term of the statistical expression

$$
\frac{1}{f(t_r)} f(|\{\alpha \leq t_r : \Pi_O(\omega_\alpha, \omega_2) \nless \varrho\}|)
$$

is

$$\frac{1}{f(\alpha_{\mathfrak{j}_\mathfrak{s}})} f\left(\left|\left\{\alpha \in \bigcup_{\mathfrak{r}=1}^{\mathfrak{j}_\mathfrak{s}} I_\mathfrak{r}: \Pi_\mathrm{O}(\omega_\alpha, \omega_2) \nprec \varrho\right\}\right| \mid\right) = \frac{1}{\sum_{\mathfrak{r}=1}^{\mathfrak{j}_\mathfrak{s}} f(\mathfrak{h}_\mathfrak{r})} \sum_{\mathfrak{r}=1}^{\mathfrak{j}_\mathfrak{s}} f(\mathfrak{h}_\mathfrak{r})\tau_\mathfrak{r} \qquad (3.1)$$

where

$$\tau_\mathfrak{r} = \frac{1}{f(\mathfrak{h}_\mathfrak{r})} f(|\{\alpha \in I_\mathfrak{r}: \Pi_\mathrm{O}(\omega_\alpha, \omega_2) \nprec \varrho\}|) \xrightarrow{\mathcal{I}} 0$$

since $\mathrm{F}_\theta(\mathcal{I}, f)^{(\digamma,\Pi_\mathrm{O})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega_2$. Let θ represent a lacunary sequence. The expression (3.1) is a regular weighted mean transform of $\tau_\mathfrak{r}$'s, and is therefore $\mathcal{I}$-convergent to zero as $\mathfrak{s} \to \infty$. Consequently, it has a subsequence converging to zero due to $\mathcal{I}$ meeting the condition (AP). However, as this forms a subsequence of

$$\left\{\frac{1}{f(\gamma)} f(|\{\alpha \le \gamma: \Pi_\mathrm{O}(\omega_\alpha, \omega_2) \nprec \varrho\}|)\right\}_{\gamma\in\mathfrak{M}}$$

it follows that

$$\left\{\frac{1}{f(\gamma)} f(|\{\alpha \le \gamma: \Pi_\mathrm{O}(\omega_\alpha, \omega_2) \nprec \varrho\}|)\right\}_{\gamma\in\mathfrak{M}} \nrightarrow 1$$

which contradicts the assumption. This contradiction establishes the validity of the theorem.

We can give the following result without proof.

Corollary 1. If $\lim\inf_\mathfrak{r} q_\mathfrak{r} > 1$, then $C_1^{(\digamma,\Pi_\mathrm{O})}[\mathcal{I}, f] \subseteq N_\theta^{(\digamma,\Pi_\mathrm{O})}$ and if $\lim_\mathfrak{r}\inf q_\mathfrak{r} < \infty$, then $N_\theta^{(\digamma,\Pi_\mathrm{O})}[\mathcal{I}, f] \subseteq C_1^{(\digamma,\Pi_\mathrm{O})}[\mathcal{I}, f]$.

Theorem 6. Let $\lambda = (\lambda_n) \in \Delta$. Then

(i) If $[V,\lambda]^{(\digamma,\Pi_\mathrm{O})}(\mathcal{I}, f) - \lim_{\alpha\to\infty} \omega_\alpha = \omega$, then $\mathcal{I}^{(\digamma,\Pi_\circ)} - \digamma_{\lambda,f} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ and the inclusion $[V,\lambda]^{(\digamma,\Pi_\mathrm{O})}(\mathcal{I}, f) \subset \mathcal{I}^{(\digamma,\Pi_\mathrm{O})} - \digamma_{\lambda,f}$ is proper for every ideal $\mathcal{I}$.

(ii) If $(\omega_\alpha) \in l_\infty(\digamma)$, the space of all bounded sequences of $(\digamma, \Pi_\mathrm{O})$ and $\mathcal{I}^{(\digamma,\Pi_\mathrm{O})} - \digamma_{\lambda,f} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$, then $[V,\lambda]^{(\digamma,\Pi_\mathrm{O})}(\mathcal{I}, f) - \lim_{\alpha\to\infty} \omega_\alpha = \omega$.

