Konuralp Journal of Mathematics, 6 (1) (2018) 92-97

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

Common Fixed Point Theorems For F-Contractions In G-Metric Spaces Using Compatible Mappings

Seher Sultan Sepet[1] **and Cafer Aydın**[2*]

[1]*Kahramanmaraş Sütçü İmam University, Faculty of Science and Art, Department of Mathematics, 46050, Kahramanmaraş-TURKEY*
[2]*Kahramanmaraş Sütçü İmam University, Faculty of Science and Art, Department of Mathematics, 46050, Kahramanmaraş-TURKEY*
[*]*Corresponding author E-mail: caydin61@gmail.com*

Abstract

Using a mapping $F:\mathbb{R}_+ \to \mathbb{R}$, Wardowski [1] introduce a new type of contraction called F-contraction and prove a new fixed point theorem concerning F-contraction. In the present article, we prove some fixed point theorems with helping compatible maps for type 1 and type 2 F-contraction in complete G-metric spaces.

Keywords: *Fixed point, G-metric space, F-contraction, Compatible mapping.*
2010 Mathematics Subject Classification: *47H10, 54H25*

1. Introduction and Preliminaries

The common fixed point theory is very important and useful in Mathematics. It can be applied in various areas, for instant, variational inequalities, optimization, and approximation theory. The Banach contraction principle [2], is the simplest and one of the most versatile elementary results in fixed point theory. Over the years, various extensions and generalizations of this principle have appeared in the literature. In 2006, Mustafa and Sims [3] introduced a new structure called G-metric space as a generalization of the usual metric spaces. Afterwards based on the notion of a G-metric space, many fixed point results for different contractive conditions have been presented, for more details see [4–7].
On the other hand, there has been a considerable interest to study common fixed point for a pair of mappings satisfying contractive conditions in various spaces. Several interesting and elegant results were obtained in this direction by various authors. It was the turning point in the fixed point field when the notion of commutativity was used by Jungck [8], to obtain common fixed point theorems in metric spaces. This result was further generalized and extended in various ways by many authors. Furthermore in 1986, Jungck [9] introduced the concept of compatible maps in metric spaces. Then Kumar [10], introduced the concept of compatible maps in G-metric space.
Recently Wardowski [1], introduced the notion of a F-contraction mapping and investigated the existence of fixed points for such mappings. Consistent with Wardowski [1], we denote by $\mathscr{F}$ the set of all functions $F:\mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing. That is, $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$ for all $\alpha, \beta \in \mathbb{R}_+$
(F2) For every sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ we have $\lim_{n\to\infty}\alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$
(F3) There exists a number $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Note that every F contraction is continuous (see [1]). The results of Wardowski have become of recent interest of many authors (see [11–18]). Now, we mention briefly some fundamental definitions.

Definition 1.1. ([3]) Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

$(G1)$ $G(x,y,z) = 0$ if $x = y = z$;
$(G2)$ $0 < G(x,x,y)$; for all $x,y \in X$, with $x \neq y$;
$(G3)$ $G(x,x,y) \leq G(x,y,z)$, for all $x,y,z \in X$, with $z \neq y$;
$(G4)$ $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables);
$(G5)$ $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, for all $x,y,z,a \in X$, (rectangle inequa-lity).

Then the function G is called a generalized metric or more specifically a G-metric on X, and the pair (X,G) is called a G-metric space.

Email addresses: **, sultanseher2020@outlook.com (Seher Sultan Sepet), caydin61@gmail.com (Cafer Aydın)**

Definition 1.2. ([3]) Let (X,G) be a G-metric space. A sequence (x_n) in X is G-convergent to x if $\lim_{n,m\to\infty} G(x,x_n,x_m)=0$; that is, for every $\varepsilon>0$, there is $N\in\mathbb{N}$ such that $G(x,x_n,x_m)<\varepsilon$, for all $n,m\geq N$.

