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Certain Results on Almost Kenmotsu $(\kappa,\mu,\nu)-$Spaces

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Abstract

The present paper deals with the study of Ricci soliton on weak symmetries of almost Kenmotsu $(\kappa,\mu,\nu)-$space and its geometric properties. Also, we obtain the condition for Ricci soliton on weakly symmetric and weakly Ricci symmetric almost Kenmotsu $(\kappa,\mu,\nu)-$space with the tensor field $\pounds_{\xi}g+2S$ is parallel to be shrinking, steady and expanding respectively.

Keywords: *Almost Kenmotsu manifolds, almost Kenmotsu $(\kappa,\mu,\nu)-$spaces, Ricci solitons, weakly symmetric manifolds.*
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1. Introduction

A Ricci soliton (g,V,λ) on a Riemannian manifold (M,g) of dimension n is a generalization of Einstein metric such that

$$\pounds_V g+2S+2\lambda g=0, \tag{1.1}$$

where $\pounds$ is the Lie derivative, S is the Ricci tensor, V is a complete vector field on M and λ is a non-zero constant [23]. Metrics satisfying (1.1) are interesting and useful in physics and are often referred as quasi-Einstein (e. g. [15], [16], [18], [22]). Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t}g=-2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scaling, and often arise as blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The initial contribution in this direction is due to Friedan who discusses some aspects of it in [22]. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. A Ricci soliton on a compact manifold has constant curvature in dimension 2 ([20], [24]). For details we refer to Chow and Knopf [17], Derdzinski [20] and Yadav et al. [21]. We can observe the properties of Ricci solitons in ([3]-[8]).

Weakly symmetric Riemannian manifolds are generalizations of locally symmetric manifolds and pseudo-symmetric manifolds. These are manifolds in which the covariant derivative of the curvature tensor R, linear expression in R. The appearing coefficients of this expression are called associated 1-forms. They satisfy in the specified types of the manifolds gradually weaker conditions. In 1992, the notion of weakly symmetric and weakly Ricci symmetric manifolds were introduced by Tamassy and Binh ([32], [33]). In [32], the authors considered weakly symmetric and weakly projective-symmetric Riemannian manifolds and obtained some geometrical results. In 1993, Tamassy and Binh considered weakly symmetric and weakly Ricci symmetric Einstein and Sasakian manifolds [32]. In 2000, De et al. [19] gave necessary conditions for the compatibility of several K-contact structures with weak symmetry and weakly Ricci symmetry. In 2002, *Özgü*r, considered weakly symmetric and weakly Ricci symmetric LP-Sasakian manifolds [29] and also by second author [30] in [2011]. In [31], *Özgü*r studied weakly symmetric Kenmotsu manifolds and found many interesting results and Aktan and Gorgulu [1] studied weak symmetries of almost $r-$paracontact Riemannian manifold of P-Sasakian type. Second author [14] also investigated the properties of weakly symmetric and weakly Ricci symmetric Riemannian manifolds.

Manifold known as Kenmotsu manifold has been introduced and studied by Kenmotsu [26] in 1972. The properties of Kenmotsu manifolds were studied by many authors such as ([2], [11], [12], [13], [26], [27], [34], [35], [36]) and others. Koufogiorgos el al. [25] introduced in the notion of $(\kappa,\mu,\nu)-$contact metric manifold defined as follow:

$$R(X,Y)\xi=\eta(Y)(kI+\mu h+\nu\varphi h)X-\eta(X)(kI+\mu h+\nu\varphi h)Y, \tag{1.2}$$

for some smooth functions k,μ and ν on M, where as Ozturk, Aktan and Murathan [28] studied almost $\alpha-$cosymplective $(k,\mu,\nu)-$space under different conditions (like $\eta-$parallelism) and gave an example in dimension three. These almost Kenmotsu manifolds whose almost

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Kenmotsu structures (φ,ξ,η,g) satisfy the condition

$$R(\xi,X)Y = k(g(Y,X)\xi-\eta(X)Y)+\mu(g(hY,X)\xi-\eta(Y)hX)+\nu(g(\varphi hY,X)\xi-\eta(Y)\varphi hX), \quad (1.3)$$

for $k,\mu,\nu\in\Re_n(M^{2n+1})$, where $\Re_n(M^{2n+1})$ be the subring of the ring of smooth functions f on M^{2n+1} for which $df\wedge\eta=0$. For details we refer [25], [10].