Proof. Let $\varrho \in \mathrm{O}$ with $0_\mathrm{O} \prec \varrho$ and $[V,\lambda]^{(\digamma,\Pi_\mathrm{O})}(\mathcal{I}, f) - \lim_{\alpha\to\infty} \omega_\alpha = \omega$. Then, we have

$$\begin{aligned}\sum_{\alpha\in I_\gamma} f\big(\Pi_\mathrm{O}(\omega_\alpha, \omega)\big) &\ge \sum_{\alpha\in I_\gamma \,\&\, f(\Pi_\mathrm{O}(\omega_\alpha,\omega))\nprec\varrho} f\big(\Pi_\mathrm{O}(\omega_\alpha, \omega)\big) \\ &\ge \varrho. f\left(\left|\left\{\alpha \in I_\gamma: \Pi_\mathrm{O}(\omega_\alpha, \omega) \nprec \varrho\right\}\right|\right)\end{aligned}$$

So for a given $\tau \in \mathrm{O}$ with $0_\mathrm{O} \prec \tau$,

$$\frac{1}{f(\lambda_\gamma)} f\left(\left|\left\{\alpha \in I_\gamma : \Pi_{\mathbb{O}}(\omega_\alpha, \omega) \nprec \varrho\right\}\right|\right) \nprec \tau \Rightarrow \frac{1}{f(\lambda_\gamma)} \sum_{\alpha \in I_\gamma} f\left(\Pi_{\mathbb{O}}(\omega_\alpha, \omega)\right) \nprec \varrho\tau,$$

i.e.,

$$\left\{\gamma \in \mathbb{N}: \frac{1}{f(\lambda_\gamma)} f\left(\left|\left\{\alpha \in I_\gamma : \Pi_{\mathbb{O}}(\omega_\alpha, \omega) \nprec \varrho\right\}\right|\right) \nprec \tau\right\}$$
$$\subseteq \left\{\gamma \in \mathbb{N}: \frac{1}{f(\lambda_\gamma)} \sum_{\alpha \in I_\gamma} f\left(\Pi_{\mathbb{O}}(\omega_\alpha, \omega)\right) \nprec \varrho\tau\right\}.$$

Since $[V,\lambda]^{(\digamma,\Pi_{\mathbb{O}})}(\mathcal{J}, f) - \lim_{\alpha\to\infty} \omega_\alpha = \omega$, so the set on the right-hand side belongs to $\mathcal{J}$ and so it follows that $\mathcal{J}^{(\digamma,\Pi_{\mathbb{O}})} - \digamma_{\lambda,f} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$.

To establish that $\mathcal{J}^{(\digamma,\Pi_{\mathbb{O}})} - \digamma_{\lambda,f} \subsetneqq [V,\lambda]^{(\digamma,\Pi_{\mathbb{O}})}(\mathcal{J}, f)$, consider a fixed $\mathfrak{L} \in \mathcal{J}$. Define (ω_α) as

$$\omega_\alpha = \begin{cases} \alpha\mathfrak{p}, & \text{for } \gamma - \left[\sqrt{\lambda_\gamma}\right] + 1 \le \alpha \le \gamma, \gamma \notin \mathfrak{L} \\ \alpha\mathfrak{p}, & \text{for } \gamma - \lambda_\gamma + 1 \le \alpha \le \gamma, \gamma \in \mathfrak{L} \\ \theta, & \text{if not.} \end{cases}$$

where $\mathfrak{p} \in \digamma$ is a fixed element satisfying $\|\mathfrak{p}\| = 1$, and θ is the null element of $\digamma$. Then $(\omega_\alpha) \notin l_\infty(\digamma)$ and for every $\varrho \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \varrho$ since

$$\frac{1}{f(\lambda_\gamma)} f\left(\left|\left\{\alpha \in I_\gamma : \Pi_{\mathbb{O}}(\omega_\alpha, 0) \nprec \varrho\right\}\right|\right) = \frac{f\left(\left[\sqrt{\lambda_\gamma}\right)\right)}{f(\lambda_\gamma)} \to 0$$

as $\gamma \to \infty$ and $\gamma \notin \mathfrak{L}$, so for every $\tau \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \tau$,