Proposition 1.1. *([3]) Let (X,G) be a G-metric space, then the following are equivalent.*

(i) (x_n) *is G-convergent to* x.
(ii) $G(x_n,x_n,x)\to 0$, *as* $n\to\infty$.
(iii) $G(x_n,x,x)\to 0$, *as* $n\to\infty$.
(iv) $G(x_m,x_n,x)\to 0$, *as* $n,m\to\infty$.

Definition 1.3. ([3]) Let (X,G) be a G-metric space, a sequence (x_n) is called G-Cauchy if for every $\varepsilon>0$, there is $N\in\mathbb{N}$ such that $G(x_n,x_m,x_l)<\varepsilon$, for all $n,m,l\geq N$; that is, $G(x_n,x_m,x_l)\to 0$ as $n,m,l\to\infty$.

Proposition 1.2. *[3] If (X,G) is a G-metric space, then the following statements are equivalent.*

(i) The sequence (x_n) *is G-Cauchy.*
(ii) For every $\varepsilon>0$*,there exists* $N\in\mathbb{N}$ *such that;* $G(x_n,x_m,x_m)<\varepsilon$*, for all* $n,m\geq N$.

2. Main Results

In this section we present our main definitions and theorems. We start with the following definition.

Definition 2.1. ([10]) Let f and g be maps from a G-metric spaces (X,G) into itself. The maps f and g are said to be compatible map if there exists a sequence (x_n) such that

$$\lim_{n\to\infty} G(fgx_n,gfx_n,gfx_n)=0$$

or

$$\lim_{n\to\infty} G(gfx_n,fgx_n,fgx_n)=0$$

whenever (x_n) is sequence in X such that

$$\lim_{n\to\infty} fx_n=\lim_{n\to\infty} gx_n=t$$

for some $t\in X$.

Definition 2.2. Let (X,G) a G-metric spaces and $f,g:X\to X$ be compatible mappings. Furthermore

$$f(X)\subseteq g(X) \tag{2.1}$$

$$f \text{ or } g \text{ is continuous.} \tag{2.2}$$

A mapping $f,g:X\to X$ is said to be type 1 F-contraction on (X,G) if there exists a number $\tau>0$ such that for all $x,y,z\in X$ satisfying $G(fx,fy,fz)>0$, the following holds:

$$\tau+F(G(fx,fy,fz))\leq F(G(gx,gy,gz)). \tag{2.3}$$

Moreover $f,g:X\to X$ is said to be type 2 F-contraction on (X,G) if there exists a number $\tau>0$ such that for all $x,y,z\in X$ and $\beta\in[0,\frac{1}{3}]$ satisfying $G(fx,fy,fz)>0$, the following holds:

$$\tau+F(G(fx,fy,fz))\leq F(\beta[G(gx,fx,fx)+G(gy,fy,fy)+G(gz,fz,fz)]). \tag{2.4}$$

Theorem 2.1. *Let (X,G) be a complete G-metric space and $f,g:X\to X$ be type 1 F-contraction. Then f and g have a unique common fixed point.*

Proof: Let x_0 be an arbitrary point in X. By (2.1) one can choose a point x_1 in X such that $fx_0=gx_1$. In general choose x_{n+1} and y_n such that

$$y_n=fx_n=gx_{n+1},\quad (n=1,2,3,\ldots). \tag{2.5}$$

Now suppose that $y_n\neq y_{n+1}$ for every $n\in\mathbb{N}$. Then $G(y_n,y_{n+1},y_{n+1})>0$. We denote

$$\gamma_n=G(y_n,y_{n+1},y_{n+1})=G(fx_n,fx_{n+1},fx_{n+1}). \tag{2.6}$$

Then $G(fx_n,fx_{n+1},fx_{n+1})>0$, so using (2.3) we obtain

$$\tau+F(G(fx_n,fx_{n+1},fx_{n+1}))\leq F(G(gx_n,gx_{n+1},gx_{n+1})).$$

Hence using (2.5) we obtain

$$F(G(y_n,y_{n+1},y_{n+1}))\leq F(G(y_{n-1},y_n,y_n))-\tau.$$

Using the above inequality we obtain

$$F(\gamma_n)\leq F(\gamma_{n-1})-\tau\leq F(\gamma_{n-2})-2\tau\leq\ldots\leq F(\gamma_0)-n\tau. \tag{2.7}$$

