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The Unique Midpoint Property in Ultrametric Spaces

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ABSTRACT

In this paper, we investigate ultrametric spaces that satisfy the Unique Midpoint Property (UMP), a condition where every pair of distinct points has a unique midpoint equidistant from both. By analyzing the interaction between the strong triangle inequality of ultrametric spaces and the requirements of the UMP, we derive fundamental constraints on the structure of such spaces. In particular, we prove that in any ultrametric space with the UMP, the distance between any two distinct points equals the distance from each to their unique midpoint. Our main result shows that any ultrametric space satisfying the UMP must be trivial in structure it can only consist of either a single point or exactly three points.

1. Introduction

Let X be a set and $d: X \times X \to \mathbb{R}$ be a function. If the function d satisfies the following properties for all $x, y, z \in X$:

(d1) $d(x,y) \geqslant 0$
$(\boldsymbol{d2})$ $d(x,y) = 0$ iff $x = y$
(d3) $d(x,y) = d(y,x)$
$(\boldsymbol{d4})$ $\max\{d(x,y), d(y,z)\} \geqslant d(x,z)$

then d is called an ultrametric function on X, and the pair (X,d) is called an ultrametric space. It is obvious that every ultrametric space is a metric space. There are also interesting properties that balls provide. Here are two important properties that we use throughout this article: Let (X,d) be an ultrametric space, $x, y \in X$ and $r_1, r_2 > 0$. If $B(x,r_1) \cap B(y,r_2) \neq \emptyset$, then $B(x,r_1) \subseteq B(y,r_2)$ or $B(y,r_2) \subseteq B(x,r_1)$. Also, if $y \in B(x,r_1)$, then $B(x,r_1) = B(y,r_1)$. Let (X,d) be a metric space. If for an arbitrary pair $x, y \in X$ there exists a unique $z \in X$ such that $d(x,z) = d(z,y)$, then the point z is called the midpoint of x and y. This space is said to have the unique midpoint property (UMP).

Since the topic of this article is the existence of the unique midpoint property in ultrametric spaces, no extra information on either the ultrametric space or the unique midpoint property is presented, but for convenience, the topic of the ultrametric space and the unique midpoint property can be followed through the articles [1], [2], [3], and [4].

2. Assumption: There Exist an Ultrametric Space with the Unique Midpoint Property

Let (X,d) be an ultrametric space with the unique midpoint property. For any given points $x, m_0 \in X$, by our assumption, there exists a unique point $m_1 \in X$ such that

$$d(x,m_1) = d(m_1,m_0)$$

and by the ultrametric axiom ($d4$), it is obtained that

$$d(x,m_1) = d(m_1,m_0) \geqslant d(x,m_0).$$

In this situation there are two cases: (case 1) $d(x,m_1) = d(m_1,m_0) > d(x,m_0)$ and (case 2) $d(x,m_1) = d(m_1,m_0) = d(x,m_0)$.

Case 1: $d(x,m_1) = d(m_1,m_0) > d(x,m_0)$.

Let $m_2 \in X$ be the unique midpoint of x and m_1, so

$$d(x,m_2) = d(m_1,m_2).$$

Then by ultrametric axiom ($d4$):

$$\max\{d(x,m_2), d(m_2,m_1)\} \geqslant d(x,m_1)$$

must hold by the case

$$d(m_1,m_2) = d(x,m_2) \geqslant d(x,m_1) > d(x,m_0)$$

and

$$d(m_1,m_2) = d(x,m_2) \geqslant d(m_0,m_1) > d(x,m_0).$$

So there are two subcases:
Subcase 1:

$$d(m_1,m_2) = d(x,m_2) = d(x,m_1) = d(m_0,m_1) > d(x,m_0).$$

By the ultrametric axiom:

$$\max\{d(m_0,m_2), d(x,m_0)\} \geqslant d(x,m_2).$$

By the Case 1:

$$max\{d(m_0,m_2), d(x,m_0)\} > d(x,m_2)$$

must hold. Therefore, the following equality is imperative:

$$\max\{d(m_0,m_2), d(x,m_0)\} = d(m_0,m_2).$$

Hence

$$d(m_0,m_2) > d(x,m_0).$$

Also:

$$\max\{d(x,m_0), d(m_2,x)\} \geqslant d(m_0,m_2) > d(x,m_0)$$
$$d(x,m_2) \geqslant d(m_0,m_2) > d(x,m_0).$$

By the uniqueness of the midpoint of x and m_0 :

$$d(x,m_2) > d(m_0,m_2) > d(x,m_0).$$

Therefore:

$$\max\{d(m_0,m_2), d(x,m_0)\} \geqslant d(x,m_2)$$
$$d(x,m_2) > d(m_0,m_2) \geqslant d(x,m_2).$$

This is a contradiction.