A non-flat differentiable manifold M^{2n+1} is called weakly symmetric if there exist a vector field P and 1-forms $\alpha,\beta,\gamma,\delta$ (not simultaneously zero) on M^{2n+1} such that

$$(\nabla_X R)(Y,Z)W = \alpha(X)R(Y,Z)W+\beta(Y)R(X,Z)W+\gamma(Z)R(Y,X)W+\delta(W)R(Y,Z)X+g(R(Y,Z)W,X)P, \quad (1.4)$$

holds for all vector fields $X,Y,Z,W\in\chi(M^{2n+1})$. A weakly symmetric manifold (M^{2n+1},g) is said to be pseudo-symmetric if $\beta=\gamma=\delta=\frac{1}{2}\alpha$ and $\alpha(X)=g(X,P)$, locally symmetric if $\alpha=\beta=\gamma=\delta=0$ and $P=0$. A weakly symmetric manifold is said to be proper if at least one of the $1-$forms $\alpha,\beta,\gamma,\delta$ is not zero or $P\neq 0$.

A differentiable manifold M^{2n+1} is called weakly Ricci-symmetric if there exists 1-forms ε,σ,ρ on M^{2n+1} such that the condition

$$(\nabla_X S)(Y,Z) = \varepsilon(X)S(Y,Z)+\sigma(Y)S(X,Z)+\rho(Z)S(X,Y), \quad (1.5)$$

holds for all vector fields $X,Y,Z,W\in\chi(M^{2n+1})$. If $\varepsilon=\sigma=\rho$, then M^{2n+1} is called pseudo Ricci-symmetric [9].

In view of (1.4), if M^{2n+1} is weakly symmetric, we have

$$(\nabla_X S)(Z,W) = \alpha(X)S(Z,W)+\beta(R(X,Z)W)+\gamma(Z)S(X,W)+\delta(W)S(X,Z)+\rho(R(X,W)Z), \quad (1.6)$$

where the $1-$form ρ is defined by $\rho(X)=g(X,P)$ for all $X\in\chi(M^{2n+1})$.

As the series of the above studies, we consider Ricci soliton on an almost Kenmotsu $(\kappa,\mu,\nu)-$ space and discuss its some geometric properties.

2. Preliminaries

Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a $(2n+1)-$dimensional almost contact Riemannian manifold, where φ is a $(1,1)$ tensor field, ξ is the structure vector field, η is a $1-$form and g is Riemannian metric. It is well that the almost contact structure (φ,ξ,η,g) satisfies

$$\eta(\xi)=1, \quad \varphi\xi=0, \quad \eta\circ\varphi=0, \quad (2.1)$$

$$\varphi^2X=-X+\eta(X)\xi, \quad \eta(X)=g(X,\xi), \quad (2.2)$$

and

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y), \quad (2.3)$$

for any vector fields X,Y on M^{2n+1}. The $2-$form Ψ on M^{2n+1} defined by $\Psi(X,Y)=g(\varphi X,Y)$, is called the fundamental $2-$form of the almost contact metric manifold M^{2n+1}. Almost contact metric manifolds such that $d\eta=0$ and $d\Psi=2\eta\wedge\Psi$ are almost Kenmotsu manifolds. Finally, a normal almost Kenmotsu manifold is called Kenmotsu manifold. An almost Kenmotsu manifold is a nice example of an almost contact manifold which is neither $K-$contact nor Sasakian manifolds. Here we recall some fundamental curvature properties of almost Kenmotsu manifolds satisfy (1.2) and (1.3) and the following properties:

