Konuralp Journal of Mathematics, 6 (1) (2018) 117-127

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

Injective and Relative Injective Zagreb Indıces of Graphs

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Abstract

Let $G=(V,E)$ be a graph. The injective neighborhood of a vertex $u\in V(G)$ denoted by $N_{in}(u)$ is defined as $N_{in}(u)=\{v\in V(G):|\Gamma(u,v)|\geq 1\}$, where $|\Gamma(u,v)|$ is the number of common neighborhoods between the vertices u and v in G. The cardinality of $N_{in}(u)$ is called the injective degree of the vertex u in G and denoted by $deg_{in}(u)$, [2]. In this paper, we introduce the injective Zagreb indices of a graph G as $M_1^{inj}(G)=\sum_{u\in V(G)}\left[deg_{in}(u)\right]^2$, $M_2^{inj}(G)=\sum_{uv\in E(G)} deg_{in}(u)deg_{in}(v)$, respectively, and the relative injective Zagreb indices as $RM_1^{inj}(G)=\sum_{u\in V(G)} deg_{in}(u)deg(u)$, $RM_2^{inj}(G)=\sum_{uv\in E(G)}\left[deg_{in}(u)deg(v)+deg(u)deg_{in}(v)\right]$, respectively. Some properties of these topological indices are obtained. Exact values for some families of graphs and some graph operations are computed.

Keywords: *First injective Zagreb index, Second injective Zagreb index, First relative injective Zagreb index, Second relative injective Zagreb index.*
2010 Mathematics Subject Classification: *05C69*

1. Introduction

In this research work, we concerned about simple graphs which are finite, undirected with no loops and multiple edges. For a graph $G=(V,E)$, we denote $p=|V(G)|$ and $q=|E(G)|$. The complement of G, denoted by $\overline{G}$, is a simple graph on the same set of vertices $V(G)$ in which two vertices u and v are adjacent if and only if they are not adjacent in G. Let u be a vertex in G. The open neighborhood and the closed neighborhood of u are denoted by $N(u)=\{v\in V: uv\in E\}$ and $N[u]=N(u)\cup\{u\}$, respectively. The degree of u is denoted by $deg(u)$, and is defined to be the number of edges incident to u, shortly $deg(u)=|N(u)|$. The distance between u and any vertex v in G denoted by $d(u,v)$ is the number of edges of the shortest path joining u and v. The eccentricity of u denoted by $e(u)$ is the maximum distance between u and any other vertex v in G, that is $e(u)=\max\{d(u,v), v\in V(G)\}$. All the definitions and terminologies about graph in this paper available in [8]. The common neighborhood graph (congraph) of G, denoted by $con(G)$, is the graph with the vertex set $V(G)$, in which two vertices are adjacent if and only if they have at least one common neighbor in the graph G. The common neighborhood (CN-neighborhood) of u denoted by $N_{cn}(u)$ is defined as $N_{cn}(u)=\{v\in V(G): uv\in E(G)$ and $|\Gamma(u,v)|\geq 1\}$, where $|\Gamma(u,v)|$ is the number of common neighborhood between u and v, [1]. The injective neighborhood (Inj-neighborhood) of u denoted by $N_{in}(u)$ is defined as $N_{in}(u)=\{v\in V(G):|\Gamma(u,v)|\geq 1\}$. The cardinality of $N_{in}(u)$ is called the injective degree (Inj-degree) of the vertex u and denoted by $deg_{in}(u)$, [2]. Note that, for $u\in V(G)$ easily we observe that, $deg_{in}(u)=p-1$ if and only if $e(u)\leq 2$ and $N_{cn}(u)=N(u)$. Also, we denote to the sum of the Inj-degrees of all vertices of a graph G by $2q^{in}$, namely $2q^{in}=\sum_{u\in V(G)} deg_{in}(u)$.
The path, wheel, cycle and complete graphs with p vertices are denoted by P_p, W_p, C_p and K_p, respectively, and $K_{r,m}$ is the complete bipartite graph on $r+m$ vertices.
The Zagreb indices have been introduced by Gutman and Trinajestic [7].

$$M_1(G)=\sum_{u\in V(G)}\left[deg(u)\right]^2=\sum_{u\in V(G)}\sum_{v\in N(u)} deg(v)=\sum_{uv\in E(G)}\left[deg(u)+deg(v)\right].$$

$$M_2(G)=\sum_{uv\in E(G)} deg(u)deg(v)=\frac{1}{2}\sum_{u\in V(G)} deg(u)\sum_{v\in N(u)} deg(v).$$

Here, $M_1(G)$ and $M_2(G)$ denote the first and the second Zagreb indices, respectively. The first and second Zagreb coindices of a graph G denoted by $\overline{M_1}(G)$ and $\overline{M_2}(G)$, respectively, had introduced in [5], which are defined as $\overline{M_1}(G)=\sum_{uv\in E(\overline{G})}\left[deg_G(u)+deg_G(v)\right]$ and $\overline{M_2}(G)=\sum_{uv\in E(\overline{G})} deg_G(u)deg_G(v)$. Some properties of the Zagreb coindices of a simple graph G and its complement and some graph operations are studied in [3]. For more details about Zagreb indices we refer to [4, 6, 11, 15, 13, 14, 9, 12, 10]. In this paper, we introduce the injective and relative injective Zagreb indices of graphs. Exact values for some families of graphs and some graph operations are obtained.

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2. Some properties of injective and Relative injective Zagreb indices of graphs

In this section, we define the first and second injective and relative injective Zagreb indices of graphs. Some properties and exact expressions of some standard graphs are found.

Definition 2.1. Let $G=(V,E)$ be a graph. Then the first and second injective Zagreb indices of G are defined by

1. $M_1^{inj}(G)=\sum_{u\in V(G)}\big[deg_{in}(u)\big]^2=\sum_{u\in V(G)}\sum_{v\in N_{in}(u)}deg_{in}(v)$.
2. $M_2^{inj}(G)=\sum_{uv\in E(G)}deg_{in}(u)deg_{in}(v)=\frac{1}{2}\sum_{u\in V(G)}deg_{in}(u)\sum_{v\in N(u)}deg_{in}(v)$.

Definition 2.2. For a graph G, the first and second relative injective Zagreb indices of G are defined by

1. $RM_1^{inj}(G)=\sum_{u\in V(G)}deg_{in}(u)deg(u)=\sum_{uv\in E(G)}\big[deg_{in}(u)+deg_{in}(v)\big]$
   $=\sum_{u\in V(G)}\sum_{v\in N(u)}deg_{in}(v)$.
2. $RM_2^{inj}(G)=\sum_{uv\in E(G)}\big[deg_{in}(u)deg(v)+deg(u)deg_{in}(v)\big]$
   $=\sum_{u\in V(G)}deg_{in}(u)\sum_{v\in N(u)}deg(v)=\sum_{u\in V(G)}deg(u)\sum_{v\in N(u)}deg_{in}(v)$.

Proposition 2.1. *For any graph G, $M_1^{inj}(G)=M_1(G)=RM_1^{inj}(G)$ if and only if $deg_{in}(v)=deg(v)$, $\forall v\in V(G)$. Furthermore, in this case $M_2^{inj}(G)=M_2(G)=\frac{1}{2}RM_2^{inj}(G)$.*

Proposition 2.2. *Let G be a (p,q) connected graph. Then $M_1^{inj}(G)=p\big(p-1\big)^2=M_1(K_p)$ if and only if $e(v)\le 2$ and $N_{cn}(v)=N(v)$, $\forall v\in V(G)$. Furthermore, in this case $M_2^{inj}(G)=q\big(p-1\big)^2$.*

Proposition 2.3. *For any triangle-free graph G with diameter less than or equal two, $M_1^{inj}(G)=M_1(\overline{G})$ and $M_2^{inj}(G)=\overline{M_2}(\overline{G})$.*

The following results for standard graphs on p vertices follow easily by direct calculations.