$$\left\{\gamma \in \mathbb{N}: \frac{1}{f(\lambda_\gamma)} f\left(\left|\left\{\alpha \in I_\gamma : \Pi_{\mathbb{O}}(\omega_\alpha, \omega) \nprec \varrho\right\}\right|\right) \nprec \tau\right\} \subset \mathfrak{L} \cup \{1,2,\dots,\mathfrak{s}\}$$

for some $\mathfrak{s} \in \mathbb{N}$. Since $\mathcal{J}$ is admissible, it follows that $\mathcal{J}^{(\digamma,\Pi_{\mathbb{O}})} - \digamma_{\lambda,f} - \lim_{\alpha\to\infty} \omega_\alpha = \theta$. Clearly,

$$\frac{1}{f(\lambda_\gamma)} \sum_{\alpha \in I_\gamma} f\left(\Pi_{\mathbb{O}}(\omega_\alpha, \omega)\right) \to \infty \ (\gamma \to \infty)$$

which implies $[V,\lambda]^{(\digamma,\Pi_{\mathbb{O}})}(\mathcal{J}, f) - \lim_{\alpha\to\infty} \omega_\alpha \neq \theta$. It is important to observe that if $\mathfrak{L} \in \mathcal{J}$ is infinite, then $\digamma_{\lambda,f}^{(\digamma,\Pi_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha \neq \theta$. This example demonstrates that $\mathcal{J}$ - $\digamma_{\lambda,f}$-convergence is a more general concept than λ-statistical convergence in octonion valued b-metric spaces.

(ii) Suppose that $\mathcal{J}^{(\digamma,\Pi_\circ)} - \digamma_{\lambda,f} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ and $(\omega_\alpha) \in l_\infty$. Then, there exists a $\mathfrak{D} > 0$ such that $\Pi_O(\omega_\alpha, \omega) \nprec \mathfrak{D}$. Given $\varrho \in O$ with $0_O \prec \varrho$, we have

$$\frac{1}{f(\lambda_\gamma)} \sum_{\alpha\in I_\gamma} f\big(\Pi_O(\omega_\alpha,\omega)\big) = \frac{1}{f(\lambda_\gamma)} \sum_{\alpha\in I_\gamma \& f(\Pi_O(\omega_\alpha,\omega))\nprec\varrho} f\big(\Pi_O(\omega_\alpha,\omega)\big)$$
$$+\frac{1}{f(\lambda_\gamma)} \sum_{\alpha\in I_\gamma \& f(\Pi_O(\omega_\alpha,\omega))\prec\varrho} f\big(\Pi_O(\omega_\alpha,\omega)\big)$$
$$\leq \frac{\mathcal{D}}{f(\lambda_\gamma)} f\big(\big|\{\alpha \in I_\gamma : \Pi_O(\omega_\alpha, 0) \nprec \varrho\}\big|\big) + \varrho.$$

Note that

$$\mathfrak{R}(\varrho) := \left\{\gamma \in \mathbb{N} : \frac{1}{f(\lambda_\gamma)} f\big(\big|\{\alpha \in I_\gamma : \Pi_O(\omega_\alpha, \omega) \nprec \varrho\}\big|\big) \nprec \frac{\varrho}{\mathfrak{D}}\right\} \in \mathcal{J}.$$

If $\gamma \in \mathfrak{R}^c$, then

$$\frac{1}{f(\lambda_\gamma)} \sum_{\alpha\in I_\gamma} f\big(\Pi_O(\omega_\alpha,\omega)\big) \prec 2\varrho.$$

Therefore,

$$\left\{\gamma \in \mathbb{N} : \frac{1}{f(\lambda_\gamma)} \sum_{\alpha\in I_\gamma} f\big(\Pi_O(\omega_\alpha,\omega)\big) \nprec 2\varrho\right\} \subset \mathfrak{R}$$

and thus it belongs to $\mathcal{J}$. This demonstrates that $[V,\lambda]^{(\digamma,\Pi_O)}(\mathcal{J}, f) - \lim_{\alpha\to\infty} \omega_\alpha = \omega$.

Theorem 7. Let $(\digamma, \Pi_O)$ be an octonion-valued b-metric spaces, $\omega \in \digamma$ be a point, and $(\omega_\alpha) \subseteq \digamma$ be a sequence and $\lambda = (\lambda_\gamma) \in \Delta$. Then

(i) $\digamma^{(\digamma,\Pi_O)}(\mathcal{J}, f) \subseteq \mathcal{J}^{(\digamma,\Pi_O)} - \digamma_{\lambda,f}$ if $\lim\inf_{\gamma\to\infty} \frac{f(\lambda_\gamma)}{f(\gamma)} > 0$.