$$(\nabla_X\varphi)Y=g(\varphi X+hX,Y)\xi-\eta(Y)(\varphi X+hX), \quad (2.4)$$

$$\nabla_X\xi=-\varphi^2X-\varphi hX, \quad (2.5)$$

$$S(X,\xi)=2nk\eta(X), \quad (2.6)$$

$$Q\zeta=2nk\zeta, \quad (2.7)$$

$$l=-k\varphi^2+\mu h+\nu\varphi h, \quad (2.8)$$

$$l\varphi-\varphi l=2\mu h\varphi+2\nu h, \quad (2.9)$$

$$h^2=(k+1)\varphi^2, k\leq -1, \quad (2.10)$$

$$\nabla_\xi h = -\mu\varphi h + (\nu - 2)h. \tag{2.11}$$

Let (g,ξ,λ) be a Ricci soliton on an almost Kenmotsu $(\kappa,\mu,\nu)-$space. Then from (2.5), we get

$$\frac{1}{2}(\pounds_\xi g)(X,Y) = g(X,Y) - \eta(X)\eta(Y) - \frac{1}{2}\{g(\varphi hX,Y) + g(\varphi hY,X)\}, \tag{2.12}$$

In view of (1.1) and (2.12), we have

$$S(X,Y) = -(1+\lambda)g(X,Y) + \eta(X)\eta(Y) + \frac{1}{2}\{g(\varphi hX,Y) + g(\varphi hY,X)\}, \tag{2.13}$$

which yields

$$S(X,\xi) = -\lambda\eta(X) + \frac{1}{2}\eta(\varphi hX), \tag{2.14}$$

$$QX = -(1+\lambda)X + \eta(X)\xi + \varphi hX, \tag{2.15}$$

$$r = -2n(\lambda+1) - \lambda, \tag{2.16}$$

where Q is the Ricci operator, i. e., $g(QX,Y) = S(X,Y)$, r is the scalar curvature of M^{2n+1} and l, h are the operators defined by $l(X) = R(X,\xi)\xi$ and $h = \frac{1}{2}\pounds_\xi\,\varphi$, where $\pounds$ is the Lie derivative operator.

3. Main Results

Theorem 3.1. *If (g,ξ,λ) be a Ricci soliton on a weakly symmetric almost Kenmotsu $(\kappa,\mu,\nu)-$space then the sum of $1-$form is zero everywhere provide that Ricci soliton to be either shrinking or expanding.*

Proof. Let M^{2n+1} is a weakly symmetric almost Kenmotsu (κ,μ,ν)-space. Then substituting $W = \xi$ in (1.6), we have

$$(\nabla_X S)(Z,\xi) = \alpha(X)S(Z,\xi) + \beta(R(X,Z)\xi) + \gamma(Z)S(X,\xi) + \delta(\xi)S(X,Z) + \rho(R(X,\xi)Z), \tag{3.1}$$

In view of (1.2), (2.13) and (2.14), equation (3.1) reduces to

$$\begin{aligned}(\nabla_X S)(Z,\xi) \;=\;& -\lambda\eta(Z)\alpha(X) + \frac{1}{2}\alpha(X)\eta(\varphi hZ) + \kappa\eta(Z)\beta(X) + \mu\eta(Z)\beta(hX) + \nu\eta(Z)\beta(\varphi hX) - \kappa\beta(Z)\eta(X)\\ & -\mu\eta(X)\beta(hZ) - \nu\eta(X)\beta(\varphi hZ) - \lambda\eta(X)\gamma(Z) + \frac{1}{2}\gamma(Z)\eta(\varphi hX) - (1+\lambda)g(X,Z)\delta(\xi)\\ & +\eta(X)\eta(Z)\delta(\xi) + \frac{1}{2}\delta(\xi)\{g(\varphi hX,Z) + g(\varphi hZ,X)\} + \rho(R(X,\xi)Z).\end{aligned} \tag{3.2}$$

Taking covariant differentiation of the Ricci tensor S along the vector field X, we have

$$(\nabla_X S)(Z,\xi) = \nabla_X S(Z,\xi) - S(\nabla_X Z,\xi) - S(Z,\nabla_X\xi).$$