Proposition 2.4. *For any path P_p with $p\ge 3$,*

1. $M_1^{inj}(P_p)=\begin{cases}2, & \text{if } p=3;\\ 4(p-3), & \text{otherwise.}\end{cases}$
2. $M_2^{inj}(P_p)=\begin{cases}0, & \text{if } p=3;\\ 3, & \text{if } p=4;\\ 4p-14, & \text{otherwise.}\end{cases}$
3. $RM_1^{inj}(P_p)=\begin{cases}2, & \text{if } p=3;\\ 4p-10, & \text{otherwise.}\end{cases}$
4. $RM_2^{inj}(P_p)=\begin{cases}4, & \text{if } p=3;\\ 8p-22, & \text{otherwise.}\end{cases}$

Proposition 2.5. *For any cycle C_p with $p\ge 3$,*

1. $M_1^{inj}(C_p)=M_2^{inj}(C_p)=\begin{cases}4, & \text{if } p=4;\\ 4p, & \text{otherwise.}\end{cases}$
2. $RM_1^{inj}(C_p)=\begin{cases}8, & \text{if } p=4;\\ 4p, & \text{otherwise.}\end{cases}$
3. $RM_2^{inj}(C_p)=\begin{cases}16, & \text{if } p=4;\\ 8p, & \text{otherwise.}\end{cases}$

Proposition 2.6. *For any complete bipartite graph $K_{r,m}$,*

1. $M_1^{inj}(K_{r,m})=r\big(r-1\big)^2+m\big(m-1\big)^2$.
2. $M_2^{inj}(K_{r,m})=rm(r-1)(m-1)$.
3. $RM_1^{inj}(K_{r,m})=rm(r+m-2)$.
4. $RM_2^{inj}(K_{r,m})=rm\big[r(r-1)+m(m-1)\big]$.

Proposition 2.7. *For any wheel graph W_p with $p\ge 4$,*

1. $M_1^{inj}(W_p)=p\big(p-1\big)^2$.
2. $M_2^{inj}(W_p)=2\big(p-1\big)^3$.
3. $RM_1^{inj}(W_p)=4\big(p-1\big)^2$.
4. $RM_2^{inj}(W_p)=\big(p-1\big)^2\big(p+8\big)$.

The injective complement of a graph G denoted by $\overline{G}^{inj}$ is the graph with the same vertices as G and any two vertices u, v are adjacent if u and v are not injective adjacent in G, [2].

Theorem 2.1. *For any graph G,*

1. $M_1^{inj}(\overline{G}^{inj}) = p\big(p-1\big)^2 - 4q^{in}(p-1) + M_1^{inj}(G)$.
2. $$M_2^{inj}(\overline{G}^{inj}) = \frac{1}{2}(2p-3)M_1^{inj}(G) - M_2\big(con(G)\big) + 2\big(q^{in}\big)^2 + \frac{1}{2}\big(p-1\big)^2\big(p(p-1)-6q^{in}\big).$$

Proof.

1. $$M_1^{inj}(\overline{G}^{inj}) = \sum_{u\in V(G)} \big(deg_{in}^{\overline{G}^{inj}}(u)\big)^2 = \sum_{u\in V(G)} \big(p-1-deg_{in}(u)\big)^2 = p\big(p-1\big)^2 - 4q^{in}(p-1) + M_1^{inj}(G).$$
2. $$\begin{aligned} M_2^{inj}(\overline{G}^{inj}) =& \frac{1}{2}\sum_{u\in V(G)} deg_{in}^{\overline{G}^{inj}}(u) \sum_{v\in N_{\overline{G}^{inj}}(u)} deg_{in}^{\overline{G}^{inj}}(v) \\ =& \frac{1}{2}\sum_{u\in V(G)} \big(p-1-deg_{in}(u)\big) \sum_{v\in N_{\overline{G}^{inj}}(u)} \big(p-1-deg_{in}(v)\big) \\ =& \frac{1}{2}\sum_{u\in V(G)} \big(p-1-deg_{in}(u)\big)\Big[\big(p-1\big)^2 - (p-2)deg_{in}(u) \\ & -2q^{in} + \sum_{v\in N_{in}(u)} deg_{in}(v)\Big] \\ =& \frac{1}{2}(2p-3)M_1^{inj}(G) - M_2\big(con(G)\big) + 2\big(q^{in}\big)^2 \\ & + \frac{1}{2}\big(p-1\big)^2\big(p(p-1)-6q^{in}\big). \end{aligned}$$

Note that, the equality $\sum_{v\in N_{\overline{G}^{inj}}(u)} deg_{in}(v) = 2q^{in} - deg_{in}(u) - \sum_{v\in N_{in}(u)} deg_{in}(v)$ is used.

3. Injective and Relative injective Zagreb indices for some graph operations

In this section, we compute the first and second injective and relative injective Zagreb indices for some graph operations.

The Cartesian product of two graphs G_1 and G_2, where $|V(G_1)| = p_1$, $|V(G_2)| = p_2$ and $|E(G_1)| = q_1$, $|E(G_2)| = q_2$ is denoted by $G_1\Box G_2$ has the vertex set $V(G_1)\times V(G_2)$ and, two vertices (u,u') and (v,v') are connected by an edge if and only if either ($[u = v$ and $u'v' \in E(G_2)]$) or ($[u' = v'$ and $uv \in E(G_1)]$). In other word, $|E(G_1\Box G_2)| = q_1p_2 + q_2p_1$. The degree of a vertex (u,u') of $G_1\Box G_2$ is as follows:

$$deg^{G_1\Box G_2}(u,u') = deg^{G_1}(u) + deg^{G_2}(u').$$

Lemma 3.1. *Let $G = G_1\Box G_2$ and let (u,u') be a vertex in G. Then*

$$deg_{in}^{G}(u,u') = deg_{in}^{G_1}(u) + deg_{in}^{G_2}(u') + deg^{G_1}(u)deg^{G_2}(u').$$

Proof. By the definition of $G = G_1\Box G_2$, one can observe that, the Inj-degree of any vertex (u,u') in G consists of three parts. The Inj-degree of (u,u') in the copy of G_1 of the projection u' in G_2 and the Inj-degree of (u,u') in the copy of G_2 of the projection u in G_1 and the Inj-degree of (u,u') in the copies of G_1 which have second projection belongs to $N_{G_2}(u')$ and the copies of G_2 which have first projection belongs to $N_{G_1}(u)$. It is clear that, the first and second parts of the Inj-degree of (u,u') are $deg_{in}^{G_1}(u)$ and $deg_{in}^{G_2}(u')$, respectively. Now for the third part, suppose $v\in N_{G_1}(u)$ be arbitrary. Then v corresponds $deg^{G_2}(u')$ vertex in the copy of G_2 of the projection v in G_1. Hence,

$$deg_{in}^{G}(u,u') = deg_{in}^{G_1}(u) + deg_{in}^{G_2}(u') + deg^{G_1}(u)deg^{G_2}(u').$$

The Cartesian product of more than two graphs is denoted by $\prod_{i=1}^{n} G_i$, in which $\prod_{i=1}^{n} G_i = G_1\Box G_2\Box\ldots\Box G_n = (G_1\Box G_2\Box\ldots\Box G_{n-1})\Box G_n$ and any two vertices $u = (u_1,u_2,\ldots,u_n)$ and $v = (v_1,v_2,\ldots,v_n)$ are adjacent in $\prod_{i=1}^{n} G_i$ if and only if $u_i = v_i$, $\forall i\neq j$ and $u_jv_j \in E(G_j)$, where $i,j = 1,2,\ldots,n$. If $G_1 = G_2 = \cdots = G_n = G$, we have the n-th Cartesian power of G and denote it by G^n.

