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On the rate of convergence of different implicit iterations in convex metric spaces

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Abstract

In this paper, we introduce a new three-step implicit iteration process and prove that it is faster than the other implicit iteration processes. We prove some convergence theorem for generalized contraction mappings in convex metric space. We also support our results by a numerical example.

Keywords: *Convergence rate, implicit iterative processes*

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1. Introduction and preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers. Let E be a nonempty subset of a metric space X, and $T : E \to E$ a mapping. We denote the set of all fixed points of T by $F(T) = \{p \in E : p = Tp\}$.
In 1970, Takahashi [19] introduced the notion of convex metric spaces and studied the fixed point theory for nonexpansive mappings in this setting. Later on, many authors ([5, 11, 14, 16, 21]) discussed the existence of the fixed point and the convergence of the iterative process for various mappings in convex metric spaces.
We recall some definitions as follows:

Definition 1. *[19]A convex structure in a metric space (X,d) is a mapping $W : X^2 \times [0,1] \to X$ satisfying, for all $x,y,u \in X$ and all $\alpha \in [0,1]$,*

$$d(W(x,y;\alpha),u) \leq \alpha d(x,u) + (1-\alpha) d(y,u). \tag{1.1}$$

A metric space X together with W is called a convex metric space.
A nonempty subset C of X is said to be convex if $W(x,y;\alpha) \in C$ for all $(x,y;\alpha) \in C \times C \times [0,1]$.

All normed spaces and their convex subsets are convex metric spaces. But there are many examples of convex metric spaces which are not embedded in any normed space(see Takahashi [19]).
In the last decades many papers have been published on the implicit iterative approximation of fixed points for certain classes of mappings in various spaces ([7], [8], [21]). In the convex metric and linear spaces setting we shall state some of these iteration processes as follows:
Let K be a nonempty convex subset of a convex metric space X and $T : K \to K$ be a given mapping. Then for $x_0 \in K$, Chugh et al. [6] introduced the following three-step iteration:

$$\begin{aligned} x_n &= W(x_{n-1}, Ty_n, \alpha_n), \\ y_n &= W(z_n, Tz_n, \beta_n), \\ z_n &= W(x_n, Tx_n, \gamma_n), \quad n \in \mathbb{N}, \end{aligned} \tag{1.2}$$

where $\{a_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$.
They also prove that the iteration (1.2) is faster than implicit Mann iteration and implicit Ishikawa iteration.

Remark 1. *(i) If we take $\gamma_n = 1$, then the iteration process (1.2) reduces to the following implicit Ishakawa iteration:*

$$\begin{aligned} x_n &= W(x_{n-1}, Ty_n, \alpha_n) \\ y_n &= W(x_n, Tx_n, \gamma_n), \quad n \in \mathbb{N}. \end{aligned} \tag{1.3}$$

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(ii) Taking $\gamma_n = \beta_n = 1$ in (1.2), we see that this process reduces to the following implicit Mann iteration (Ćirić et al., [8, 9]):

$$x_n = W(x_{n-1}, Ty_n, \alpha_n), \quad n \in \mathbb{N}. \tag{1.4}$$

Motivated by the above works, we define a new implicit iteration process with higher rate of convergence when compared with three-step iteration (1.2), implicit Ishikawa (1.3) and implicit Mann (1.4) iterative processes. Our iteration process as follows:

$$\begin{aligned} x_n &= W(Tx_{n-1}, Ty_n, \alpha_n), \\ y_n &= W(z_n, Tz_n, \beta_n), \\ z_n &= W(x_n, Tx_n, \gamma_n), \ n \in \mathbb{N}, \end{aligned} \tag{1.5}$$

where $\{a_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$.