By the use of (2.5) and (2.14) above equation takes the form

$$\begin{aligned}(\nabla_X S)(Z,\xi) \;=\;& -\lambda g(\nabla_X Z,\xi) - \lambda g(Z,\nabla_X\xi) + \frac{1}{2}\{g((\nabla_X\varphi)hZ,\xi) + g(\varphi(\nabla_X h)Z,\xi) + g(\varphi h(\nabla_X Z),\xi) + g(\varphi hZ,\nabla_X\xi)\}\\ & +\lambda g(\nabla_X Z,\xi) - \frac{1}{2}g(\varphi h\nabla_X Z,\xi) - S(Z,X) - \lambda\eta(Z)\eta(X) + \frac{1}{2}g(\varphi hZ,\xi)\eta(X) + S(Z,\varphi hX),\end{aligned} \tag{3.3}$$

Comparing the right hand sides of (3.2) and (3.4), we obtain

$$\begin{aligned}&-\lambda\eta(Z)\alpha(X) + \frac{1}{2}\alpha(X)\eta(\varphi hZ) + \kappa\eta(Z)\beta(X) + \mu\eta(Z)\beta(hX) + \nu\eta(Z)\beta(\varphi hX) - \kappa\beta(Z)\eta(X) - \mu\eta(X)\beta(hZ) - \nu\eta(X)\beta(\varphi hZ)\\ &\quad -\lambda\eta(X)\gamma(Z) + \frac{1}{2}\gamma(Z)\eta(\varphi hX) - (1+\lambda)g(X,Z)\delta(\xi) + \eta(X)\eta(Z)\delta(\xi) + \frac{1}{2}\delta(\xi)\{g(\varphi hX,Z) + g(\varphi hZ,X)\}\\ &\quad +\rho(R(X,\xi)Z) = -\lambda g(\nabla_X Z,\xi) - \lambda g(Z,\nabla_X\xi) + \frac{1}{2}\{g((\nabla_X\varphi)hZ,\xi) + g(\varphi(\nabla_X h)Z,\xi) + g(\varphi h(\nabla_X Z),\xi) + g(\varphi hZ,\nabla_X\xi)\}\\ &\quad +\lambda g(\nabla_X Z,\xi) - \frac{1}{2}g(\varphi h\nabla_X Z,\xi) - S(Z,X) - \lambda\eta(Z)\eta(X) + \frac{1}{2}g(\varphi hZ,\xi)\eta(X) + S(Z,\varphi hX).\end{aligned} \tag{3.4}$$

Setting $X = Z = \xi$ in (3.4) and on simplification, we yield

$$\lambda\{\alpha(\xi) + \gamma(\xi) + \delta(\xi)\} = 0,$$

which implies that the vanishing of the 1-form $\alpha+\gamma+\delta$ over the vector field ξ necessary in order that M^{2n+1} be a Ricci soliton on weakly symmetric almost Kenmotsu $(\kappa,\mu,\nu)-$space. Now we can easily show that, as similar to the previous calculation, $\lambda\{\alpha(X) + \gamma(X) + \delta(X)\} = 0$ holds for arbitrary vector field X on M^{2n+1}, which gives the statement of the theorem. □

Özturk, Aktan and Papantoniou [28] proved that on an almost Kenmotsu $(\kappa,\mu,\nu)-$space of dimension greater than or equal to 5, the function κ,μ,ν only vary in the direction of ξ, *i.e.*, $X(\kappa)=X(\mu)=X(\nu)=0$ for every vector field X orthogonal to ξ. By considering this fact and Theorem 3.1, we can state the result as the corollaries:

Corollary 3.2. *Let M be an almost Kenmotsu $(\kappa,\mu,\nu)-$space of dimension greater than or equal to 5, the function κ,μ,ν only vary in the direction of ξ, i.e., $X(\kappa)=X(\mu)=X(\nu)=0$ for every vector field X orthogonal to ξ, then there does not exists weakly symmetric almost Kenmotsu $(\kappa,\mu,\nu)-$space $M^{2n+1},(\kappa\leq-1)$, if $\alpha+\gamma+\delta$ is not everywhere zero.*