Lemma 3.2. *Let $G = \prod_{i=1}^{n} G_i$ and let $u = (u_1,u_2,\ldots,u_n)$ be a vertex in G. Then*

$$deg_{in}^{G}(u) = \sum_{i=1}^{n} deg_{in}^{G_i}(u_i) + \sum_{i=1}^{n-1} deg^{G_i}(u_i) \sum_{j=i+1}^{n} deg^{G_j}(u_j).$$

Proof. By Lemma 3.1 and the induction principle, we have

$$\begin{aligned} deg_{in}^{G}(u) =& deg_{in}^{\prod_{i=1}^{n-1} G_i}(u_1,u_2,\ldots,u_{n-1}) + deg_{in}^{G_n}(u_n) + deg^{\prod_{i=1}^{n-1} G_i}(u_1,u_2,\ldots,u_{n-1})deg^{G_n}(u_n) \\ =& \sum_{i=1}^{n-1} deg_{in}^{G_i}(u_i) + \sum_{i=1}^{n-2} deg^{G_i}(u_i) \sum_{j=i+1}^{n-1} deg^{G_j}(u_j) + deg_{in}^{G_n}(u_n) \\ & + deg^{G_n}(u_n)\sum_{i=1}^{n-1} deg^{G_i}(u_i) \\ =& \sum_{i=1}^{n} deg_{in}^{G_i}(u_i) + \sum_{i=1}^{n-1} deg^{G_i}(u_i) \sum_{j=i+1}^{n} deg^{G_j}(u_j). \end{aligned}$$

Theorem 3.1. *Let $G = G_1 \Box G_2$. Then the first and second injective Zagreb indices of G are given by,*

1. $M_1^{inj}(G) = p_2M_1^{inj}(G_1) + p_1M_1^{inj}(G_2) + 4q_2RM_1^{inj}(G_1) + 4q_1RM_1^{inj}(G_2) + M_1(G_1)M_1(G_2) + 8q_1^{in}q_2^{in}.$
2. $M_2^{inj}(G) = p_2M_2^{inj}(G_1) + p_1M_2^{inj}(G_2) + 2q_2RM_2^{inj}(G_1) + 2q_1RM_2^{inj}(G_2) + q_2M_1^{inj}(G_1) + q_1M_1^{inj}(G_2) + \left(2q_2^{in} + M_1(G_2)\right)RM_1^{inj}(G_1) + \left(2q_1^{in} + M_1(G_1)\right)RM_1^{inj}(G_2) + M_1(G_1)M_2(G_2) + M_1(G_2)M_2(G_1).$

Proof.

1.
$$
\begin{aligned}
M_1^{inj}(G) &= \sum_{(u,u')\in V(G)} \left(deg_{in}^{G}(u,u')\right)^2\\
&= \sum_{u\in V(G_1)}\sum_{u'\in V(G_2)} \left(deg_{in}^{G_1}(u) + deg_{in}^{G_2}(u') + deg^{G_1}(u)deg^{G_2}(u')\right)^2\\
&= p_2M_1^{inj}(G_1) + p_1M_1^{inj}(G_2) + 4q_2RM_1^{inj}(G_1) + 4q_1RM_1^{inj}(G_2)\\
&\quad + M_1(G_1)M_1(G_2) + 8q_1^{in}q_2^{in}.
\end{aligned}
$$

2.
$$
\begin{aligned}
M_2^{inj}(G) &= \sum_{(u,u')(v,v')\in E(G)} deg_{in}^{G}(u,u')deg_{in}^{G}(v,v')\\
&= \sum_{u\in V(G_1)}\sum_{u'v'\in E(G_2)} deg_{in}^{G}(u,u')deg_{in}^{G}(u,v')\\
&\quad + \sum_{u'\in V(G_2)}\sum_{uv\in E(G_1)} deg_{in}^{G}(u,u')deg_{in}^{G}(v,u')\\
&= \sum_{u\in V(G_1)}\sum_{u'v'\in E(G_2)} \Big[\left(deg_{in}^{G_1}(u) + deg_{in}^{G_2}(u') + deg^{G_1}(u)deg^{G_2}(u')\right)\\
&\quad \left(deg_{in}^{G_1}(u) + deg_{in}^{G_2}(v') + deg^{G_1}(u)deg^{G_2}(v')\right)\Big]\\
&\quad + \sum_{u'\in V(G_2)}\sum_{uv\in E(G_1)} \Big[\left(deg_{in}^{G_1}(u) + deg_{in}^{G_2}(u') + deg^{G_1}(u)deg^{G_2}(u')\right)\\
&\quad \left(deg_{in}^{G_1}(v) + deg_{in}^{G_2}(u') + deg^{G_1}(v)deg^{G_2}(u')\right)\Big]\\
&= p_2M_2^{inj}(G_1) + p_1M_2^{inj}(G_2) + 2q_2RM_2^{inj}(G_1) + 2q_1RM_2^{inj}(G_2)\\
&\quad + q_2M_1^{inj}(G_1) + q_1M_1^{inj}(G_2) + \left(2q_2^{in} + M_1(G_2)\right)RM_1^{inj}(G_1)\\
&\quad + \left(2q_1^{in} + M_1(G_1)\right)RM_1^{inj}(G_2) + M_1(G_1)M_2(G_2) + M_1(G_2)M_2(G_1).
\end{aligned}
$$

Theorem 3.2. *Let $G = G_1 \Box G_2$. Then the first and second relative injective Zagreb indices of G are given by,*

1. $RM_1^{inj}(G) = p_2RM_1^{inj}(G_1) + p_1RM_1^{inj}(G_2) + p_2M_1(G_1) + p_1M_1(G_2) + 4(q_2q_1^{in} + q_1q_2^{in}).$
2. $RM_2^{inj}(G) = p_2RM_2^{inj}(G_1) + p_1RM_2^{inj}(G_2) + 4\left[q_2RM_1^{inj}(G_1) + q_1RM_1^{inj}(G_2)\right] + 2\left[q_2^{in}M_1(G_1) + q_1^{in}M_1(G_2)\right] + 4\left[q_2M_2(G_1) + q_1M_2(G_2)\right] + 2M_1(G_1)M_1(G_2).$

Proof.

1.
$$
\begin{aligned}
RM_1^{inj}(G) &= \sum_{(u,u')\in V(G)} deg_{in}^{G}(u,u')deg^{G}(u,u')\\
&= \sum_{u\in V(G_1)}\sum_{u'\in V(G_2)} \Big[\left(deg_{in}^{G_1}(u) + deg_{in}^{G_2}(u') + deg^{G_1}(u)deg^{G_2}(u')\right)\\
&\quad \left(deg^{G_1}(u) + deg^{G_2}(u')\right)\Big]\\
&= p_2RM_1^{inj}(G_1) + p_1RM_1^{inj}(G_2) + p_2M_1(G_1) + p_1M_1(G_2)\\
&\quad + 4(q_2q_1^{in} + q_1q_2^{in}).
\end{aligned}
$$