Remark 2. *(i) If we take $\gamma_n = 1$, then the iteration process (1.5) reduces to the following implicit S-iteration:*

$$\begin{aligned} x_n &= W(Tx_{n-1}, Ty_n, \alpha_n) \\ y_n &= W(x_n, Tx_n, \beta_n), \ n \in \mathbb{N}. \end{aligned} \tag{1.6}$$

(ii) The process (1.5) is independent of (1.3) and (1.4): neither of them reduce to the other.

Recently, Yildirim and Abbas [20] introduced the S-iteration process and they gave some convergence results for a contractive type mapping in the framework of W-hyperbolic spaces.
Now, let us recall the following definitions:
Suppose that there exist real numbers a, b, c satisfying $0 < a < 1, 0 < b, c < 1/2$ such that, for each pair $x, y \in E$, at least one of the following is true:

$$\begin{cases} (z_1) & d(Tx, Ty) \le ad(x,y), \\ (z_2) & d(Tx, Ty) \le b(d(x, Tx) + d(y, Ty)), \\ (z_3) & d(Tx, Ty) \le c(d(x, Ty) + d(y, Tx)). \end{cases} \tag{1.7}$$

Such a mapping is called a Zamfirescu mapping. Zamfirescu [22] obtained an important generalization of Banach fixed point theorem using Zamfirescu mapping. The condition (z_1) of (1.7) is the well known contraction condition or Banach's contraction condition introduced by Banach [2]. Any mapping satisfying the condition (z_2) of (1.7) is called a Kannan mapping, while the mapping satisfying the condition (z_3) is called Chatterjea operator.
It was shown in [4], the contractive condition (1.7) gives

$$\begin{cases} (b_1) & d(Tx, Ty) \le \delta d(x,y) + 2\delta d(x, Tx) \text{ if one uses } (z_2), \text{ and} \\ (b_2) & d(Tx, Ty) \le \delta d(x,y) + 2\delta d(x, Ty) \text{ if one uses } (z_3), \end{cases} \tag{1.8}$$

for all $x, y \in E$, where $\delta := \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}, \delta \in [0,1)$.
A mapping satisfying condition (b_1) or (b_2) is called a quasi-contractive mapping. This class of mappings is general than the class of Zamfirescu mappings.
In 1999, Osilike and Udomene [17] introduced a mapping T satisfying the following contractive condition:

$$d(Tx, Ty) \le \delta d(x,y) + Ld(x, Tx), \tag{1.9}$$

for all $x, y \in E$, where $L \ge 0$ and $\delta \in [0,1)$. This class of mappings is general than the above class of mappings.
Further in this direction, Imoru and Olantiwo [10] gave the following definition:

Definition 2. *A self mapping T on E is called a contractive-like mapping if there exists a constant $\delta \in [0,1)$ and a strictly increasing and continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that the following holds*

$$d(Tx, Ty) \le \delta d(x,y) + \varphi(d(x, Tx)), \tag{1.10}$$

for each $x, y \in E$.

In order to compare two fixed point iteration processes $\{a_n\}$ and $\{b_n\}$ that converge to a certain fixed point p of a given mapping T, Berinde [3] introduced the following formulation.

Definition 3. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive numbers that converge to a, respectively b. If*

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = 0,$$

then it can be said that $\{a_n\}$ converges to a faster than $\{b_n\}$ converges to b.

Definition 4. *Let $\{x_n\}$ and $\{u_n\}$ be two fixed point iteration processes in a metric space X such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} u_n = p$, where p is a fixed point of a self mapping T on X. Suppose that*

$$d(x_n, p) \le a_n \text{ and } d(u_n, p) \le b_n, \ n \in \mathbb{N}.$$

where $\{a_n\}$ and $\{b_n\}$ are two null sequences of positive numbers. If $\{a_n\}$ converges faster than $\{b_n\}$, then we say $\{x_n\}$ converges faster than $\{u_n\}$ to p.

We need the following lemma in order to prove our main results.

2. Main Results

We start by proving the following important result.