From (2.7), we obtain

$$\lim_{n\to\infty} F(\gamma_n)=-\infty \tag{2.8}$$

that together with $(F2)$ gives

$$\lim_{n\to\infty}\gamma_n = 0. \tag{2.9}$$

From $(F3)$ there exists $k\in(0,1)$ such that

$$\lim_{n\to\infty}\gamma_n^k F(\gamma_n) = 0. \tag{2.10}$$

By (2.7), the following holds for all $n\in\mathbb{N}$:

$$\gamma_n^k(F(\gamma_n)-F(\gamma_0)) \le -\gamma_n^k n\tau \ \le 0 \tag{2.11}$$

Letting $n\to\infty$ in (2.11) and using (2.9) and (2.10), we obtain

$$\lim_{n\to\infty} n\gamma_n^k = 0. \tag{2.12}$$

Now, let us observe that from (2.12) there exists $n_1\in\mathbb{N}$ such that $n\gamma_n^k\le 1$ for all $n\ge n_1$. In order to show that (y_n) is a Cauchy sequence consider $\forall m,n\in\mathbb{N}$ such that $m>n>n_2$. From the definition of the G-metric spaces we get

$$\begin{aligned}
G(y_n,y_m,y_m) \le& G(y_n,y_{n+1},y_{n+1})+G(y_{n+1},y_{n+2},y_{n+2})+\\
&\cdots+G(y_{m-1},y_m,y_m)\\
=&\gamma_n+\gamma_{n+1}+\ldots+\gamma_{m-1}\\
=&\sum_{i=n}^{m-1}\gamma_i\\
\le&\sum_{i=n}^{\infty}\gamma_i\\
\le&\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}.
\end{aligned}$$

From the convergence of the series $\sum_{i=1}^{\infty}\frac{1}{i^{\frac{1}{k}}}$ and the above inequality, we receive that (y_n) is a Cauchy sequence. From the completeness of X there exists $u\in X$ such that $\lim_{n\to\infty}y_n=u$ and $\lim_{n\to\infty}y_n=\lim_{n\to\infty}fx_n=\lim_{n\to\infty}gx_{n+1}=u$. Since f or g is continuous, suppose that g is continuous therefore $\lim_{n\to\infty}gfx_n=gu$. Further f and g are compatible, therefore

$$G(gfx_n,fgx_n,fgx_n)=0,$$

implies

$$\lim_{n\to\infty}fgx_n=gu.$$

Suppose that

$$G(fgx_n,fx_n,fx_n)>0.$$

Then from (2.3), we obtain

$$\tau+F(G(fgx_n,fx_n,fx_n))\le F(G(ggx_n,gx_n,gx_n)) \tag{2.13}$$

proceeding limit as $n\to\infty$, we obtain

$$\tau+F(G(gu,u,u))\le F(G(gu,u,u)),$$

a contradiction. Then $G(fgx_n,fx_n,fx_n)=0$ we obtain $G(gu,u,u)=0$ that is, $gu=u$. Thus u is a fixed point of g. Now suppose that $G(fx_n,fu,fu)>0$. Thus from (2.3) we obtain

$$\tau+F(G(fx_n,fu,fu))\le F(G(gx_n,gu,gu)) \tag{2.14}$$

proceeding limit as $n\to\infty$, we obtain

$$\begin{aligned}
\tau+F(G(u,fu,fu))&\le F(G(u,u,u))\\
&=F(0).
\end{aligned}$$

Thus from $(F2)$ we get

$$\tau+F(G(u,fu,fu))\le-\infty$$

a contraction. Then $G(fx_n,fu,fu)=0$, we obtain $G(u,fu,fu)=0$ that is, $fu=u$. Thus u is a common fixed point of f and g. We assume that $u\ne w$ and w is another common fixed point of f and g. Then $G(fu,fw,fw)>0$ and from (2.3) we get

$$\tau+F(G(fu,fw,fw))\le F(G(gu,gw,gw))$$

that is,

$$\tau+F(G(u,w,w))\le F(G(u,w,w))$$

a contradiction. Then we obtain $G(fu,fw,fw)=0$ that is, $u=w$. Therefore, the fixed point of f and g is unique.