2. $$RM_2^{inj}(G) = \sum_{(u,u')\in V(G)} deg_{in}^G(u,u') \sum_{(v,v')\in N_G(u,u')} deg^G(v,v')$$
$$= \sum_{u\in V(G_1)} \sum_{\substack{(u,u')\in V(G)\\ u'\in V(G_2)}} deg_{in}^G(u,u') \sum_{\substack{(u,v')\in N_G(u,u')\\ v'\in N_{G_2}(u')}} deg^G(u,v')$$
$$+ \sum_{u'\in V(G_2)} \sum_{\substack{(u,u')\in V(G)\\ u\in V(G_1)}} deg_{in}^G(u,u') \sum_{\substack{(v,u')\in N_G(u,u')\\ v\in N_{G_1}(u)}} deg^G(v,u')$$
$$= \sum_{u\in V(G_1)} \sum_{u'\in V(G_2)} \left(deg_{in}^{G_1}(u) + deg_{in}^{G_2}(u') + deg^{G_1}(u)deg^{G_2}(u')\right)$$
$$\sum_{v'\in N_{G_2}(u')} \left(deg^{G_1}(u) + deg^{G_2}(v')\right)$$
$$+ \sum_{u'\in V(G_2)} \sum_{u\in V(G_1)} \left(deg_{in}^{G_1}(u) + deg_{in}^{G_2}(u') + deg^{G_1}(u)deg^{G_2}(u')\right)$$
$$\sum_{v\in N_{G_1}(u)} \left(deg^{G_1}(v) + deg^{G_2}(u')\right)$$
$$= p_2RM_2^{inj}(G_1) + p_1RM_2^{inj}(G_2) + 4\left[q_2RM_1^{inj}(G_1) + q_1RM_1^{inj}(G_2)\right]$$
$$+ 2\left[q_2^{in}M_1(G_1) + q_1^{in}M_1(G_2)\right] + 4\left[q_2M_2(G_1) + q_1M_2(G_2)\right]$$
$$+ 2M_1(G_1)M_1(G_2).$$

The composition $G = G_1[G_2]$ of two graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$, where $|V(G_1)| = p_1$, $|E(G_1)| = q_1$ and $|V(G_2)| = p_2$, $|E(G_2)| = q_2$ is the graph with vertex set $V(G_1)\times V(G_1)$ and any two vertices (u,u') and (v,v') are adjacent whenever u is adjacent to v in G_1 or $u = v$ and u' is adjacent to v' in G_2. Thus $|E(G_1[G_2])| = q_1p_2^2 + q_2p_1$. The degree of a vertex (u,u') of $G_1[G_2]$ is as follows:
$$deg^{G_1[G_2]}(u,u') = p_2deg^{G_1}(u) + deg^{G_2}(u').$$

Lemma 3.3. *Let $G = G_1[G_2]$ and let (u,u') be a vertex in G. Then*
$deg_{in}^G(u,u') = p_2\left[deg_{in}^{G_1}(u) + deg^{G_1}(u)\right] + p_2 - 1.$

Theorem 3.3. *Let $G = G_1[G_2]$. Then the first and second injective Zagreb indices of G are given by,*

1. $$M_1^{inj}(G) = p_2^3\left[M_1^{inj}(G_1) + M_1(G_1) + 2RM_1^{inj}(G_1)\right] + 4p_2^2(p_2-1)\left(q_1^{in} + q_1\right)$$
$$+ p_1p_2\left(p_2-1\right)^2.$$
2. $$M_2^{inj}(G) = p_2^3\left[M_2^{inj}(G_1) + M_2(G_1) + RM_2^{inj}(G_1)\right] + 4p_2q_2(p_2-1)\left(q_1^{in} + q_1\right)$$
$$+ p_2^2q_2\left[M_1^{inj}(G_1) + M_1(G_1) + 2RM_1^{inj}(G_1)\right] + \left(p_2-1\right)^2(p_1q_2 + p_2q_1)$$
$$+ p_2^2(p_2-1)\left[M_1(G_1) + RM_1^{inj}(G_1)\right].$$

Proof.

1. $$M_1^{inj}(G) = \sum_{(u,u')\in V(G)} \left(deg_{in}^G(u,u')\right)^2$$
$$= \sum_{u\in V(G_1)} \sum_{u'\in V(G_2)} \left[p_2\left(deg_{in}^{G_1}(u) + deg^{G_1}(u)\right) + p_2 - 1\right]^2$$
$$= p_2^3\left[M_1^{inj}(G_1) + M_1(G_1) + 2RM_1^{inj}(G_1)\right] + 4p_2^2(p_2-1)\left(q_1^{in} + q_1\right)$$
$$+ p_1p_2\left(p_2-1\right)^2.$$
2. $$M_2^{inj}(G) = \sum_{(u,u')(v,v')\in E(G)} deg_{in}^G(u,u')deg_{in}^G(v,v')$$
$$= \sum_{u\in V(G_1)} \sum_{u'v'\in E(G_2)} deg_{in}^G(u,u')deg_{in}^G(u,v')$$
$$+ \sum_{u'\in V(G_2)} \sum_{uv\in E(G_1)} deg_{in}^G(u,u')deg_{in}^G(v,u')$$
$$= \sum_{u\in V(G_1)} \sum_{u'v'\in E(G_2)} \left[p_2\left(deg_{in}^{G_1}(u) + deg^{G_1}(u)\right) + p_2 - 1\right]^2$$
$$+ \sum_{u'\in V(G_2)} \sum_{uv\in E(G_1)} \Big[\left[p_2\left(deg_{in}^{G_1}(u) + deg^{G_1}(u)\right) + p_2 - 1\right]$$
$$\left[p_2\left(deg_{in}^{G_1}(v) + deg^{G_1}(v)\right) + p_2 - 1\right]\Big]$$
$$= p_2^3\left[M_2^{inj}(G_1) + M_2(G_1) + RM_2^{inj}(G_1)\right] + 4p_2q_2(p_2-1)\left(q_1^{in} + q_1\right)$$
$$+ p_2^2q_2\left[M_1^{inj}(G_1) + M_1(G_1) + 2RM_1^{inj}(G_1)\right] + \left(p_2-1\right)^2(p_1q_2 + p_2q_1)$$
$$+ p_2^2(p_2-1)\left[M_1(G_1) + RM_1^{inj}(G_1)\right].$$

Theorem 3.4. *Let $G = G_1[G_2]$. Then the first and second relative injective Zagreb indices of G are given by,*

1. $RM_1^{inj}(G) = p_2^3\left[RM_1^{inj}(G_1)+M_1(G_1)\right]+4p_2q_2\left(q_1^{in}+q_1\right)+2(p_2-1)\left(p_2^2q_1+p_1q_2\right)$.
2. $RM_2^{inj}(G) = p_2^3\left[RM_2^{inj}(G_1)+2M_2(G_1)\right]+p_2\left(2q_2(p_2+1)+p_2(p_2-1)\right)M_1(G_1) + 2p_2q_2(p_2+1)RM_1^{inj}(G_1)+\left(2p_2(q_1^{in}+q_1)+p_1(p_2-1)\right)M_1(G_2) + 4q_1q_2\left(p_2^2-1\right)$.

Proof.

1. $$RM_1^{inj}(G) = \sum_{(u,u')\in V(G)} deg_{in}^G(u,u')deg^G(u,u')$$
$$= \sum_{u\in V(G_1)}\sum_{u'\in V(G_2)}\Big[\left[p_2\left(deg_{in}^{G_1}(u)+deg^{G_1}(u)\right)+p_2-1\right]\left[p_2deg^{G_1}(u)+deg^{G_2}(u')\right]\Big]$$
$$= p_2^3\left[RM_1^{inj}(G_1)+M_1(G_1)\right]+4p_2q_2\left(q_1^{in}+q_1\right)+2(p_2-1)\left(p_2^2q_1+p_1q_2\right).$$
2. $$RM_2^{inj}(G) = \sum_{(u,u')\in V(G)} deg_{in}^G(u,u')\sum_{(v,v')\in N_G(u,u')} deg^G(v,v')$$
$$= \sum_{u\in V(G_1)}\sum_{u'\in V(G_2)}\left[p_2\left(deg_{in}^{G_1}(u)+deg^{G_1}(u)\right)+p_2-1\right]\sum_{v'\in N_{G_2}(u')}\left(p_2deg^{G_1}(u)+deg^{G_2}(v')\right)$$
$$+\sum_{u'\in V(G_2)}\sum_{u\in V(G_1)}\left[p_2\left(deg_{in}^{G_1}(u)+deg^{G_1}(u)\right)+p_2-1\right]\sum_{v\in N_{G_1}(u)}\left(p_2deg^{G_1}(v)+deg^{G_2}(u')\right)$$
$$= p_2^3\left[RM_2^{inj}(G_1)+2M_2(G_1)\right]+p_2\left(2q_2(p_2+1)+p_2(p_2-1)\right)M_1(G_1) + 2p_2q_2(p_2+1)RM_1^{inj}(G_1)+\left(2p_2(q_1^{in}+q_1)+p_1(p_2-1)\right)M_1(G_2) + 4q_1q_2\left(p_2^2-1\right).$$