Theorem 1. *Let E be a nonempty closed convex subset of a convex metric space X and $T:E\to E$ be a quasi-contractive operator satisfying (1.10) with $F(T)\neq\emptyset$. Then, for $x_0\in E$, $\{x_n\}$ defined by (1.5) with $\sum(1-\alpha_n)=\infty$, converges to the fixed point of T.*

Proof. Let $p\in F(T)$. From (1.5) and (1.10), we have

$$\begin{aligned} d(x_n,p) &= d(W(Tx_{n-1},Ty_n,\alpha_n),p) \\ &\leq \alpha_n d(Tx_{n-1},p)+(1-\alpha_n)d(Ty_n,p) \\ &\leq \alpha_n[\delta d(x_{n-1},p)+\varphi(d(p,Tp))] \\ &\quad +(1-\alpha_n)[\delta d(y_n,p)+\varphi(d(p,Tp))], \end{aligned} \tag{2.1}$$

$$\begin{aligned} d(y_n,p) &= d(W(z_n,Tz_n,\beta_n),p) \\ &\leq \beta_n d(z_n,p)+(1-\beta_n)d(Tz_n,p) \\ &\leq \beta_n d(z_n,p)+(1-\beta_n)[\delta d(z_n,p)+\varphi(d(p,Tp))], \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} d(z_n,p) &= d(W(x_n,Tx_n,\gamma_n),p) \\ &\leq \gamma_n d(x_n,p)+(1-\gamma_n)d(Tx_n,p) \\ &\leq \gamma_n d(x_n,p)+(1-\gamma_n)[\delta d(x_n,p)+\varphi(d(p,Tp))] \end{aligned} \tag{2.3}$$

As $\varphi(d(p,Tp))=0$, so (2.1), (2.2) and (2.3) become

$$d(x_n,p)\leq \alpha_n\delta d(x_{n-1},p)+(1-\alpha_n)\delta d(y_n,p), \tag{2.4}$$

$$\begin{aligned} d(y_n,p) &\leq \beta_n d(z_n,p)+(1-\beta_n)\delta d(z_n,p) \\ &= [\beta_n+(1-\beta_n)\delta]\,d(z_n,p), \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} d(z_n,p) &\leq \gamma_n d(x_n,p)+(1-\gamma_n)\delta d(x_n,p) \\ &= [\gamma_n+(1-\gamma_n)\delta]\,d(x_n,p). \end{aligned} \tag{2.6}$$

Combining (2.4), (2.5) and (2.6), we obtain that

$$d(x_n,p)\leq \alpha_n\delta d(x_{n-1},p)+(1-\alpha_n)\delta[\beta_n+(1-\beta_n)\delta][\gamma_n+(1-\gamma_n)\delta]\,d(x_n,p)$$

which implies that

$$[1-(1-\alpha_n)\delta[\beta_n+(1-\beta_n)\delta][\gamma_n+(1-\gamma_n)\delta]]\,d(x_n,p)\leq \alpha_n\delta d(x_{n-1},p).$$

That is,

$$d(x_n,p)\leq \frac{\alpha_n\delta}{1-(1-\alpha_n)\delta[\beta_n+(1-\beta_n)\delta][\gamma_n+(1-\gamma_n)\delta]}d(x_{n-1},p). \tag{2.7}$$

We set

$$\Delta_n=\frac{\alpha_n\delta}{1-(1-\alpha_n)\delta[\beta_n+(1-\beta_n)\delta][\gamma_n+(1-\gamma_n)\delta]}.$$