Corollary 3.3. *Let M be an almost Kenmotsu $(\kappa,\mu,\nu)-$space of dimension greater than or equal to 5, the function κ,μ,ν only vary in the direction of ξ, i. e., $X(\kappa)=X(\mu)=X(\nu)=0$ for every vector field X orthogonal to ξ, then there exist no weakly symmetric Ricci soliton almost Kenmotsu $(\kappa,\mu,\nu)-$space $M^{2n+1},(\kappa\leq-1)$, if the Ricci soliton is steady in nature.*

Theorem 3.4. *Let (g,ξ,λ) be a Ricci soliton on a weakly Ricci symmetric almost Kenmotsu $(\kappa,\mu,\nu)-$space, then the sum of $1-$forms is zero, i.e., $\varepsilon+\sigma+\rho=0$, everywhere provided that the Ricci soliton to be either shrinking or expanding.*

Proof. Assume that M^{2n+1} is a weakly Ricci symmetric almost Kenmotsu (κ,μ,ν)-space. Putting $Z=\xi$ in (1.5) and by use of (2.14), we have

$$(\nabla_X S)(Y,\xi)=\varepsilon(X)\{-\lambda\eta(Y)+{}^1\!/_2\eta(\varphi hY)\}+\sigma(Y)\{-\lambda\eta(X)+{}^1\!/_2\eta(\varphi hX)\}+\rho(\xi)S(X,Y). \tag{3.5}$$

Again replacing Z with Y in (3.3) and comparing the right hand sides of the equation (3.3) and (3.5), we get

$$\begin{aligned}&\varepsilon(X)\{-\lambda\eta(Y)+{}^1\!/_2\eta(\varphi hY)\}+\sigma(Y)\{-\lambda\eta(X)+{}^1\!/_2\eta(\varphi hX)\}+\rho(\xi)S(X,Y)\\ &=-\lambda g(\nabla_XY,\xi)-\lambda g(Y,\nabla_X\xi)+{}^1\!/_2\{g(\nabla_X\varphi)hY,\xi)+g(\varphi(\nabla_Xh)Y,\xi)+g(\varphi h(\nabla_XY),\xi)+g(\varphi hY,\nabla_X\xi)\}\\ &+\lambda g(\nabla_XY,\xi)-{}^1\!/_2g(\varphi h\nabla_XY,\xi)-S(Y,X)-\lambda\eta(Y)\eta(X)+\frac{1}{2}g(\varphi hY,\xi)\eta(X)+S(Y,\varphi hX).\end{aligned} \tag{3.6}$$

Taking $X=Y=\xi$ in (3.6) and using (2.1), (2.5), (2.10) and (2.14), we get

$$\lambda\{\varepsilon(\xi)+\sigma(\xi)+\rho(\xi)\}=0. \tag{3.7}$$

Again putting $X=\xi$ in (3.6), we have

$$\lambda\sigma(Y)\eta(Y)=-\sigma(\xi)\{-\lambda\eta(Y)+{}^1\!/_2\eta(\varphi hY)\}. \tag{3.8}$$

Replacing Y with X, we yield

$$\lambda\sigma(X)\eta(X)=-\sigma(\xi)\{-\lambda\eta(X)+{}^1\!/_2\eta(\varphi hX)\}. \tag{3.9}$$

If we take $Y=\xi$ in (3.6), we obtain

$$\lambda\varepsilon(X)\eta(X)=-\alpha(\xi)\{-\lambda\eta(X)+{}^1\!/_2\eta(\varphi hX)\} \tag{3.10}$$

and

$$\lambda\rho(X)\eta(X)=-\rho(\xi)\{-\lambda\eta(X)+{}^1\!/_2\eta(\varphi hX)\}. \tag{3.11}$$

The summation of (3.9), (3.10) and (3.11), using (3.7), give
$\lambda\eta(X)\{\sigma(X)+\varepsilon(X)+\rho(X)\}=0$, for all $X\in\chi(M^{2n+1})$. Let us suppose that the Ricci soliton is either shrinking or expanding, then the last result shows that either $\eta(X)=0$ or $\sigma(X)+\varepsilon(X)+\rho(X)=0$. In general $\eta(X)\neq0$ on almost Kenmotsu manifolds, therefore $\sigma(X)+\varepsilon(X)+\rho(X)=0$ and hence the statement of the theorem. □