The join G_1+G_2 of two graphs G_1 and G_2 with disjoint vertex sets $|V(G_1)| = p_1$, $|V(G_2)| = p_2$ and edge sets $|E(G_1)| = q_1$, $|E(G_2)| = q_2$ is the graph on the vertex set $V(G_1)\cup V(G_2)$ and the edge set $E(G_1)\cup E(G_2)\cup\{u_1u_2 : u_1\in V(G_1), u_2\in V(G_2)\}$. Hence, the join of two graphs is obtained by connecting each vertex of one graph to each vertex of the other graph, while keeping all edges of both graphs. The degree of any vertex $u\in G_1+G_2$ is given by

$$deg^{G_1+G_2}(u) = \begin{cases} deg^{G_1}(u)+p_2, & \text{if } u\in V(G_1);\\ deg^{G_2}(u)+p_1, & \text{if } u\in V(G_2).\end{cases}$$

Let $G = \sum_{i=1}^n G_i$ be the join of the graphs G_i, $i = 1,2,\ldots,n$ and denote to the set of isolated vertices in each G_i by I_i. Then by the definition of G the degree of any vertex $u\in V(G)$ is given as in the following lemma.

Lemma 3.4. *Let $G = \sum_{i=1}^n G_i$ and $u\in V(G)$. Then*

1. *If $n = 2$, then* $deg_{in}^G(u) = \begin{cases} p_1+p_2-1-|I_2|, & \text{if } u\in I_1\subseteq V(G_1);\\ p_1+p_2-1-|I_1|, & \text{if } u\in I_2\subseteq V(G_2);\\ p_1+p_2-1, & \text{otherwise.}\end{cases}$
2. *If $n\geq 3$, then* $deg_{in}^G(u) = -1+\sum_{i=1}^n p_i$.

Theorem 3.5. *Let $G = G_1+G_2$. Then the first and second injective Zagreb indices of G are given by,*

1. $M_1^{inj}(G) = (p_1+p_2)\left(p_1+p_2-1\right)^2+|I_1||I_2|\left(|I_1|+|I_2|\right)-4|I_1||I_2|(p_1+p_2-1)$.
2. $M_2^{inj}(G) = (q_1+q_2+p_1p_2)\left(p_1+p_2-1\right)^2+|I_1|^2|I_2|^2 - |I_1||I_2|(p_1+p_2)(p_1+p_2-1)$.

Proof.

1. $$M_1^{inj}(G) = \sum_{u\in V(G)}\left[deg_{in}^G(u)\right]^2 = \sum_{u\in I_1}\left(p_1+p_2-1-|I_2|\right)^2+\sum_{u\in I_2}\left(p_1+p_2-1-|I_1|\right)^2 + \sum_{u\in V(G)-(I_1\cup I_2)}\left(p_1+p_2-1\right)^2$$
$$= (p_1+p_2)\left(p_1+p_2-1\right)^2+|I_1||I_2|\left(|I_1|+|I_2|\right)-4|I_1||I_2|(p_1+p_2-1).$$

2. $$\begin{aligned}M_2^{inj}(G) =&\frac{1}{2}\sum_{u\in V(G)} deg_{in}^G(u)\sum_{v\in N_G(u)} deg_{in}^G(v)\\
=&\frac{1}{2}\sum_{u\in V(G_1)} deg_{in}^G(u)\Big[\sum_{v\in N_{G_1}(u)} deg_{in}^G(v)+\sum_{v\in V(G_2)} deg_{in}^G(v)\Big]\\
&+\frac{1}{2}\sum_{u\in V(G_2)} deg_{in}^G(u)\Big[\sum_{v\in N_{G_2}(u)} deg_{in}^G(v)+\sum_{v\in V(G_1)} deg_{in}^G(v)\Big]\\
=&\frac{1}{2}\sum_{u\in I_1}(p_1+p_2-1-|I_2|)\Big[\sum_{v\in I_2}(p_1+p_2-1-|I_1|)+\sum_{v\in V(G_2)-I_2}(p_1+p_2-1)\Big]\\
&+\frac{1}{2}\sum_{u\in V(G_1)-I_1}(p_1+p_2-1)\Big[\sum_{v\in N_{G_1}(u)}(p_1+p_2-1)+\sum_{v\in I_2}(p_1+p_2-1-|I_1|)\\
&+\sum_{v\in V(G_2)-I_2}(p_1+p_2-1)\Big]+\frac{1}{2}\sum_{u\in I_2}(p_1+p_2-1-|I_1|)\Big[\sum_{v\in I_1}(p_1+p_2-1-|I_2|)\\
&+\sum_{v\in V(G_1)-I_1}(p_1+p_2-1)\Big]+\frac{1}{2}\sum_{u\in V(G_2)-I_2}(p_1+p_2-1)\Big[\sum_{v\in N_{G_2}(u)}(p_1+p_2-1)\\
&+\sum_{v\in I_1}(p_1+p_2-1-|I_2|)+\sum_{v\in V(G_1)-I_1}(p_1+p_2-1)\Big]\\
=&(q_1+q_2+p_1p_2)(p_1+p_2-1)^2+|I_1|^2|I_2|^2-|I_1||I_2|(p_1+p_2)(p_1+p_2-1).\end{aligned}$$

Theorem 3.6. *Let $G = G_1 + G_2$. Then the first and second relative injective Zagreb indices of G are given by,*

1. $RM_1^{inj}(G) = 2(q_1+q_2+p_1p_2)(p_1+p_2-1)-|I_1||I_2|(p_1+p_2)$.
2. $RM_2^{inj}(G) =(p_1+p_2-1)\big[M_1(G_1)+M_1(G_2)+4(p_1q_2+p_2q_1)+p_1p_2(p_1+p_2)\big]$
$-2|I_1||I_2|(q_1+q_2+p_1p_2)$.

Proof.

1. $$\begin{aligned}RM_1^{inj}(G) =&\sum_{u\in V(G)} deg_{in}^G(u)deg^G(u)\\
=&\sum_{u\in I_1} p_2(p_1+p_2-1-|I_2|)+\sum_{u\in I_2} p_1(p_1+p_2-1-|I_1|)\\
&+(p_1+p_2-1)\Big[\sum_{u\in V(G_1)-I_1}\big(deg^{G_1}(u)+p_2\big)+\sum_{u\in V(G_2)-I_2}\big(deg^{G_2}(u)+p_1\big)\Big]\\
=&2(q_1+q_2+p_1p_2)(p_1+p_2-1)-|I_1||I_2|(p_1+p_2).\end{aligned}$$
2. $$\begin{aligned}RM_2^{inj}(G) =&\sum_{u\in V(G)} deg_{in}^G(u)\sum_{v\in N_G(u)} deg^G(v)\\
=&\sum_{u\in V(G_1)} deg_{in}^G(u)\Big[\sum_{v\in N_{G_1}(u)} deg^G(v)+\sum_{v\in V(G_2)} deg^G(v)\Big]\\
&+\sum_{u\in V(G_2)} deg_{in}^G(u)\Big[\sum_{v\in N_{G_2}(u)} deg^G(v)+\sum_{v\in V(G_1)} deg^G(v)\Big]\\
=&\sum_{u\in I_1}(p_1+p_2-1-|I_2|)\Big[p_1|I_2|+\sum_{v\in V(G_2)-I_2}\big(deg^{G_2}(v)+p_1\big)\Big]\\
&+\sum_{u\in V(G_1)-I_1}(p_1+p_2-1)\Big[\sum_{v\in N_{G_1}(u)}\big(deg^{G_1}(v)+p_2\big)+p_1|I_2|\\
&+\sum_{v\in V(G_2)-I_2}\big(deg^{G_2}(v)+p_1\big)\Big]+\sum_{u\in I_2}(p_1+p_2-1-|I_1|)\\
&\Big[p_2|I_1|+\sum_{v\in V(G_1)-I_1}\big(deg^{G_1}(v)+p_2\big)\Big]+\sum_{u\in V(G_2)-I_2}(p_1+p_2-1)\\
&\Big[\sum_{v\in N_{G_2}(u)}\big(deg^{G_2}(v)+p_1\big)+p_2|I_1|+\sum_{v\in V(G_1)-I_1}\big(deg^{G_1}(v)+p_2\big)\Big]\\
=&(p_1+p_2-1)\big[M_1(G_1)+M_1(G_2)+4(p_1q_2+p_2q_1)+p_1p_2(p_1+p_2)\big]\\
&-2|I_1||I_2|(q_1+q_2+p_1p_2).\end{aligned}$$

