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Sensitivity of Schur Stability of the $k-th$ Order Difference Equation System $y(n+k) = Cy(n)$

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Abstract

In this study, it is investigated that the Schur stable difference equation systems $y(n+k) = Cy(n)$ under which perturbations remains Schur stable. Some continuity theorems of the first order systems in the literature are re-expressed for the $k-th$ order system $y(n+k) = Cy(n)$. All the results obtained are also supplemented by numerical examples.

Keywords: *Condition number; Difference equations; Schur stability; Sensitivity; Perturbation systems.*
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1. Introduction

Consider the difference equation system

$$x(n+1) = Ax(n), x(0) = x_0, \tag{1.1}$$

where $A \in M_N(\mathbb{R})$ and $x(n)$ is $N-$ dimensional vector.
Schur stability parameter $\omega(A)$ for the systems (1.1) is defined as

$$\omega(A) = ||H||;\ H = \sum_{k=0}^{\infty}(A^*)^k A^k,\ A^*HA–H+I = 0,\ H = H^* > 0,$$

where I is unit matrix, A^* is adjoint of the matrix A, $||A|| = \max_{||x||=1}||Ax||$ is spectral norm of the matrix A, furthermore the norm $||x||$ is Euclidean norm for the vector $x = (x_1, x_2, ...x_N)^T$ [1]– [7]. $\omega(A) < \infty$ then difference system (1.1) is Schur stable. Otherwise, we set $\omega(A) = \infty$. Moreover, let ω^* be the practical Schur stability parameter of the system (1.1), where $1 \leq \omega^* \in \mathbb{R}$ and the user choose the value ω^* in view of their concrete physical problem. If $\omega(A) \leq \omega^*$ then the matrix A is ω^*- Schur stable matrix. Otherwise, the matrix A is ω^*- Schur unstable matrix [1], [2], [5]– [7].
In this study, some results on the sensitivity of Schur Stability for the system (1.1) are given in section 2 and the new results are provided to show the durability of the Schur stability of $k-th$ order difference equation system $y(n+k) = Cy(n)$ in section 3. The examples are given for the obtained results.

2. Some Results on The Sensitivity of Schur Stability in the Literature

Predicting the behaviour of solution of a problem and knowing under which conditions similar properties are protected under perturbations is important for the problem not to cause any chaos. The question ”how much perturbation is ignorable for preserving the characteristic properties?” is known as the sensitivity problem. In this section, we give a result in the literature on the sensitivity of the Schur stability of the systems with constant coefficients.
Let the perturbation system of (1.1)

$$u(n+1) = (A+B)u(n), u(0) = u_0 \tag{2.1}$$

where $B \in M_N(\mathbb{R})$. The following continuity theorems show that how much perturbation is permissible without disturbing the Schur stability.

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Theorem 1. ***i.*** *Let suppose that A is a Schur stable matrix, that is* $\omega(A)<\infty$. *If*

$$||B|| < \sqrt{||A||^2+\frac{1}{\omega(A)}}-||A||$$

is satisfied, then $A+B$ *is Schur stable.Moreover, the inequality*

$$|\omega(A+B)-\omega(A)| \leq \frac{(2||A||+||B||)||B||\omega^2(A)}{1-(2||A||+||B||)||B||\omega(A)}$$

holds (Corollary 1 in [4]).