(ii) If $\lim\inf_{\gamma\to\infty} \frac{f(\lambda_\gamma)}{f(\gamma)} = 0$, $\mathcal{J}$-strongly (by which we mean that $\exists$ a subsequence $(\gamma(i))_{i=1}^{\infty}$, for which $\frac{\lambda_{\gamma(i)}}{\gamma(i)} < \frac{1}{i} \forall i$ and $\{\gamma(i) : i \in \mathbb{N}\} \notin \mathcal{J}$) then $\digamma^{(\digamma,\Pi_\circ)}(\mathcal{J}, f) \subsetneqq \mathcal{J}^{(\digamma,\Pi_\circ)} - \digamma_{\lambda,f}$.

Proof. (i) Given $\varrho \in O$ with $0_O \prec \varrho$, we have

$$\frac{1}{f(\gamma)} f(|\{\alpha \leq \gamma : \Pi_O(\omega_\alpha, \omega) \nprec \varrho\}|) \geq \frac{1}{f(\gamma)} f\big(\big|\{\alpha \in I_\gamma : \Pi_O(\omega_\alpha, \omega) \nprec \varrho\}\big|\big)$$
$$\geq \frac{f(\lambda)}{f(\gamma)} \frac{1}{f(\lambda_\gamma)} f\big(\big|\{\alpha \in I_\gamma : \Pi_O(\omega_\alpha, \omega) \nprec \varrho\}\big|\big)$$

If $\liminf_{\gamma\to\infty}\frac{f(\lambda_\gamma)}{f(\gamma)} = \mathfrak{v}$ then from definition $\left\{\gamma \in \mathbb{N}: \frac{f(\lambda_\gamma)}{f(\gamma)} < \frac{\mathfrak{v}}{2}\right\}$ is finite. For $\tau \in \mathrm{O}$ with $0_{\mathrm{O}} \prec \tau$

$$\left\{\gamma \in \mathbb{N}: \frac{1}{f(\lambda_\gamma)} f\left(\left|\left\{\alpha \in I_\gamma: \Pi_{\mathrm{O}}(\omega_\alpha,\omega) \nprec \varrho\right\}\right|\right) \nprec \tau\right\}$$
$$\subseteq \left\{\gamma \in \mathbb{N}: \frac{1}{f(\gamma)} f(|\{\alpha \le \gamma: \Pi_{\mathrm{O}}(\omega_\alpha,\omega) \nprec \varrho\}|) \nprec \frac{\mathfrak{v}}{2}\tau\right\}$$
$$\cup \left\{\gamma \in \mathbb{N}: \frac{f(\lambda)}{f(\gamma)} < \frac{\mathfrak{v}}{2}\right\}.$$

Since $\mathcal{J}$ is admissible, the set on the right-hand side belongs to $\mathcal{J}$, completing the proof of (i).

(ii) Construct a sequence (ω_α) such that

$$\omega_\alpha = \begin{cases} \mathfrak{q}, & \text{if } \alpha \in I_{\gamma(i)}, i = 1,2,\dots \\ \theta, & \text{if not}, \end{cases}$$

where $\mathfrak{q} \in \digamma$, $\|\mathfrak{q}\| = 1$ and θ is the zero element of $\digamma$. Then (ω_α) is statistically convergent, placing it in $\digamma^{(\digamma,\Pi_{\mathrm{O}})}(\mathcal{J}, f)$ (Since $\mathcal{J}$ is admissible). But $(\omega_\alpha) \notin [V,\lambda]^{(\digamma,\Pi_{\mathrm{O}})}$ and so by Theorem 6(ii)$(\omega_\alpha) \notin \mathcal{J}^{(\digamma,\Pi_{\mathrm{O}})} - \digamma_{\lambda,f}$.

Theorem 8. If $\lambda \in \Delta$ be such that $\lim_{\gamma\to\infty}\frac{f(\gamma-\lambda_\gamma)}{f(\gamma)} = 1$, then $\mathcal{J}^{(\digamma,\Pi_{\mathrm{O}})} - \digamma_{\lambda,f} \subset \digamma^{(\digamma,\Pi_{\mathrm{O}})}(\mathcal{J}, f)$.