Corollary 3.1. *If $G = G_1 + G_2$ such that at least G_1 or G_2 is an isolated-free graph, then*

1. $M_1^{inj}(G) = (p_1+p_2)\big(p_1+p_2-1\big)^2$.
2. $M_2^{inj}(G) = (q_1+q_2+p_1p_2)\big(p_1+p_2-1\big)^2$.
3. $RM_1^{inj}(G) = 2(q_1+q_2+p_1p_2)(p_1+p_2-1)$.

4. $RM_2^{inj}(G) = (p_1+p_2-1)\big[M_1(G_1)+M_1(G_2)+4(p_1q_2+p_2q_1)+p_1p_2(p_1+p_2)\big]$.

In the following two theorems we have a result of the injective and relative injective Zagreb indices for the join $G=\sum_{i=1}^{n} G_i$ with $n \geq 3$.

Theorem 3.7. *Let $G=\sum_{i=1}^{n} G_i$ with $n\geq 3$. Then*

1. $M_1^{inj}(G) = \Big(-1+\sum_{i=1}^{n} p_i\Big)^2 \sum_{i=1}^{n} p_i$.
2. $M_2^{inj}(G) = \Big(-1+\sum_{i=1}^{n} p_i\Big)^2 \Big(\sum_{i=1}^{n} q_i + \sum_{i=1}^{n-1} p_i \sum_{j=i+1}^{n} p_j\Big)$.

Proof.

1. $$\begin{aligned} M_1^{inj}(G) &= \sum_{u\in V(G)} \big[deg_{in}^G(u)\big]^2 \\ &= \sum_{i=1}^{n}\sum_{u\in V(G_i)} \Big(-1+\sum_{i=1}^{n} p_i\Big)^2 \\ &= \Big(-1+\sum_{i=1}^{n} p_i\Big)^2 \sum_{i=1}^{n} p_i. \end{aligned}$$
2. $$\begin{aligned} M_2^{inj}(G) &= \frac{1}{2}\sum_{u\in V(G)} deg_{in}^G(u) \sum_{v\in N_G(u)} deg_{in}^G(v) \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{u\in V(G_i)} deg_{in}^G(u)\Big[\sum_{v\in N_{G_i}(u)} deg_{in}^G(v) + \sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{v\in V(G_j)} deg_{in}^G(v)\Big] \\ &= \frac{1}{2}\Big(-1+\sum_{i=1}^{n} p_i\Big)^2 \sum_{i=1}^{n}\sum_{u\in V(G_i)}\Big[deg^{G_i}(u)+\sum_{\substack{j=1\\ j\neq i}}^{n} p_j\Big] \\ &= \Big(-1+\sum_{i=1}^{n} p_i\Big)^2\Big(\sum_{i=1}^{n} q_i + \sum_{i=1}^{n-1} p_i \sum_{j=i+1}^{n} p_j\Big). \end{aligned}$$

Theorem 3.8. *For $G=\sum_{i=1}^{n} G_i$ with $n\geq 3$, we have*

1. $RM_1^{inj}(G) = 2\Big(-1+\sum_{i=1}^{n} p_i\Big)\Big(\sum_{i=1}^{n} q_i + \sum_{i=1}^{n-1} p_i \sum_{j=i+1}^{n} p_j\Big)$.
2. $$RM_2^{inj}(G) = \Big(-1+\sum_{i=1}^{n} p_i\Big)\Big[\sum_{i=1}^{n} M_1(G_i) + 4\sum_{i=1}^{n-1} q_i \sum_{j=i+1}^{n} p_j + 4\sum_{i=1}^{n-1} p_i \sum_{j=i+1}^{n} q_j + \sum_{i=1}^{n} p_i \sum_{\substack{j=1\\ j\neq i}}^{n} p_j \sum_{\substack{k=1\\ k\neq j}}^{n} p_k\Big].$$

Proof.

1. $$\begin{aligned} RM_1^{inj}(G) &= \sum_{u\in V(G)} deg_{in}^G(u) deg^G(u) \\ &= \sum_{i=1}^{n}\sum_{u\in V(G_i)}\Big(-1+\sum_{k=1}^{n} p_k\Big)\Big(deg^{G_i}(u)+\sum_{\substack{j=1\\ j\neq i}}^{n} p_j\Big) \\ &= \Big(-1+\sum_{i=1}^{n} p_i\Big)\sum_{i=1}^{n}\Big(2q_i + p_i\sum_{\substack{j=1\\ j\neq i}}^{n} p_j\Big) \\ &= 2\Big(-1+\sum_{i=1}^{n} p_i\Big)\Big(\sum_{i=1}^{n} q_i + \sum_{i=1}^{n-1} p_i \sum_{j=i+1}^{n} p_j\Big). \end{aligned}$$

2. $$\begin{aligned} RM_2^{inj}(G) &= \sum_{u\in V(G)} deg_{in}^G(u) \sum_{v\in N_G(u)} deg^G(v) \\ &= \sum_{i=1}^{n}\sum_{u\in V(G_i)} deg_{in}^G(u)\Big[\sum_{v\in N_{G_i}(u)} deg^G(v) + \sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{v\in V(G_j)} deg^G(v)\Big] \\ &= \Big(-1+\sum_{i=1}^{n} p_i\Big)\sum_{i=1}^{n}\sum_{u\in V(G_i)}\Big[\sum_{v\in N_{G_i}(u)}\Big(deg^{G_i}(v)+\sum_{\substack{j=1\\ j\neq i}}^{n} p_j\Big) \\ &\quad + \sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{v\in V(G_j)}\Big(deg^{G_j}(v)+\sum_{\substack{k=1\\ k\neq j}}^{n} p_k\Big)\Big] \\ &= \Big(-1+\sum_{i=1}^{n} p_i\Big)\sum_{i=1}^{n}\Big[M_1(G_i)+2q_i\sum_{\substack{j=1\\ j\neq i}}^{n} p_j + p_i\sum_{\substack{j=1\\ j\neq i}}^{n}\Big(2q_j+p_j\sum_{\substack{k=1\\ k\neq j}}^{n} p_k\Big)\Big] \\ &= \Big(-1+\sum_{i=1}^{n} p_i\Big)\Big[\sum_{i=1}^{n} M_1(G_i)+4\sum_{i=1}^{n-1} q_i\sum_{j=i+1}^{n} p_j+4\sum_{i=1}^{n-1} p_i\sum_{j=i+1}^{n} q_j \\ &\quad + \sum_{i=1}^{n} p_i\sum_{\substack{j=1\\ j\neq i}}^{n} p_j\sum_{\substack{k=1\\ k\neq j}}^{n} p_k\Big]. \end{aligned}$$

Note that: The equality $\sum_{i=1}^{n} x_i \sum_{\substack{j=1\\ j\neq i}}^{n} y_j = 2\sum_{i=1}^{n-1} x_i \sum_{j=i+1}^{n} y_j$, is used.