Theorem 2.2. *Let (X,G) be a complete G-metric space and $f,g : X \to X$ be type 2 F-contraction. Then f and g have a unique common fixed point.*

Proof: Let x_0 be an arbitrary point in X. By (2.1) one can choose a point x_1 in X such that $fx_0 = gx_1$. In general choose x_{n+1} and y_n such that

$$y_n = fx_n = gx_{n+1} \quad (n = 1,2,3,...). \tag{2.15}$$

Suppose now that $y_n \neq y_{n+1}$, for every $n \in \mathbb{N}$. Then $G(y_n,y_{n+1},y_{n+1}) > 0$. We denote

$$\gamma_n = G(y_n,y_{n+1},y_{n+1}) = G(fx_n, fx_{n+1}, fx_{n+1}). \tag{2.16}$$

Then $G(fx_n, fx_{n+1}, fx_{n+1}) > 0$ using (2.4) we obtain

$$\begin{aligned}\tau + F(G(fx_n, fx_{n+1}, fx_{n+1})) \leq & F(\beta[G(gx_n, fx_n, fx_n) + G(gx_{n+1}, fx_{n+1}, fx_{n+1})+ \\ & G(gx_{n+1}, fx_{n+1}, fx_{n+1})]) \\ = & F(\beta[G(gx_n, fx_n, fx_n) + 2G(gx_{n+1}, fx_{n+1}, fx_{n+1})]).\end{aligned}$$

Using (2.15) we obtain

$$F(G(y_n,y_{n+1},y_{n+1})) \leq F(\beta[G(y_{n-1},y_n,y_n) + 2G(y_n,y_{n+1},y_{n+1})])).$$

Now suppose that

$$G(y_{n-1},y_n,y_n) < G(y_n,y_{n+1},y_{n+1}).$$

Then

$$\tau + F(G(y_n,y_{n+1},y_{n+1})) < F(3\beta G(y_n,y_{n+1},y_{n+1})))$$

a contradiction. Then we get

$$G(y_{n-1},y_n,y_n) \geq G(y_n,y_{n+1},y_{n+1}).$$

Thus we obtain

$$\begin{aligned}\tau + F(G(y_n,y_{n+1},y_{n+1})) &\leq F(3\beta G(y_{n-1},y_n,y_n)])) \\ &\leq F(G(y_{n-1},y_n,y_n)).\end{aligned}$$

By using the above inequality

$$F(\gamma_n) \leq F(\gamma_{n-1}) - \tau \leq F(\gamma_{n-2}) - 2\tau \leq ... \leq F(\gamma_0) - n\tau. \tag{2.17}$$

By using (2.17), we obtain

$$\lim_{n\to\infty} F(\gamma_n) = -\infty \tag{2.18}$$

that together with $(F2)$ gives

$$\lim_{n\to\infty} \gamma_n = 0. \tag{2.19}$$

From $(F3)$ there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} \gamma_n^k F(\gamma_n) = 0. \tag{2.20}$$

By (2.17), the following holds for all $n \in \mathbb{N}$:

$$\gamma_n^k (F(\gamma_n) - F(\gamma_0)) \leq -\gamma_n^k n\tau \ \leq 0 \tag{2.21}$$