Then

$$\begin{aligned} 1-\Delta_n &= 1-\frac{\alpha_n\delta}{1-(1-\alpha_n)\delta[\beta_n+(1-\beta_n)\delta][\gamma_n+(1-\gamma_n)\delta]} \\ &= \frac{1-(1-\alpha_n)\delta[\beta_n+(1-\beta_n)\delta]-\alpha_n\delta}{1-(1-\alpha_n)\delta[\beta_n+(1-\beta_n)\delta][\gamma_n+(1-\gamma_n)\delta]} \\ &\geq 1-(1-\alpha_n)\delta[\beta_n+(1-\beta_n)\delta][\gamma_n+(1-\gamma_n)\delta]-\alpha_n\delta \end{aligned}$$

which further implies that

$$\begin{aligned} \Delta_n &\leq (1-\alpha_n)\delta[\beta_n+(1-\beta_n)\delta][\gamma_n+(1-\gamma_n)\delta]+\alpha_n\delta \\ &= (1-\alpha_n)\delta[1-(1-\delta)(1-\beta_n)][\gamma_n+(1-\gamma_n)\delta]+\alpha_n\delta \\ &\leq (1-\alpha_n)\delta+\alpha_n \\ &= 1-(1-\alpha_n)(1-\delta). \end{aligned} \tag{2.8}$$

Now by (2.7) and (2.8), we have

$$\begin{aligned} d(x_n,p) &\leq [1-(1-\alpha_n)(1-\delta)]\,d(x_{n-1},p) \\ &\leq \prod_{i=1}^{n}[1-(1-\alpha_i)(1-\delta)]\,d(x_0,p). \end{aligned} \tag{2.9}$$

Since $a>0$, $1+a\leq e^a$, (2.9) gives that

$$\begin{aligned} d(x_n,p) &\leq \exp\left\{\sum_{i=1}^{n}(1-\alpha_i)(1-\delta)\right\}d(x_0,p) \\ &\leq \exp\left\{\sum_{n=1}^{\infty}(1-\alpha_n)(1-\delta)\right\}d(x_0,p) \end{aligned} \tag{2.10}$$

Using fact that $0\leq\delta<1$ and $\sum_{i=1}^{\infty}(1-\alpha_n)=\infty$, we conclude that $\lim_{n\to\infty}d(x_n,p)=0$. Therefore, $\{x_n\}$ converges to p. □

Now, we will give our result on the rate of convergence for implicit iterations.

Theorem 2. *Let E be a nonempty closed convex subset of a convex metric space X and $T:E\to E$ be a quasi-contractive operator satisfying (1.10) with $F(T)\neq\emptyset$. Then, for $x_0\in E$, $\{x_n\}$ defined by (1.5) with $\sum(1-\alpha_n)=\infty$, converges to the fixed point of T faster than implicit three-step (1.2) iterative process.*

Proof. Let p a fixed point of T. From (1.2) and (1.10), we have that

$$\begin{aligned} d(x_n,p) &= d(W(x_{n-1},Ty_n,\alpha_n),p) \\ &\leq \alpha_n d(x_{n-1},p)+(1-\alpha_n)d(Ty_n,p) \\ &\leq \alpha_n d(x_{n-1},p)+(1-\alpha_n)[\delta d(y_n,p)+\varphi(d(p,Tp))], \end{aligned} \tag{2.11}$$

$$\begin{aligned} d(y_n,p) &= d(W(z_n,Tz_n,\beta_n),p) \\ &\leq \beta_n d(z_n,p)+(1-\beta_n)d(Tz_n,p) \\ &\leq \beta_n d(z_n,p)+(1-\beta_n)[\delta d(z_n,p)+\varphi(d(p,Tp))], \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} d(z_n,p) &= d(W(x_n,Tx_n,\gamma_n),p) \\ &\leq \gamma_n d(x_n,p)+(1-\gamma_n)d(Tx_n,p) \\ &\leq \gamma_n d(x_n,p)+(1-\gamma_n)[\delta d(x_n,p)+\varphi(d(p,Tp))]. \end{aligned} \tag{2.13}$$

Using $\varphi(d(p,Tp))=0$, (2.11), (2.12) and (2.13), we obtain

$$\begin{aligned} d(x_n,p) &\leq \alpha_n d(x_{n-1},p)+(1-\alpha_n)\delta[\beta_n+\delta(1-\beta_n)] \\ &\quad [\gamma_n+\delta(1-\gamma_n)]\,d(x_n,p). \end{aligned} \tag{2.14}$$