ii. *Let A be a* ω^*-*Schur stable matrix* $(\omega(A)\leq\omega^*)$. *If*

$$||B|| < \sqrt{||A||^2+\frac{\omega^*-\omega(A)}{\omega^*\omega(A)}}-||A||$$

satisfied, then $A+B$ *is* ω^*- *Schur stable (Theorem 5 in [4]).*

3. Sensitivity of Schur Stability of the $k-th$ Order Difference Equation Systems $y(n+k)=Cy(n)$

Consider the $k-th$ order equation system

$$y(n+k)=Cy(n), n=0,1,2,\ldots \tag{3.1}$$

where $C\in M_N(\mathbb{R})$ and $y(n)$ is $N-$ dimensional vector with initial conditions $y(0)=y_0, y(1)=y_1,\ldots,y(k-1)=y_{k-1}$.
$k-th$ order system (3.1) is equivalent to the following first order system

$$x(n+1)=Ax(n), x(0)=x_0,$$

where $x(n)=\begin{pmatrix} y(n)\\ y(n+1)\\ \vdots\\ y(n+k-1)\end{pmatrix}$ is a $k\times N$ dimensional vector and $A=\begin{pmatrix} 0_{(k-1)N\times N} & I_{(k-1)N\times(k-1)N}\\ C_{N\times N} & 0_{N\times(k-1)N}\end{pmatrix}$ is $k\times N$ dimensional square matrix [8].

Let give the following theorems 2 and 3 for the system (3.1) for $k=2$.

Theorem 2. *Let assume that (1.1) and (3.1) for* $k=2$ *are the Schur stable systems, the matrix X and Y are the solution of discret Lyapunov matrix equation (DLME) for the systems (1.1) and (3.1), respectively. For fundamental matrix Y and the condition number of C, the following equations are provided:*

$$X=\begin{pmatrix} 2Y-I & 0\\ 0 & 2Y\end{pmatrix}, \omega(A)=2\omega(C).$$

Proof. The solution matrix of the DLME $C^*YC-Y+I=0$ is known as $Y=\sum_{k=0}^{\infty}(C^*)^kC^k$. On the other hand the solution matrix of DLME $A^*XA-X+I=0$ is expressed as follows:

$$\begin{aligned}
X &= \sum_{k=0}^{\infty}(A^*)^kA^k\\
&= I+\begin{pmatrix} C^*C & 0\\ 0 & I\end{pmatrix}+\begin{pmatrix} C^*C & 0\\ 0 & C^*C\end{pmatrix}+\begin{pmatrix} (C^*)^2C^2 & 0\\ 0 & C^*C\end{pmatrix}\\
&+\begin{pmatrix} (C^*)^2C^2 & 0\\ 0 & (C^*)^2C^2\end{pmatrix}+\begin{pmatrix} (C^*)^3C^3 & 0\\ 0 & (C^*)^2C^2\end{pmatrix}+\begin{pmatrix} (C^*)^3C^3 & 0\\ 0 & (C^*)^3C^3\end{pmatrix}\\
&+\begin{pmatrix} (C^*)^4C^4 & 0\\ 0 & (C^*)^3C^3\end{pmatrix}+\begin{pmatrix} (C^*)^4C^4 & 0\\ 0 & (C^*)^4C^4\end{pmatrix}+\ldots\\
&= \begin{pmatrix} I+2(C^*C+(C^*)^2C^2+\ldots) & 0\\ 0 & 2(I+C^*C+(C^*)^2C^2+\ldots)\end{pmatrix}=\begin{pmatrix} I+2\sum_{k=1}^{\infty}(C^*)^kC^k & 0\\ 0 & 2\sum_{k=1}^{\infty}(C^*)^kC^k\end{pmatrix}.
\end{aligned}$$

Therefore

$$X=\begin{pmatrix} 2Y-I & 0\\ 0 & 2Y\end{pmatrix}$$

is obtained. Because of the Schur stability of C, the condition number of A is

$$\omega(A)=||X||=2||Y||=2\omega(C),$$

where the condition number of C is $\omega(C)=||Y||$. □

Example 1. Consider the second order difference equation system $y(n+2)=Cy(n)$, where $C=\begin{pmatrix} 0.25 & 0.2\\ 0.4 & 0.2\end{pmatrix}$ with the condition number $\omega(C)=1.40092$. This system is equivalent to the first order difference equation system

$$x(n+1)=Ax(n), \tag{3.2}$$

where $A=\begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0.25 & 0.2 & 0 & 0\\ 0.4 & 0.2 & 0 & 0\end{pmatrix}$. It is seen that $\omega(A)=2\omega(C)=2.80184$ for the system (3.2).