Letting $n \to \infty$ in (2.21) and using (2.19) and (2.20), we obtain

$$\lim_{n\to\infty} n\gamma_n^k = 0. \tag{2.22}$$

Now, let us observe that from (2.22) there exists $n_1 \in \mathbb{N}$ such that $n\gamma_n^k \leq 1$ for all $n \geq n_1$. In order to show that (y_n) is a Cauchy sequence consider $\forall m,n \in \mathbb{N}$ such that $m > n > n_2$. From the definition of the G-metric spaces and (2.22) we get

$$\begin{aligned}G(y_n,y_m,y_m) \leq & G(y_n,y_{n+1},y_{n+1}) + G(y_{n+1},y_{n+2},y_{n+2})+ \\ & \cdots + G(y_{m-1},y_m,y_m) \\ = & \gamma_n + \gamma_{n+1} + ... + \gamma_{m-1} \\ = & \sum_{i=n}^{m-1} \gamma_i \\ \leq & \sum_{i=n}^{\infty} \gamma_i \\ \leq & \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.\end{aligned}$$

From the convergence of the series $\sum_{i=1}^{\infty}\frac{1}{i^{\frac{1}{k}}}$ and the above inequality we receive that (y_n) is a Cauchy sequence. From the completeness of X there exists $u\in X$ such that $\lim_{n\to\infty}y_n=u$ and $\lim_{n\to\infty}y_n=\lim_{n\to\infty}fx_n=\lim_{n\to\infty}gx_{n+1}=u$.
Since f or g is continuous, suppose that g is continuous therefore $\lim_{n\to\infty}gfx_n=gu$. Further f and g are compatible, therefore

$$G(gfx_n,fgx_n,fgx_n)=0,$$

implies

$$\lim_{n\to\infty}fgx_n=gu.$$

Suppose that

$$G(fgx_n,fx_n,fx_n)>0.$$

Then from (2.4), we obtain

$$\tau+F(G(fgx_n,fx_n,fx_n))\le F(\beta[G(ggx_n,fgx_n,fgx_n)+2G(gx_n,fx_n,fx_n)] \tag{2.23}$$

proceeding limit as $n\to\infty$, we obtain

$$\begin{aligned}\tau+F(G(gu,u,u))&\le F(\beta[G(gu,gu,gu)+2G(u,u,u))\\&=F(0).\end{aligned}$$

Thus from $(F2)$ we get

$$\tau+F(G(gu,u,u))\le-\infty$$

a contraction. Then $G(fgx_n,fx_n,fx_n)=0$ so we obtain $G(gu,u,u)=0$ that is, $gu=u$. Thus u is a fixed point of g. Now suppose that $G(fx_n,fu,fu)>0$. Thus from (2.4) we obtain

$$\tau+F(G(fx_n,fu,fu))\le F(\beta[G(gx_n,fx_n,fx_n)+2G(gu,fu,fu)] \tag{2.24}$$

proceeding limit as $n\to\infty$, we obtain

$$\begin{aligned}\tau+F(G(u,fu,fu))&\le F(\beta[G(u,u,u)+2G(u,fu,fu)])\\&=F(2\beta G(u,fu,fu).\end{aligned}$$

a contraction. Then $G(fx_n,fu,fu)=0$, we obtain $G(u,fu,fu)=0$ that is, $fu=u$. Thus u is a common fixed point of f and g. We assume that $u\neq w$ and w is another common fixed point of f and g. Then $G(fu,fw,fw)>0$ and from (2.4) we get

$$\tau+F(G(fu,fw,fw))\le F\beta[G(gu,fu,fu)+2G(gw,fw,fw)]$$

that is,

$$\begin{aligned}\tau+F(G(u,w,w))&\le F(\beta[G(u,u,u)+2G(w,w,w)])\\&=F(0).\end{aligned}$$

From $(F2)$ we obtain

$$\tau+F(G(u,w,w))\le-\infty$$

a contradiction. Then we obtain $G(fu,fw,fw)=0$ that is, $u=w$. Therefore, the common fixed point of f and g is unique.