That is,

$$\begin{aligned} d(x_n,p) &\leq \frac{\alpha_n}{1-(1-\alpha_n)\delta[\beta_n+\delta(1-\beta_n)][\gamma_n+\delta(1-\gamma_n)]} \\ &\quad d(x_{n-1},p) \\ &\leq b_n \end{aligned} \tag{2.15}$$

where

$$b_n=\left(\frac{\alpha_n}{1-(1-\alpha_n)\delta[\beta_n+\delta(1-\beta_n)][\gamma_n+\delta(1-\gamma_n)]}\right)^n d(x_0,p). \tag{2.16}$$

From (2.7) at the proof of above theorem, we know that

$$d(x_n,p)\leq\frac{\alpha_n\delta}{1-(1-\alpha_n)\delta[\beta_n+(1-\beta_n)\delta][\gamma_n+(1-\gamma_n)\delta]}d(x_{n-1},p).$$

Let

$$\begin{aligned} d(x_n,p) &\leq \frac{\alpha_n\delta}{1-(1-\alpha_n)\delta[\beta_n+(1-\beta_n)\delta][\gamma_n+(1-\gamma_n)\delta]}d(x_{n-1},p) \\ &\leq a_n \end{aligned}$$

where

$$a_n=\left(\frac{\alpha_n\delta}{1-(1-\alpha_n)\delta[\beta_n+(1-\beta_n)\delta][\gamma_n+(1-\gamma_n)\delta]}\right)^n d(x_0,p). \tag{2.17}$$

Moreover, it is clear that $\lim_{n\to\infty}\frac{a_n}{b_n}=0$. Indeed, since

$$\begin{aligned}
[\gamma_n+(1-\gamma_n)\delta] &< 1 \Rightarrow [\beta_n+(1-\beta_n)\delta][\gamma_n+(1-\gamma_n)\delta]<1\\
&\Rightarrow (1-\alpha_n)\delta[\beta_n+(1-\beta_n)\delta][\gamma_n+(1-\gamma_n)\delta]<1\\
&\Rightarrow 1-(1-\alpha_n)\delta[\beta_n+(1-\beta_n)\delta][\gamma_n+(1-\gamma_n)\delta]>0,
\end{aligned}$$

we get

$$\begin{aligned}
\alpha_n\delta &< \alpha_n\delta \Rightarrow \frac{\alpha_n\delta}{1-(1-\alpha_n)\delta[\beta_n+\delta(1-\beta_n)][\gamma_n+\delta(1-\gamma_n)]}\\
&< \frac{\alpha_n}{1-(1-\alpha_n)\delta[\beta_n+\delta(1-\beta_n)][\gamma_n+\delta(1-\gamma_n)]}\\
&\Rightarrow \left(\frac{\alpha_n\delta}{1-(1-\alpha_n)\delta[\beta_n+\delta(1-\beta_n)][\gamma_n+\delta(1-\gamma_n)]}\right)^n\\
&< \left(\frac{\alpha_n}{1-(1-\alpha_n)\delta[\beta_n+\delta(1-\beta_n)][\gamma_n+\delta(1-\gamma_n)]}\right)^n.
\end{aligned}$$

This implies that

$$\begin{aligned}
\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{\left(\frac{\alpha_n\delta}{1-(1-\alpha_n)\delta[\beta_n+\delta(1-\beta_n)][\gamma_n+\delta(1-\gamma_n)]}\right)^n d(x_0,p)}{\left(\frac{\alpha_n}{1-(1-\alpha_n)\delta[\beta_n+\delta(1-\beta_n)][\gamma_n+\delta(1-\gamma_n)]}\right)^n d(x_0,p)}\\
&= 0.
\end{aligned}$$

So, the iteration process (1.5) is faster than the implicit three-step iteration (1.2). □

Recently Chugh et al. [6] proved that the iteration process (1.2) converges faster than implicit Ishikawa (1.3) and implicit Mann (1.4) iterative processes. This statement implies that the following result.