Let the perturbation system of (3.1) be

$$z(n+k)=(C+D)z(n) \tag{3.3}$$

where D is N dimensional square matrix.

Theorem 3. ***i.*** *Let suppose that $||C|| > 1$ and C is a Schur stable matrix (system (3.1) for $k = 2$ is Schur stable). For the perturbation matrix D which satisfied the inequality $||D|| < \sqrt{||C||^2 + \frac{1}{2\omega(C)}} - ||C||$, $C+D$ is Schur stable matrix (system (3.3) for $k = 2$ is Schur stable).*
ii. *Let $||C|| > 1$ and C is a ω^*-Schur stable matrix (system (3.1) for $k = 2$ is ω^*-Schur stable) i.e. $\omega(C) \leq \omega^*$. If D satisfies the inequality*

$$||D|| < \sqrt{||C||^2 + \frac{\omega^* - \omega(C)}{2\omega^*\omega(C)}} - ||C||$$

then the matrix $C+D$ is ω^*-Schur stable, too.

Proof. **i.** The system (1.1) is Schur stable because of the Schur stability of the system (3.1) for $k = 2$. For the coefficient matrix A of the system (1.1), $||A|| = max\lambda(A^*A) = max\{||C||, 1\}$ and for the coefficient matrices of the systems (2.1) and (3.3), $||B|| = ||D||$. From Theorem 1.i it can be written

$$\omega(C) \leq \frac{\omega(C)}{1-2(||C||+||D||)||D||\omega(C)}.$$

It must be satisfied that

$$1-2(||C||+||D||)||D||\omega(C) > 0$$

for the Schur stability of the matrix $C+D$, that is $\omega(C+D) < \infty$. So, if the inequality $2(||C||+||D||)||D||\omega(C) < \frac{1}{2}$ is solved according to $||D||$, then

$$||D|| < \sqrt{||C||^2 + \frac{1}{2\omega(C)}} - ||C||$$

is obtained.
ii. The ω^* Schur stability of the matrix $C+D$ depends on the provision of the inequality $\omega(C+D) \leq \omega^*$, for the perturbation matrix D. From Theorem 1.i, it is obtained that

$$\omega(C+D) \leq \frac{\omega(C)}{1-2(||C||+||D||)||D||\omega(C)} \leq \omega^*.$$

And if the inequality

$$\frac{\omega(C)}{1-2(||C||+||D||)||D||\omega(C)} \leq \omega^*.$$

is solved according to $||D||$, the inequality

$$||D|| < \sqrt{||C||^2 + \frac{\omega^* - \omega(C)}{2\omega^*\omega(C)}} - ||C||$$

is obtained. □

Example 2. Consider the system $x(n+2) = Cx(n)$ where $C = \begin{pmatrix} 1 & -0.5 \\ 0.75 & -1 \end{pmatrix}$. For this system we obtain $||C|| = 1.63278$ and $\omega(C) = 6.01596$. So, this system is Schur stable. The matrix C persists on its Schur stability if perturbated with

$-$ $||D|| < 0.0252558$.
We let perturbate the system with the matrix $D = \begin{pmatrix} 0.025 & 0 \\ 0 & -0.025 \end{pmatrix}$, $||D|| = 0.025$.
While the system is perturbated by D, we have $\omega(C+D) = 6.89495$, hence the system is Schur stable.
We have applied Theorem 3.ii. to these system. Herein, if $\omega^* = 10$ in the given system;
$-$ $10-$Schur stability boundary is 0.0101085.
Since $\omega(C) \leq 10$, the matrix C is $10-$Schur stable. Also, it follows from Theorem 3.ii., for any perturbation matrix satisfying $||D|| \leq 0.0101085$, we know that matrix $C+D$ is $10-$Schur stable. Let the perturbation matrix

$$D = \begin{pmatrix} 0.0101085 & 0 \\ 0 & -0.0101085 \end{pmatrix}, ||D|| = 0.0101085.$$

We have $C+D$, thus we see that $\omega(C+D) = 6.33836 < 10$ holds, therefore, the matrix $C+D$ is $10-$ Schur stable.
In the system given by equation 3.1, the following results are obtained for any k order. Their proofs are similar to the proofs given above, so they will not be given here.