Example 2.1. Let $X=[-1,1]$ and let $G:X\times X\times X\to\mathbb{R}^+$ defined as follows:

$$G(x,y,z)=\max\{|x-y|,|y-z|,|x-z|\},$$

for all x,y,z in X. Then (X,G) is a G-metric space. Define $f(x)=\frac{x}{8}$ and $g(x)=\frac{x}{2}$. Here we note that, f is continuous and $f(X)\subseteq g(X)$. If $F(\alpha)=\ln\alpha$, $\alpha\in(0,\infty)$, and $\tau=\ln 4$ then

$$G(fx,fy,fz)\le\frac{1}{4}G(gx,gy,gz)$$

which indicates the presence of the type 1 F-contraction mapping.

Acknowledgement

The main results of this paper were presented in part at the International Conference on Mathematics and Mathematics Education, May 11-13, 2017, Şanlıurfa, Turkey.

References

[1] D. Wardowski, Fixed Point of a new type of contactive mappings in complete metric spaces, Fixed Point Theory and Applications, 94, 6 pp., (2012), DOI:10.1186/1687-1812-2012-94.
[2] S. Banach, Surles operations dansles ensembles abstracits et leur application aux equations integrales, Fund Math., 3, (1922), 133-181.
[3] Z. Mustafa and B. Sims, A new approach to a generalized metric spaces, J. Nonlinear Convex Anal., 7, (2), (2006), 289-297.
[4] M. Abbas, A. R. Khan and T. Nazir, Coupled common fixed point results in two generalized metric spaces, Applied Mathematics and Computation, Vol:217 No.13 pp. (2011), 6328-6336.
[5] M. Abbas, T. Nazir and D. Doric, Common fixed point of mappings satisfying (E.A) property in generalized metric spaces, Applied Mathematics and Computation, Vol:218, No.14 pp., (2012), 7665-7670.
[6] Z. Mustafa, W. Shatanawi and M. Bataineh, Existence of fixed point results in G-metric spaces, Int. J. Math. and Math. Sci., page 10, (2009), DOI: 10.1155/2009/283028.
[7] H.S. Ding and E. Karapınar, A note on some coupled fixed point theorems on G-metric space, Journal of Inequalities and Applications, Vol:2012, article 170, (2012), DOI: 10.1186/1029-242X-2012-170.
[8] G. Jungck, Commuting mappings and fixed point, Amer. Math. Monthly, Vol:83, No.4 (1976), 261-263.
[9] G. Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci., 9, (1986), 771-779.
[10] M. Kumar, Compatible Maps in G-Metric spaces, Int. Journal of Math. Anaysis, Vol:6, No.29 (2012), 1415-1421.
[11] M. Cosentino, and P. Vetro, Fixed point results for F-contractive mappings of Hardy-Rogers-type, Filomat Vol:28, No.4 pp., (2014), 715-722.
[12] J. Ahmad, A. Rawashdeh and A. Azam, New fixed point theorems for generalized F-contractions in complete metric space, Fixed Point Theory and Applications, 80, (2015), DOI: 10.1186/s13663-015-0333-2.
[13] R. Batra and S. Vashistha, Fixed points of an F-contraction on metric spaces with a graph, Int. J. Comput. Math., Vol:91, (12), (2014), 2483-2490.
[14] S. Chong, J. K. Kim, L. Wong, and J. Tang, Existence of fixed points for generalized F-contractions and generalized F-suzuki contractions in metric spaces, Global Journal of pure and applied Mathematics, ISSN 0973-1768, Vol:12, No.6 (2016), pp. 4867-4882.
[15] G. Mınak, A. Helvacı, and I. Altun, Ćirić type generalized F-contractions on complete metric spaces and fixed point results, Filomat, 28, (6), (2014), 1143-1151.
[16] H. Piri and P. Kumam, Some fixed point theorems concerning F-contraction in complete metric spaces, Fixed Point Theory Appl., 210, (2014), DOI: 10.1186/1687-1812-2014,210.
[17] D. Wardowski and N. Van Dung, Fixed points of F -weak contractions on complete metric spaces, Demonstr. Math., Vol:XLVII No.1, (2014), 146-155.
[18] R. Batra, S. Vashistha and R. Kumar, A coincidence point theorem for F-contractions on metric spaces equipped with an altered distance, J. Math. Comput. Sci., 4, (5), (2014), 826-833.