Corollary 1. *Let E be a nonempty closed convex subset of a convex metric space X and $T:E\to E$ be a quasi-contractive operator satisfying (1.10) with $F(T)\neq\emptyset$. Then, for $x_0\in E$, $\{x_n\}$ defined by (1.5) with $\sum_{n=1}^{\infty}(1-\alpha_n)=\infty$, converges to the fixed point of T faster than implicit Mann (1.4) and implicit Ishikawa (1.3) iterative processes.*

Now, we support our above analytical proof by a numerical example.

Example 1. *Let $E=[0,1]$, $Tx=\frac{x}{2}$, $x\neq 0$ and $\alpha_n=\beta_n=\gamma_n=\frac{1}{n}$ for $n\in\mathbb{N}$, then for implicit three-step iteration (1.2), we obtain that*

$$\begin{aligned}
x_n &= \frac{1}{n}x_{n-1}+\left(1-\frac{1}{n}\right)Ty_n=\frac{1}{n}x_{n-1}+\frac{1}{2}\left(1-\frac{1}{n}\right)y_n\\
y_n &= \frac{1}{n}z_n+\left(1-\frac{1}{n}\right)Tz_n=\frac{1}{n}z_n+\left(1-\frac{1}{n}\right)\frac{z_n}{2}=\frac{1}{2}\left(1+\frac{1}{n}\right)z_n\\
z_n &= \frac{1}{n}x_n+\left(1-\frac{1}{n}\right)Tx_n=\frac{1}{n}x_n+\left(1-\frac{1}{n}\right)\frac{x_n}{2}=\frac{1}{2}\left(1+\frac{1}{n}\right)x_n
\end{aligned}$$

This implies that

$$\begin{aligned}
x_n &= \frac{1}{n}x_{n-1}+\frac{1}{2}\left(1-\frac{1}{n}\right)\left[\frac{1}{2}\left(1+\frac{1}{n}\right)\frac{1}{2}\left(1+\frac{1}{n}\right)x_n\right]\\
&= \frac{1}{n}x_{n-1}+\frac{1}{8}\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^2 x_n.
\end{aligned}$$

That is,

$$\left[1-\frac{1}{8}\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^2\right]x_n=\frac{1}{n}x_{n-1}$$

and so

$$\begin{aligned}
x_n &= \frac{\frac{1}{n}}{1-\frac{1}{8}\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^2}x_{n-1}=\frac{n^2}{7n^3-n^2+n-1}x_{n-1} \qquad (2.18)\\
&= \prod_{i=1}^{n}\left[\frac{i^2}{7i^3-i^2+i-1}\right]x_0.
\end{aligned}$$

Using similar method, for our implicit iteration process (1.5), we have

$$\begin{aligned} x_n &= \frac{1}{n}Tx_{n-1} + \left(1-\frac{1}{n}\right)Ty_n \\ &= \frac{1}{n}\frac{x_{n-1}}{2} + \frac{1}{2}\left(1-\frac{1}{n}\right)y_n \\ y_n &= \frac{1}{n}z_n + \left(1-\frac{1}{n}\right)Tz_n \\ &= \frac{1}{n}z_n + \left(1-\frac{1}{n}\right)\frac{z_n}{2} = \frac{1}{2}\left(1+\frac{1}{n}\right)z_n \\ z_n &= \frac{1}{n}x_n + \left(1-\frac{1}{n}\right)Tx_n \\ &= \frac{1}{n}x_n + \left(1-\frac{1}{n}\right)\frac{x_n}{2} = \frac{1}{2}\left(1+\frac{1}{n}\right)x_n \end{aligned} \tag{2.19}$$