The corona product $G_1\circ G_2$ of two graphs G_1 and G_2, where $|V(G_1)|=p_1$, $|V(G_2)|=p_2$ and $|E(G_1)|=q_1$, $|E(G_2)|=q_2$ is the graph obtained by taking $|V(G_1)|$ copies of G_2 and joining each vertex of the i-th copy with vertex $u\in V(G_1)$. Obviously, $|V(G_1\circ G_2)|=p_1(p_2+1)$ and $|E(G_1\circ G_2)|=q_1+p_1(q_2+p_2)$. It follows from the definition of the corona product $G_1\circ G_2$, the degree of each vertex $u\in G_1\circ G_2$ is given by

$$deg^{G_1\circ G_2}(u)=\begin{cases} deg^{G_1}(u)+p_2, & \text{if } u\in V(G_1);\\ deg^{G_2}(u)+1, & \text{if } u\in V(G_2).\end{cases}$$

Lemma 3.5. *Let $G=G_1\circ G_2$ and $u\in V(G)$. Then*

$$deg_{in}^{G}(u)=\begin{cases} deg_{in}^{G_1}(u)+p_2deg^{G_1}(u)+p_2-|I_2|, & u\in V(G_1);\\ p_2-1+deg^{G_1}(v), & u\in I_2\subseteq V(G_2);\\ p_2+deg^{G_1}(v), & u\in V(G_2)-I_2,\end{cases}$$

where $I_2\subseteq V(G_2)$ is the set of isolated vertices of G_2 and $v\in V(G_1)$ is adjacent to u.

Theorem 3.9. *Let $G=G_1\circ G_2$. Then*

1. $$\begin{aligned} M_1^{inj}(G) =& M_1^{inj}(G_1)+2p_2RM_1^{inj}(G_1)+p_2(p_2+1)M_1(G_1)+p_1\big(p_2-|I_2|\big)^2 \\ &+4(p_2-|I_2|)(q_1^{in}+p_2q_1)+p_2^2(p_1p_2+4q_1)-|I_2|(2p_1p_2+4q_1-p_1). \end{aligned}$$
2. $$\begin{aligned} M_2^{inj}(G) =& M_2^{inj}(G_1)+p_2\big[RM_2^{inj}(G_1)+p_2M_2(G_1)\big]+q_2\big(p_1p_2^2+4p_2q_1+M_1(G_1)\big) \\ &+(2p_2-|I_2|)\big[RM_1^{inj}(G_1)+p_2M_1(G_1)\big]+q_1(p_2-|I_2|)(3p_2-|I_2|) \\ &+(p_2^2-|I_2|)\big[2q_1^{in}+2p_2q_1+p_1(p_2-|I_2|)\big]. \end{aligned}$$

Proof.

1. $$\begin{aligned} M_1^{inj}(G) &= \sum_{u\in V(G)}\big[deg_{in}^G(u)\big]^2 = \sum_{u\in V(G_1)}\big[deg_{in}^G(u)\big]^2+\sum_{v\in V(G_1)}\sum_{u\in V(G_2)}\big[deg_{in}^G(u)\big]^2 \\ &= \sum_{u\in V(G_1)}\big[deg_{in}^{G_1}(u)+p_2deg^{G_1}(u)+p_2-|I_2|\big]^2+ \\ &\quad \sum_{v\in V(G_1)}\sum_{u\in I_2}\big[p_2-1+deg^{G_1}(v)\big]^2+\sum_{v\in V(G_1)}\sum_{u\in V(G_2)-I_2}\big[p_2+deg^{G_1}(v)\big]^2 \\ &= M_1^{inj}(G_1)+2p_2RM_1^{inj}(G_1)+p_2(p_2+1)M_1(G_1)+p_1\big(p_2-|I_2|\big)^2 \\ &\quad +4(p_2-|I_2|)(q_1^{in}+p_2q_1)+p_2^2(p_1p_2+4q_1)-|I_2|(2p_1p_2+4q_1-p_1). \end{aligned}$$

2. $$\begin{aligned}M_2^{inj}(G) =&\frac{1}{2}\sum_{u\in V(G)} deg_{in}^G(u)\sum_{v\in N_G(u)} deg_{in}^G(v)\\
=&\frac{1}{2}\sum_{u\in V(G_1)} deg_{in}^G(u)\Big[\sum_{v\in N_{G_1}(u)} deg_{in}^G(v)+\sum_{v\in V(G_2)} deg_{in}^G(v)\Big]\\
&+\frac{1}{2}\sum_{v\in V(G_1)}\sum_{u\in V(G_2)} deg_{in}^G(u)\Big[\sum_{w\in N_{G_2}(u)} deg_{in}^G(w)+deg_{in}^G(v)\Big]\\
=&\frac{1}{2}\sum_{u\in V(G_1)}\big[deg_{in}^{G_1}(u)+p_2deg^{G_1}(u)+p_2-|I_2|\big]\Big[\sum_{v\in N_{G_1}(u)}\big[deg_{in}^{G_1}(v)+p_2deg^{G_1}(v)\\
&+p_2-|I_2|\big]+\sum_{u\in I_2}\big[p_2-1+deg^{G_1}(v)\big]+\sum_{u\in V(G_2)-I_2}\big[p_2+deg^{G_1}(v)\big]\Big]\\
&+\frac{1}{2}\sum_{v\in V(G_1)}\sum_{u\in I_2}\big[p_2-1+deg^{G_1}(v)\big]\big[deg_{in}^{G_1}(v)+p_2deg^{G_1}(v)+p_2-|I_2|\big]\\
&+\frac{1}{2}\sum_{v\in V(G_1)}\sum_{u\in V(G_2)-I_2}\big[p_2+deg^{G_1}(v)\big]\Big[deg^{G_2}(u)\big[p_2+deg^{G_1}(v)\big]\\
&+deg_{in}^{G_1}(v)+p_2deg^{G_1}(v)+p_2-|I_2|\Big]\\
=&M_2^{inj}(G_1)+p_2\big[RM_2^{inj}(G_1)+p_2M_2(G_1)\big]+q_2\big(p_1p_2^2+4p_2q_1+M_1(G_1)\big)\\
&+(2p_2-|I_2|)\big[RM_1^{inj}(G_1)+p_2M_1(G_1)\big]+q_1(p_2-|I_2|)(3p_2-|I_2|)\\
&+(p_2^2-|I_2|)\big[2q_1^{in}+2p_2q_1+p_1(p_2-|I_2|)\big].\end{aligned}$$

Theorem 3.10. *Let $G=G_1\circ G_2$. Then*

1. $$\begin{aligned}RM_1^{inj}(G) =&RM_1^{inj}(G_1)+p_2M_1(G_1)+(p_1p_2+2q_1)(2p_2+2q_2-|I_2|)\\&+2p_2(q_1^{in}+p_2q_1)-p_1|I_2|.\end{aligned}$$
2. $$\begin{aligned}RM_2^{inj}(G) =&RM_2^{inj}(G_1)+p_2RM_1^{inj}(G_1)+2p_2M_2(G_1)+\big[p_2(p_2+2)-|I_2|\big]M_1(G_1)\\&+(p_1p_2+2q_1)\big[M_1(G_2)+2q_2-|I_2|\big]+(p_2-|I_2|)\big[2p_2q_1+p_1(p_2+2q_2)\big]\\&+2(p_2+2q_2)\big(q_1^{in}+p_2q_1\big)+p_2^2\big(p_1p_2+4q_1\big).\end{aligned}$$

Proof.

