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Axial Vibration of a Nanoring Rod Using Nonlocal Finite Element Method

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Abstract

This paper provides a thorough analysis of the axial vibration behavior of nanoring rods based on nonlocal elasticity theory, highlighting its relevance to nanoscale systems. The equation governing the axial vibration of nanoscale rods under nonlocal effects is formulated. By applying appropriate transformations to this equation, the frequency equation is derived. Additionally, a nonlocal finite element formulation for the rod is developed using the weighted residual method.

Keywords: Axial vibration, Nanoring rod, Nonlocal elasticity, Finite element method

1. Introduction

Recent studies have demonstrated that the size effect in micro and nanoscale materials plays a critical role in determining their mechanical properties. This is due to the significant increase in the surface-to-volume ratio as the size of the material decreases. Consequently, the influence of surface atoms becomes more pronounced, leading to changes in the material's strength, elasticity, thermal conductivity, and optical properties. The effect of these changes in material properties has not been addressed by classical theories. To account for this, various theories incorporating the size effect have been developed. The most widely recognized of these is the "Nonlocal Elasticity Theory", which has been adopted as the solution method in the present study. The emergence of this theory and along with several related studies are presented below.

The theory of nonlocal elasticity using the laws of global equilibrium and the second law of thermodynamics was developed by Eringen [1]. In the following years, Eringen searched for solutions to various problems using the nonlocal elasticity theory and demonstrated its effectiveness [2-5]. A study has been conducted combining nanotechnology with the nonlocal elasticity theory. One version of the nonlocal elasticity theory has been utilized to develop a nonlocal Bernoulli/Euler beam model [6]. The wave propagation on carbon nanotubes is investigated by Euler-Bernoulli and Timoshenko beam models using nonlocal elasticity theory [7]. The Euler-Bernoulli, Timoshenko, Reddy, and Levinson beam theories have been reformulated using Eringen's nonlocal elasticity theory. Analytical solutions for bending, vibration and buckling problems are provided. The influence of nonlocal behavior on displacements, buckling loads, and natural frequencies is also discussed [8]. Axial free vibration analysis of nanoscale rods based on the theory of nonlocal elasticity has been carried out [9]. Axial and torsional vibration analyses of nanorods composed of carbon nanotubes or microtubules were performed using different beam theories [10-20]. Nonlocal free vibration of axial rods embedded in elastic medium is examined by utilizing Love–Bishop rod theory. Size–

dependent finite element formulation is presented based on Weighted Residual Method [21]. The time-dependent torsional vibration of a single-walled carbon nanotube (SWCNT) under linear and harmonic external torque is investigated based on Eringen's theory of nonlocal elasticity and Hamilton's principle is used to derive the boundary conditions [22]. Nonlinear torsional vibrations of nanorods in an elastic medium under three-dimensional thermal stresses are investigated. The equation of motion is extended with scale effect using nonlocal theory [23]. The applicability of nonlocal elasticity theory has been discussed for the case where the material properties of functionally graded porous nanotubes vary in the radial direction according to a rule of mixture [24].

This paper presents a finite element solution for the axial vibration of ring cross-section rods based on nonlocal elasticity theory. The first section discusses the importance of size effects and provides examples of previous studies related to nonlocal elasticity theory that takes size effects into account. In the second section, the formulation of this theory is explained. The third section derives the governing equation for the axial vibration of ring rods. The fourth section presents the nonlocal finite element formulation.

2. Nonlocal Elasticity Theory (NL)

The nonlocal stress tensor at point x is expressed as [5]:

$$\sigma_{kl}(x) = \int_V K(|x' - x|, \alpha)\tau_{kl}(x')dV(x'), \tag{1}$$

where $K|x - x'|$ represents the distance in Euclidean form and defines the strain effect of the stress value of the elastic body at point x at point x'. α is a material constant that depends on the ratio ($\frac{e_0 a}{l}$). Here, e_0 is specific to the atomic structure and is determined experimentally; a represents the internal characteristic length of the atomic structure (granular distance or the distance between C-C molecules), and L denotes the external characteristic length. $\tau_{kl}(x')$ is the local fourth-order elasticity tensor of the body at point x' and V is the volume occupied by the elastic body. The nonlocal constitutive formulation is [5]:

$$\left[1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\right]\sigma_{kl} = \tau_{kl} \tag{2}$$

2.1. Axial Vibration Analysis of Ring Rods

In this section, the equation of motion for free vibration of axial micro and nanoscale rods with ring cross-section will be obtained by nonlocal elasticity theory. For this purpose, the displacement at any point *x* and at any instant *t* in the rod section is considered as *u* (Fig.1).