Corollary 1. ***i.*** *Schur stable $k-th$ order linear difference equation system $y(n+k) = Cy(n)$ can be converted the first order system $x(n+1) = Ax(n)$ with matrix $A = \begin{pmatrix} 0 & I \\ C & 0 \end{pmatrix}$ and also $\omega(A) = k\omega(C)$ is provided.*
ii.*The matrix $C+D$ is Schur stable for the perturbation matrix D satisfied the inequality $||D|| < \sqrt{||C||^2 + \frac{1}{k\omega(C)}} - ||C||$, where $||C|| > 1$ and $||C||$ is a Schur stable matrix;*
iii. *The matrix $C+D$ is ω^*-Schur stable for the perturbation matrix D satisfied the inequality*

$$||D|| \leq \sqrt{||C||^2 + \frac{\omega^* - \omega(C)}{k\omega^*\omega(C)}} - ||C||,$$

where $||C|| > 1$ and C is a ω^*-Schur stable matrix ($\omega(C) \leq \omega^*$).

Example 3. Consider the fourth order difference equation system $y(n+4) = Cy(n)$, where $A = \begin{pmatrix} 0.7 & 0.1 & 0 & -1 \\ 0 & -0.9 & 0 & 0 \\ 0.6 & 0 & 0.3 & 0 \\ 0 & 2 & 0 & -0.1 \end{pmatrix}$ with the condition number $\omega(C) = 45.2706$.
i. This system is equivalent to the first order difference equation system

$$x(n+1) = \begin{pmatrix} 0_{12\times 4} & I_{12} \\ C & 0_{4\times 12} \end{pmatrix} = Ax(n). \tag{3.4}$$

It is seen that $\omega(A) = 4\omega(C) = 181.0824$ for the system (3.4).
ii. The matrix C persists on its Schur stability if perturbated with
$-$ $||D|| < 0.00125375$.

We let perturbate the system with the matrix

$$D=\begin{pmatrix} 0.00125 & 0 & 0 & 0 \\ 0 & -0.00125 & 0 & 0 \\ 0 & 0 & 0.00125 & 0 \\ 0 & 0 & 0 & -0.00125 \end{pmatrix}, ||D|| = 0.00125.$$

While the system is perturbated by D, we have $\omega(C+D) = 45.8647$, hence the system is Schur stable.
ii. We have applied Theorem (3.ii.) to these system. Herein, if $\omega^* = 50$ in the given system;
$-\ 50-$ Schur stability boundary is 0.000118621.
Since $\omega(C) \leq 50$, the matrix C is $50-$ Schur stable. Also, it follows from Theorem (3.ii.), for any perturbation matrix satisfying $||D|| \leq 0.000118621$, we know that matrix $C+D$ is $50-$ Schur stable. Let the perturbation matrix

$$D=\begin{pmatrix} 0.0001186 & 0 & 0 & 0 \\ 0 & -0.0001186 & 0 & 0 \\ 0 & 0 & 0.0001186 & 0 \\ 0 & 0 & 0 & -0.0001186 \end{pmatrix}, ||D|| = 0.0001186.$$

We have $C+D$, thus we see that $\omega(C+D) = 45.3263 < 50$ holds, therefore, the matrix $C+D$ is $50-$ Schur stable.
Note.The numerical examples have been computed by using matrix vector calculator MVC [9].

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