In view of Theorem 3.4 and the results of Özturk et al. (for instance, see [28]), we state the corollary:

Corollary 3.5. *Let M be an almost Kenmotsu $(\kappa,\mu,\nu)-$space of dimension greater than or equal to 5, the function κ,μ,ν only vary in the direction of ξ, i.e., $X(\kappa)=X(\mu)=X(\nu)=0$ for every vector field X orthogonal to ξ, then there does not exist Ricci soliton on weakly Ricci symmetric almost Kenmotsu $(\kappa,\mu,\nu)-$space $M^{2n+1},(\kappa\leq-1)$, if the sum of the $1-$forms, i.e. $\varepsilon+\sigma+\rho$, is not everywhere zero.*

Theorem 3.6. *In a weakly $\varphi-$symmetric almost Kenmotsu $(\kappa,\mu,\nu)-$space, the curvature tensor assumes the form (3.13).*

Proof. We assume that M^{2n+1} is a weakly φ- symmetric almost Kenmotsu (κ,μ,ν)-space, *i.e.*, $\varphi^2((\nabla_XR)(Y,Z)W)=0$, $\forall\, X,Y,Z,W\in\chi(M^{2n+1})$. Then by virtue of (1.4) and (2.2), we get

$$\begin{aligned}-(\nabla_XR)(Y,Z)W+\eta((\nabla_XR)(Y,Z)W)\xi&=\alpha(X)\{-R(Y,Z)W+\eta(R(Y,Z)W)\xi\}+\beta(Y)\{-R(X,Z)W+\eta(R(X,Z)W)\xi\\ &+\gamma(Z)\{-R(Y,X)W+\eta(R(Y,Z)W)\xi\}+\delta(W)\{-R(Y,Z)X+\eta(R(Y,Z)X)\xi\}+g(R(Y,Z)W,X)\{-P+\eta(P)\xi\}.\end{aligned} \tag{3.12}$$

Setting $W=\xi$ in (3.12) and then use of (1.2) gives

$$\delta(\xi)R(Y,Z)X=(kI+\mu h+\nu\varphi h)\left\{\begin{array}{l}-\alpha(X)\{\eta(Z)Y-\eta(Y)Z\}-\beta(Y)\{\eta(Z)X-\eta(X)Z\}\\ -\gamma(Z)\{\eta(X)Y-\eta(Y)X\}+\delta(\xi)\{\eta(Y)g(X,Z)\\ -\eta(Z)g(X,Y)\}\xi+\{\eta(Z)g(X,Y)-\eta(Y)g(X,Z)\}\varphi^2P\end{array}\right\}+(\nabla_XR)(Y,Z)\xi-\eta((\nabla_XR)(Y,Z)\xi)\xi.$$

With the help of (1.2) and (2.5), we can find

$$\begin{aligned}(\nabla_X R)(Y,Z)\xi &= \{(X\kappa)I+(X\mu)h+\mu(\nabla_X h)+(X\nu)\varphi h+\nu(\nabla_X\varphi h)\}\{\eta(Z)Y-\eta(Y)Z\}\\ &+(\kappa I+\mu h+\nu\varphi h)\{g(X,Z)Y-g(X,Y)Z+g(\varphi hX,Y)Z-g(\varphi hX,Z)Y\}\\ &-R(Y,Z)X+R(Y,Z)\varphi hX.\end{aligned}$$

In consequence of above relations, we have

$$\begin{aligned}(1+\delta(\xi))R(Y,Z)X &= \{(X\kappa)I+(X\mu)h+\mu(\nabla_X h)+(X\nu)\varphi h+\nu(\nabla_X\varphi h)\}\{\eta(Z)Y-\eta(Y)Z\}\\ &+(kI+\mu h+\nu\varphi h)\{-\alpha(X)\{\eta(Z)Y-\eta(Y)Z\}-\beta(Y)\{\eta(Z)X-\eta(X)Z\}\\ &-\gamma(Z)\{\eta(X)Y-\eta(Y)X\}+\delta(\xi)\{\eta(Y)g(X,Z)-\eta(Z)g(X,Y)\}\xi\\ &+g(X,Z)Y-g(X,Y)Z+g(\varphi hX,Y)Z-g(\varphi hX,Z)Y\\ &+\{\eta(Z)g(X,Y)-\eta(Y)g(X,Z)\}\varphi^2P\}+R(Y,Z)\varphi hX,\end{aligned}\tag{3.13}$$

provided $1+\delta(\xi)\neq 0$. Hence theorem is proved. □

Theorem 3.7. *A weakly $\varphi-$Ricci symmetric almost Kenmotsu $(\kappa,\mu,\nu)-$space is an $\eta-$Einstein manifold.*