1. $$\begin{aligned}RM_1^{inj}(G) =&\sum_{u\in V(G)} deg_{in}^G(u)deg^G(u)\\
=&\sum_{u\in V(G_1)} deg_{in}^G(u)deg^G(u)+\sum_{v\in V(G_1)}\sum_{u\in V(G_2)} deg_{in}^G(u)deg^G(u)\\
=&\sum_{u\in V(G_1)}\big(deg_{in}^{G_1}(u)+p_2deg^{G_1}(u)+p_2-|I_2|\big)\big(deg^{G_1}(u)+p_2\big)\\
&+\sum_{v\in V(G_1)}\Big[-|I_2|+\sum_{u\in V(G_2)}\big(p_2+deg^{G_1}(v)\big)\big(deg^{G_2}(u)+1\big)\Big]\\
=&RM_1^{inj}(G_1)+p_2M_1(G_1)+(p_1p_2+2q_1)(2p_2+2q_2-|I_2|)\\
&+2p_2(q_1^{in}+p_2q_1)-p_1|I_2|.\end{aligned}$$
2. $$\begin{aligned}RM_2^{inj}(G) =&\sum_{u\in V(G)} deg_{in}^G(u)\sum_{v\in N_G(u)} deg^G(v)\\
=&\sum_{u\in V(G_1)} deg_{in}^G(u)\Big[\sum_{v\in N_{G_1}(u)} deg^G(v)+\sum_{v\in V(G_2)} deg^G(v)\Big]\\
&+\sum_{v\in V(G_1)}\sum_{u\in V(G_2)} deg_{in}^G(u)\Big[\sum_{w\in N_{G_2}(u)} deg^G(w)+deg^G(v)\Big]\\
=&\sum_{u\in V(G_1)}\big(deg_{in}^{G_1}(u)+p_2deg^{G_1}(u)+p_2-|I_2|\big)\Big[\sum_{v\in N_{G_1}(u)}\big(deg^{G_1}(v)+p_2\big)\\
&+\sum_{u\in V(G_2)}\big(deg^{G_2}(v)+1\big)\Big]+\sum_{v\in V(G_1)}\sum_{u\in I_2}\big(p_2-1+deg^{G_1}(v)\big)\big(deg^{G_1}(v)+p_2\big)\\
&+\sum_{v\in V(G_1)}\sum_{u\in V(G_2)-I_2}\big(p_2+deg^{G_1}(v)\big)\Big[deg^{G_1}(v)+p_2+\sum_{w\in N_{G_2}(u)}\big(deg^{G_2}(w)+1\big)\Big]\\
=&RM_2^{inj}(G_1)+p_2RM_1^{inj}(G_1)+2p_2M_2(G_1)+\big[p_2(p_2+2)-|I_2|\big]M_1(G_1)\\
&+(p_1p_2+2q_1)\big[M_1(G_2)+2q_2-|I_2|\big]+(p_2-|I_2|)\big[2p_2q_1+p_1(p_2+2q_2)\big]\\
&+2(p_2+2q_2)\big(q_1^{in}+p_2q_1\big)+p_2^2\big(p_1p_2+4q_1\big).\end{aligned}$$

Example 3.1. For any cycle C_{p_1} and any path P_{p_2} with $p_2\geq 2$

1. $$M_1^{inj}(C_{p_1}\circ P_{p_2})=\begin{cases}4(p_2^3+13p_2^2+10p_2+1), & \text{if } p_1=4;\\ p_1(p_2^3+13p_2^2+16p_2+4), & \text{otherwise.}\end{cases}$$

2. $M_2^{inj}(C_{p_1} \circ P_{p_2}) = \begin{cases} 4(4p_2^3+19p_2^2+8p_2-3), & \text{if } p_1 = 4; \\ 4p_1p_2(p_2+1)(p_2+4), & \text{otherwise.} \end{cases}$
3. $RM_1^{inj}(C_{p_1} \circ P_{p_2}) = \begin{cases} 4(6p_2-1)(p_2+2), & \text{if } p_1 = 4; \\ 6p_1p_2(p_2+2), & \text{otherwise.} \end{cases}$
4. $RM_2^{inj}(C_{p_1} \circ P_{p_2}) = \begin{cases} 4(p_2^3+25p_2^2+19p_2-14), & \text{if } p_1 = 4; \\ p_1(p_2^3+25p_2^2+24p_2-12), & \text{otherwise.} \end{cases}$

Example 3.2. For any two cycles C_{p_1} and C_{p_2},

1. $M_1^{inj}(C_{p_1} \circ C_{p_2}) = \begin{cases} 4(p_2^3+13p_2^2+10p_2+1), & \text{if } p_1 = 4; \\ p_1(p_2^3+13p_2^2+16p_2+4), & \text{otherwise.} \end{cases}$
2. $M_2^{inj}(C_{p_1} \circ C_{p_2}) = \begin{cases} 4(4p_2^3+19p_2^2+12p_2+1), & \text{if } p_1 = 4; \\ p_1(4p_2^3+21p_2^2+20p_2+4), & \text{otherwise.} \end{cases}$
3. $RM_1^{inj}(C_{p_1} \circ C_{p_2}) = \begin{cases} 4(6p_2+1)(p_2+2), & \text{if } p_1 = 4; \\ 2p_1(3p_2+1)(p_2+2), & \text{otherwise.} \end{cases}$
4. $RM_2^{inj}(C_{p_1} \circ C_{p_2}) = \begin{cases} 4(p_2^3+25p_2^2+33p_2+4), & \text{if } p_1 = 4; \\ p_1(p_2^3+25p_2^2+38p_2+8), & \text{otherwise.} \end{cases}$

References

[1] A. Alwardi, B. Arsi´c, I. Gutman, N. D. Soner, The common neighborhood graph and its energy, Iran. J. Math. Sci. Inf. 7(2) (2012) 1—8.
[2] Anwar Alwardi, R. Rangarajan and Akram Alqesmah, On the Injective domination of graphs, In communication.
[3] A.R. Ashrafi, T. Došlić, A. Hamzeha, The Zagreb coindices of graph operations, Discrete Applied Mathematics 158 (2010) 1571–1578.
[4] J. Braun, A. Kerber, M. Meringer, C. Rucker, Similarity of molecular descriptors: the equivalence of Zagreb indices and walk counts, MATCH Commun. Math. Comput. Chem. 54 (2005) 163–176.
[5] T. Došlić, Vertex-Weighted Wiener Polynomials for Composite Graphs, Ars Math. Contemp. 1 (2008) 66–80.
[6] I. Gutman, K.C. Das, The first Zagreb index 30 years after, MATCH Commun. Math. Comput. Chem. 50 (2004) 83–92.
[7] I. Gutman, N. Trinajstic, Graph theory and molecular orbitals, Total π-electron energy of alternant hydrocarbons, Chem. Phys. Lett. 17 (1972) 535–538.
[8] F. Harary, Graph theory, Addison-Wesley, Reading Mass (1969).
[9] M. H. Khalifeh, H. Yousefi-Azari, A.R. Ashrafi, The first and second Zagreb indices of some graph operations, Discrete Applied Mathematics 157 (2009) 804–811.
[10] Modjtaba Ghorbani, Mohammad A. Hosseinzadeh, A new version of Zagreb indices, Filomat 26 (1) (2012) 93–100.
[11] S. Nikolić, G. Kovačević, A. Miličević, N. Trinajstić, The Zagreb indices 30 years after, Croat. Chem. Acta 76 (2003) 113–124.
[12] Rundan Xing, Bo Zhou and Nenad Trinajstic, On Zagreb Eccentricity Indices, Croat. Chem. Acta 84 (4) (2011) 493—497.
[13] B. Zhou, I. Gutman, Further properties of Zagreb indices, MATCH Commun. Math. Comput. Chem. 54 (2005) 233–239.
[14] B. Zhou, I. Gutman, Relations between Wiener, hyper-Wiener and Zagreb indices, Chem. Phys. Lett. 394 (2004) 93–95.
[15] B. Zhou, Zagreb indices, MATCH Commun. Math. Comput. Chem. 52 (2004) 113–118.