The dynamic equilibrium in the *x* direction for this rod is as follows:

$$u(x,t) = u(x,t) \tag{3}$$

As can be inferred from this, only σ_{xx} stress occurs in the rod, and Eq (2) is rearranged as follows:

$$\sigma_{xx} - (e_0 a)^2 \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E\varepsilon_{xx} \tag{4}$$

here, E is the elasticity modulus, and ε_{xx} is the deformation occurring in the x-direction. Multiplying z on both sides of Eq. (4) and integrating over the cross-sectional area (A) of the rod, we obtain

$$\left(N^{nl} - (e_0 a)^2 \frac{\partial^2 N^{nl}}{\partial x^2}\right) = EA\varepsilon_{xx} = EA\frac{\partial u}{\partial x} \tag{5}$$

Axial vibration of N^{nl} continuous systems is as follows:

$$\frac{\partial N^{nl}}{\partial x} = m\frac{\partial^2 u}{\partial t^2} \tag{6}$$

here, $m = \rho A$ equals and defines the mass per unit length. Substituting Eq. (6) into (5), we get

$$EA\frac{\partial^2 u}{\partial x^2} - \rho A\frac{\partial^2 u}{\partial t^2} + (e_0 a)^2 \rho A\frac{\partial^4 u}{\partial x^2 \partial t^2} = 0 \tag{7}$$

The equation of motion for the free vibration of an axially nano rod with a circular cross-section, including the size effect, is obtained. When $e_0 a = 0$ in Eq. (7), the equation of motion for the classical rod is obtained.

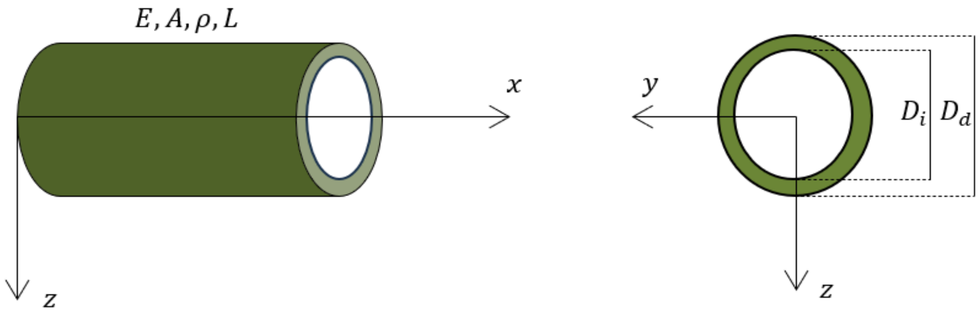

Fig. 1. Axial vibration of ring rod

2.1.1. Continuous System

The solution of the main Eq. (7) for nonlocal axial vibration is analyzed using the following transformation,

$$u(x,t) = U(x)T(t), \quad T(t) = \sin(\omega t - \theta) \tag{8}$$

The derivatives of the u term in Eq. (7) are as follows:

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 U}{dx^2}\sin(\omega t - \theta)$$
$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 \sin(\omega t - \theta)\, U \tag{9}$$
$$\frac{\partial^4 u}{\partial x^2 \partial t^2} = -\omega^2 \frac{d^2 U}{dx^2}\sin(\omega t - \theta)$$

The expressions in Eq. (9) are substituted into Eq. (7), yielding the following form,

$$U'' + \alpha^2 U = 0 \tag{10}$$

$$\alpha^2 = \frac{\rho A \omega^2}{EA - (e_0 a)^2 \rho A \omega^2} \tag{11}$$

The solution of Eq. (10) is performed using the $U(x) = Ce^{kx}$ transformation, yielding the following result,

$$U(x) = Acos\alpha x + Bsin\alpha x \tag{12}$$

This equation determines the mode shape and frequency of the bar. The coefficients *A* and *B* in the equation are obtained from the boundary conditions.