Proof. Contracting (3.13) along the vector field Y, we get

$$\begin{aligned}(1+\delta(\xi))S(Z,X) &= 2n\{(X\kappa)I+(X\mu)h+\mu(\nabla_X h)+(X\nu)\varphi h+\nu(\nabla_X\varphi h)\}\eta(Z)\\ &+(kI+\mu h+\nu\varphi h)\{-2n\alpha(X)\eta(Z)-\beta(X)\eta(Z)+\beta(Z)\eta(X)\\ &-2n\gamma(Z)\eta(X)+\delta(\xi)g(X,Z)-\delta(\xi)\eta(X)\eta(Z)+2ng(X,Z)\\ &-2ng(\varphi hX,Z)+\eta(Z)g(X,\varphi^2P)\}+S(Z,\varphi hX),\end{aligned}\tag{3.14}$$

provided $1+\delta(\xi)\neq 0$. Replacing Y by φY and X by φX in (3.14) and using (2.5) and (2.6) in it, we get

$S(Y,W)=ag(Y,W)+b\eta(Y)\eta(W),$

where a and b are smooth functions and connected by the relation $a+b=2nk$. Thus the statement of the theorem. □

On the other hand if the triplet (g,ξ,λ) be a Ricci soliton on an almost Kenmotsu $(\kappa,\mu,\nu)-$space. Then from (2.13) and (3.14), we state the results as the corollaries.

Corollary 3.8. *A weakly $\varphi-$Ricci symmetric Ricci soliton almost Kenmotsu $(\kappa,\mu,\nu)-$space is an $\eta-$Einstein manifold with $\lambda=a-b-1$.*

Corollary 3.9. *A weakly $\varphi-$Ricci symmetric and weakly $\varphi-$Ricci symmetric Ricci soliton are equivalent on an almost Kenmotsu (κ,μ,ν)-space.*

Aktan et al. [2] studied weakly symmetric and weakly Ricci symmetric on an almost Kenmotsu $(\kappa,\mu,\nu)-$spaces and found some geometrical results. They proved that the Ricci tensor S on weakly symmetric almost Kenmotsu $(\kappa,\mu,\nu)-$spaces assumes the form

$$S(X,Z)=\frac{1}{\delta(\xi)}\left\{\begin{array}{l}2nX(\kappa)\eta(Z)+2n\kappa g(Z,\nabla_X\xi)-S(Z,\nabla_X\xi)-2n\kappa\alpha(X)-\beta(\kappa)\eta(Z)X-\kappa\eta(Z)\beta(X)-\beta(\mu)\eta(Z)hX\\ -\mu\eta(Z)\beta(hX)-\beta(\nu)\eta(Z)\varphi hX-\nu\eta(Z)\beta(\varphi hX)+\beta(\kappa)\eta(X)Z+\kappa\eta(X)\beta(Z)+\beta(\mu)\eta(X)hZ\\ +\mu\eta(X)\beta(hZ)+\beta(\nu)\eta(X)\varphi hZ+\nu\eta(X)\beta(\varphi hZ)-2n\kappa\gamma(Z)\eta(X)+\rho(\kappa)(g(X,Z)\xi-\eta(Z)X)\\ +\kappa(g(X,Z)\rho(\xi)-\eta(Z)\rho(X))+\rho(\mu)(g(hZ,X)\xi-\eta(Z)(hX))+\mu(g(hZ,X)\rho(\xi)-\eta(Z)\rho(hX))\\ +\rho(\nu)(g(\varphi hZ,X)\xi-\eta(Z)(\varphi hX))+\nu(g(\varphi hZ,X)\rho(\xi)-\eta(Z)\rho(\varphi hX))\end{array}\right\}\tag{3.15}$$