From (2.19), we get

$$x_n = \frac{1}{2n}x_{n-1} + \frac{1}{8}\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^2 x_n.$$

This implies that

$$\left[1-\frac{1}{8}\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^2\right]x_n = \frac{1}{2n}\frac{x_{n-1}}{2}$$

and hence

$$\begin{aligned} x_n &= \frac{\frac{1}{2n}}{1-\frac{1}{8}\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^2}\frac{x_{n-1}}{2} \\ &= \frac{1}{2}\frac{n^2}{7n^3-n^2+n-1}x_{n-1} \\ &= \prod_{i=1}^{n}\left[\frac{1}{2}\frac{i^2}{7i^3-i^2+i-1}\right]x_0. \end{aligned} \tag{2.20}$$

Using (2.18) and (2.20), we obtain

$$\begin{aligned} \frac{x_n\,(\text{our iteration }(1.5))}{x_n\,(\text{the iteration }(1.2))} &= \prod_{i=1}^{n}\left[\frac{1}{2}\frac{i^2}{7i^3-i^2+i-1}\right]\left[\frac{7i^3-i^2+i-1}{i^2}\right] \\ &= \prod_{i=1}^{n}\frac{1}{2}. \end{aligned}$$

Also,

$$0 \le \lim_{n\to\infty}\prod_{i=1}^{n}\frac{1}{2} = \lim_{n\to\infty}\left(\frac{1}{2}\right)^{n-1} = 0.$$

Therefore

$$\lim_{n\to\infty}\left|\frac{x_n\,(\text{our iteration }(1.5))}{x_n\,(\text{the iteration }(1.2))}\right| = 0.$$

It follows from Definition 3 that the our iteration (1.5) converges faster than implicit three-step iteration (1.2) to $p=0$ which is the fixed point of T.

The following table presents a comparison of rate of convergence of the our implicit iteration process with the other implicit iteration processes for the mapping given in Example1 for initial point $x_0 = 1$.

n	**IMI**	**III**	**ISI**
1	1.000000000000000	1.000000000000000	1.000000000000000
2	0.666666666666667	0.615384615384615	0.307692307692308
3	0.333333333333333	0.263736263736264	0.065934065934066
4	0.133333333333333	0.086117963668984	0.010764745458623
5	0.044444444444444	0.022662622018154	0.001416413876135
6	0.012698412698413	0.004989935123263	0.000155935472602
7	0.003174603174603	0.000944041780077	0.000014750652814
8	0.000705467372134	0.000156525061982	0.000001222852047
9	0.000141093474427	0.000023093861604	0.000000090210397
10	0.000025653358987	0.000003068951708	0.000000005994046
11	0.000004275559831	0.000000370972185	0.000000000362278
12	0.000000657778436	0.000000041123937	0.000000000020080
13	0.000000093968348	0.000000004209537	0.000000000001028
14	0.000000012529113	0.000000000400228	0.000000000000049
15	0.000000001566139	0.000000000035523	0.000000000000002

n	**ITSI**	**OII**
1	1.000000000000000	1.000000000000000
2	0.581818181818182	0.290909090909091
3	0.227667984189723	0.056916996047431
4	0.066685359213466	0.008335669901683
5	0.015580691405015	0.000973793212813
6	0.003025784979531	0.000094555780610
7	0.000502588013549	0.000007852937712
8	0.000072917274848	0.000000569666210
9	0.000009389982930	0.000000036679621
10	0.000001086960837	0.000000002122970
11	0.000000114267820	0.000000000111590
12	0.000000011001799	0.000000000005372
13	0.000000000977038	0.000000000000239
14	0.000000000080517	0.000000000000010
15	0.000000000006189	0.000000000000000

Remark 3. *From the example above, we see that our iteration OII (our implicit iteration) is faster than the IMI (implicit Mann iteration), III (implicit Ishikawa iteration), ISI (implicit S-iteration) and ITSI (implicit three-step iteration) under the same control conditions.*

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