Proof. Let $\varsigma > 0$ be given. Since $\lim_{\gamma\to\infty}\frac{f(\gamma-\lambda_\gamma)}{f(\gamma)} = 1$, we can choose $\mathfrak{s} \in \mathbb{N}$ such that $\left|\frac{f(\gamma-\lambda_\gamma)}{f(\gamma)} - 1\right| < \frac{\varsigma}{2}$, for all $\gamma \ge \mathfrak{s}$. Now observe that, for $\varrho \in \mathrm{O}$ with $0_{\mathrm{O}} \prec \varrho$,

$$\frac{1}{f(\gamma)} f(|\{\alpha \le \gamma: \Pi_{\mathrm{O}}(\omega_\alpha,\omega) \nprec \varrho\}|)$$
$$= \frac{1}{f(\gamma)} f\left(\left|\left\{\alpha \le \gamma - \lambda_\gamma: \Pi_{\mathrm{O}}(\omega_\alpha,\omega) \nprec \varrho\right\}\right|\right)$$
$$+ \frac{1}{f(\gamma)} f\left(\left|\left\{\alpha \in I_\gamma: \Pi_{\mathrm{O}}(\omega_\alpha,\omega) \nprec \varrho\right\}\right|\right)$$
$$\le \frac{f(\gamma-\lambda_\gamma)}{f(\gamma)} + \frac{1}{f(\lambda_\gamma)} f\left(\left|\left\{\alpha \in I_\gamma: \Pi_{\mathrm{O}}(\omega_\alpha,\omega) \nprec \varrho\right\}\right|\right)$$
$$\le \frac{\varsigma}{2} + \frac{1}{f(\lambda_\gamma)} f\left(\left|\left\{\alpha \in I_\gamma: \Pi_{\mathrm{O}}(\omega_\alpha,\omega) \nprec \varrho\right\}\right|\right),$$

for all $\gamma \ge \mathfrak{s}$. Hence

$$\left\{\gamma \in \mathbb{N}: \frac{1}{f(\gamma)} f(|\{\alpha \le \gamma: \Pi_{\mathrm{O}}(\omega_\alpha,\omega) \nprec \varrho\}|) \nprec \varsigma\right\}$$
$$\subseteq \left\{\gamma \in \mathbb{N}: \frac{1}{f(\lambda_\gamma)} f\left(\left|\left\{\alpha \in I_\gamma: \Pi_{\mathrm{O}}(\omega_\alpha,\omega) \nprec \varrho\right\}\right|\right) \nprec \frac{\varsigma}{2}\right\} \cup \{1,2,3,\dots,\mathfrak{s}\}.$$

If $\mathcal{J}^{(\digamma,\Pi_O)}-\digamma_\lambda-\lim_{\alpha\to\infty}\omega_\alpha=\omega$ then $F(\mathcal{J},f)^{(\digamma,\Pi_O)}-\lim_{\alpha\to\infty}\omega_\alpha=\omega$.

4. Conclusion

Important ideas in summability theory have been thoroughly examined in generalized metric spaces in recent years. This work uses the modulus function to introduce the notions of strong $\mathcal{J}$-Cesaro summability, $\mathcal{J}$-lacunary statistical convergence, and $\mathcal{J}$-statistical convergence in octonion-valued b metric spaces, which are a generalization of metric spaces. Analysis is also done on the relationships between these ideas. Compared to comparable research in the literature, the findings of this study are more thorough. The generalized convergence of double sequences obeying the modulus function in octonion valued b-metric spaces may be examined using the results of this work. Also, as a future work, the concepts of deferred statistical convergence, deferred statistical boundedness and deferred strong Cesaro convergence in octonion-valued b-metric spaces, which are a generalisation of metric spaces, can be introduced using the modulus function.

Authorship contribution statement

Ö. Kişi, S. Çetin and M. Gürdal: Conceptualization, research, writing, editing, and review.. The published version of the work was examined and approved by all authors.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Ethics Committee Approval and/or Informed Consent Information

As the authors of this study, we declare that we do not have any ethics committee approval and/or informed consent statement.

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