provided $\delta(\xi)\neq 0$. We suppose that h is a $(0,2)$ type symmetric parallel tensor field on an almost Kenmotsu $(\kappa,\mu,\nu)-$space $M^{2n+1},(\kappa\leq -1)$, such that

$$\mathrm{h}(X,Z)=(\pounds_\xi g)(X,Z)+2S(X,Z).\tag{3.16}$$

Setting $X=Z=\xi$ in (3.16) and then using (2.12) and (3.15), we observe that

$$\mathrm{h}(\xi,\xi)=\frac{4n}{\delta(\xi)}\{\xi(\kappa)-\kappa\{\alpha(\xi)+\gamma(\xi)\}\},\tag{3.17}$$

If (g,ξ,λ) be a Ricci soliton on an almost Kenmotsu $(\kappa,\mu,\nu)-$space. Then from (1.1), we get

$$h(\xi,\xi)=-2\lambda.\tag{3.18}$$

In view of (3.17) and (3.18), we yield

$$\lambda=\frac{2n}{\delta(\xi)}\{\kappa\{\alpha(\xi)+\gamma(\xi)\}-\xi(\kappa)\}.\tag{3.19}$$

Thus, we can state the following:

Theorem 3.10. *If the tensor field $\pounds_\xi g+2S$ on a weakly symmetric almost Kenmotsu $(\kappa,\mu,\nu)-$space $M^{2n+1},(\kappa\leq -1)$, with $\delta(\xi)\neq 0$ is parallel, then the Ricci soliton (g,ξ,λ) is shrinking; steady and expanding according as $\xi(k)\geq 0,\delta(\xi)>0$; $\xi(k)=k\{\alpha(\xi)+\gamma(\xi)\}$ and $\xi(k)=0,\delta(\xi)<0$ respectively.*

It is also observed in [2] that the form of Ricci tensor S of a weakly Ricci symmetric almost Kenmotsu $(\kappa,\mu,\nu)-$space takes the form

$$S(X,Y)=\frac{1}{\rho(\xi)}\{2nX(\kappa)\eta(Y)+2n\kappa g(Y,\nabla_X\xi)-S(Y,\nabla_X\xi)-2n\kappa\varepsilon(X)\eta(Y)-2n\kappa\sigma(Y)\eta(X)\}, \tag{3.20}$$

provided $\rho(\xi)\neq 0$. Again let h is a $(0,2)$ type symmetric parallel tensor field on an almost Kenmotsu $(\kappa,\mu,\nu)-$space $M^{2n+1},(\kappa\leq -1)$, such that

$$\mathrm{h}(\mathrm{X},\mathrm{Y})=(\pounds_\xi g)(X,Y)+2S(X,Y). \tag{3.21}$$

Here, we observe that $X=Y=\xi$, in view of (2.12) and (3.20), equation(3.21) takes the form

$$\mathrm{h}(\xi,\xi)=\frac{4n}{\rho(\xi)}\{\xi(\kappa)-\kappa\{\varepsilon(\xi)+\sigma(\xi)\}\}. \tag{3.22}$$

In view of (3.18) and (3.22), we get

$$\lambda=\frac{2n}{\rho(\xi)}\{\kappa\{\varepsilon(\xi)+\sigma(\xi)\}-\xi(\kappa)\}. \tag{3.23}$$

Thus, we can state the following:

Theorem 3.11. *If the tensor field $\pounds_\xi g+2S$ on a weakly Ricci symmetric almost Kenmotsu $(\kappa,\mu,\nu)-$space $M^{2n+1},(\kappa\leq -1)$, with $\rho(\xi)\neq 0$ is parallel, then Ricci soliton (g,ξ,λ) is shrinking; steady and expanding according as $\xi(k)\geq 0,\rho(\xi)>0$; $\xi(k)=k\{\alpha(\xi)+\gamma(\xi)\}$ and $\xi(k)=0,\rho(\xi)<0$ respectively.*

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