Subcase 2:

$$d(m_1,m_2) = d(x,m_2) > d(m_0,m_1) = d(x,m_1) > d(x,m_0).$$

By the ultrametric property:

$$\max\{d(x,m_2), d(x,m_0)\} \geqslant d(m_0,m_2)$$
$$d(x,m_2) \geqslant d(m_0,m_2).$$

By the uniqueness of the midpoint of x and m_0 :

$$d(x,m_2) > d(m_0,m_2).$$

Also by the ultrametric property:

$$\max\{d(m_0,m_2), d(m_0,m_1)\} \geqslant d(m_1,m_2).$$

If $d(m_0,m_2) \geqslant d(m_1,m_2)$, then we get:

$$d(x,m_2) > d(m_0,m_2) \geqslant d(m_1,m_2) = d(x,m_2)$$

which is a contradiction.
If

$$d(m_0,m_1) \geqslant d(m_1,m_2)$$

then we get:

$$d(m_1,m_2) > d(m_0,m_1) \geqslant d(m_1,m_2)$$

which is another contradiction. Thus, it is shown that Case 1 never occurs. Hence Case 2 must be satisfied.

Theorem 1. Let $B(x,r)$ be an open ball and $x, m_0 \in B(x,r)$. If the midpoint m_1 of x and m_0 is an element of $B(x,r)$, then $B(x,r)$ cannot contain any element other than x, m_0 and m_1.

Proof. Assume that there exists $z \in B(x,r)$ such that $z \neq \{x, m_0, m_1\}$. It is clear that

$$d(x,m_0) = d(m_0,m_1) = d(m_1,x)$$

since Case 2 holds. Moreover, by the uniqueness of the midpoint:

$$d(x,z) \neq d(m_0,z) \neq d(m_1,z).$$

So without loss of generality:

$$d(x,z) < d(m_0,z) < d(m_1,z).$$

By the ultrametric axiom (d_4) :

$$\max\{d(x,z), d(m_0,z)\} \geqslant d(x,m_0)$$
$$d(m_0,z) \geqslant d(x,m_0).$$

There are two cases:

Case 1: $d(m_0,z) > d(x,m_0)$. By the ultrametric axiom (d_4) :

$$\max\{d(m_0,z), d(m_0,m_1)\} \geqslant d(m_1,z),$$

$$d(m_0,z) \geqslant d(m_1,z).$$

It is a contradiction.

Case 2: $d(m_0,z) = d(x,m_0)$. This means that: $d(m_0,z) = d(x,m_0) = d(m_0,m_1) = d(m_1,x)$. By the ultrametric axiom (d_4) :

$$\max\{d(m_0,z), d(m_0,m_1)\} \geqslant d(m_1,z),$$

$$d(m_0,z) \geqslant d(m_1,z).$$

Theorem 2. Let (X,d) be an ultrametric space with UMP. An element can be the midpoint of at most one pair of elements.
Proof. Assume a point m is the midpoint of x_1 and x_2. There are two cases:

Case 1: Let $x_3 \in X$ and m be the midpoint of x_2 and x_3. There exists $r_1, r_2 > 0$ such that $B(x_2, r_1)$ contains x_1, x_2, m and $B(x_2, r_2)$ contains x_2, x_3, m. So $B(x_2, r_0)$ contains x_1, x_2, x_3, m where $r_0 := \max\{r_1, r_2\}$, but this is impossible by Theorem 1.

Case 2: Let $x_3, x_4 \in X$ and m be the midpoint of x_3 and x_4. There exists $r_1, r_2 > 0$ such that $B(x_2, r_1)$ contains x_1, x_2, m and $B(x_3, r_2)$ contains x_3, x_4, m. It is clear that:

$$B(x_2, r_1) = B(m, r_1)$$

and

$$B(x_3, r_2) = B(m, r_2).$$

Hence $B(m, r_0)$ contains x_1, x_2, x_3, m where $r_0 := \max\{r_1, r_2\}$, but this is a contradiction.

Theorem 3. Any open ball containing at least three elements must contain the midpoint of those elements.

Proof. Let $x, y \in B(x,r)$ and let m be the midpoint of x and y such that $m \notin B(x,r)$, so $d(x,m) \geqslant r$.

By Case 2 , the equality

$$d(x,m) = d(x,y) = d(y,m)$$

must hold. Since $x, y \in B(x,r)$, we have $B(x,r) = B(y,r)$, hence $d(x,y) < r$. However, by the above equality, $d(x,y) \geqslant r$. This is a contradiction. Therefore m must be contained by $B(x,r)$.

Remark: As can be easily deduced from Theorem 3, a two-element ball cannot exist in an ultrametric space with UMP. Namely, let us assume that there is a two-element ball. Let $x, y \in B(x,r)$, then the equality $d(x,y) = d(y,x) = d(x,x)$ must hold by Case 2, so $x = y$.

Theorem 4. Let $B(x,r)$ be any open ball with finite elements. $|B(x,r)| = 1$ or $|B(x,r)| = 3$.

Proof. Assume that $B(x,r)$ contains at least 4 elements. It follows that the number of ways to choose two distinct elements from $B(x,r)$ is strictly greater than the size of $B(x,r)$ itself, that is,

$$|C(|B(x,r)|, 2)| > |B(x,r)|.$$

So there exists at least one element that must be the midpoint of at least two pairs of elements. This contradicts Theorem 2.

3. Conclusions

Let us take the elements x, y and their midpoint m from an ultrametric space satisfying the unique midpoint property. A ball $B(x, r)$ containing these three points contains no other points by Theorem 1. If there is an element $z \in X$ outside this ball, then the ball $B(x, d(x, y) + 1)$ contains the element z, a contradiction. Then an ultrametric space satisfying UMP consists of either one point or three points.

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