2.2. Nonlocal Finite Element Method (NL-FEM) for Axial Vibration

Finite element method is based on defining approximate functions to obtain the exact solution with approximate values. The exact solution of the axial vibration, as given in Eq. (7), requires that the right-hand side of the equation be equal to zero. When the axial displacement *u* is solved using an approximate method, the right-hand side of the equation does not equal zero. In this case, the finite element method attempts to make the average weighted residual zero in the rod of length L. To do this, the expression is multiplied by the weight function $w(x,y)$ and the integral over the length is taken,

$$I = \int_0^L w\left[EA\frac{\partial^2 u}{\partial x^2} - \rho A\frac{\partial^2 u}{\partial t^2} + (e_0 a)^2 \rho A \frac{\partial^4 u}{\partial x^2 \partial t^2}\right] dx \tag{13}$$

Eq. (13) can be divided three integral expressions,

$$I = \int_0^L wEA\frac{\partial^2 u}{\partial x^2}dx - \int_0^L w\rho A\frac{\partial^2 u}{\partial t^2}dx + \int_0^L w(e_0 a)^2 \rho A\frac{\partial^4 u}{\partial x^2 \partial t^2}dx \tag{14}$$

The method of partial integration is applied to the divided integrals,

$$\int_0^L wEA\frac{\partial^2 u}{\partial x^2}dx = w\,EA\frac{\partial u}{\partial x}\bigg|_0^L - \int_0^L EA\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}dx$$

$$\int_0^L w(e_0a)^2\rho A\frac{\partial^4 u}{\partial x^2\partial t^2}dx = \ w(e_0a)^2\rho A\frac{\partial^3 u}{\partial x\partial t^2}\bigg|_0^L - \int_0^L (e_0a)^2\rho A\frac{\partial w}{\partial x}\frac{\partial^3 u}{\partial x\partial t^2}dx \tag{15}$$

Substituting Eqs. (15) into Eq. (14), the weak formulation of differential equation of nonlocal axial vibration is acquired as:

$$\int_0^L \left[-EA\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} - (e_0a)^2\rho A\frac{\partial w}{\partial x}\frac{\partial^3 u}{\partial x\partial t^2} - \ w\rho A\frac{\partial^2 u}{\partial t^2}\right]dx \tag{16}$$

Under boundary conditions, the weighted average of the displacement is expressed as follows:

$$w\left(EA\frac{\partial u}{\partial x} + (e_0a)^2\rho A\frac{\partial^3 u}{\partial x\partial t^2}\right)\bigg|_0^L = 0 \tag{17}$$

The finite element formulation for the displacement at each node of a two-node bar element is as follows:

$$u = N_1u_i + N_2u_j \tag{18}$$

here,

$$N_1 = 1 - \frac{x}{L}, \quad N_2 = \frac{x}{L} \tag{19}$$

are the shape functions for of i and j ends. The appropriate weight function can be selected as $w = \varphi^T$. Moreover, the following equations are used to arrange Eq. (16):

$$w = \varphi^T, \qquad \frac{\partial w}{\partial x} = B^T, \qquad \frac{\partial u}{\partial x} = Bu, \quad B = D^kN, \qquad \frac{\partial^2 u}{\partial t^2} = N\ddot{u} \tag{20}$$

where $D^k = \partial(*)/\partial x$ is kinematic operator, N is shape function vector. Eq. (16) can be rewritten as:

$$\int_0^L EA(B^TB)udx + \int_0^L \rho A[(e_0a)^2B^TB + \ \varphi^TN]\ddot{u}dx = 0 \tag{21}$$

$$\int_0^L EA\begin{Bmatrix} N_1' \\ N_2' \end{Bmatrix}[N_1' \quad N_2']\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} dx + \rho A\left[(e_0 a)^2\begin{Bmatrix} N_1' \\ N_2' \end{Bmatrix}[N_1' \quad N_2'] + \varphi^T N\begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix}[N_1 \quad N_2]\right] dx\begin{Bmatrix} \ddot{u}_i \\ \ddot{u}_j \end{Bmatrix} \tag{22}$$

Eq. (22) in general form is as follows:

$$Ku + (Q_c + Q_{nl})\ddot{u} = 0 \tag{23}$$

here,

$$K = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad Q_c = \frac{EAL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad Q_{nl} = \frac{EA(e_0 a)^2}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{24}$$

K is the axial stiffness matrix of the finite element, Q_c is the classical mass inertia and Q_{nl} is the nonlocal mass inertia matrix. The transformation $u = e^{iwt}$ is applied to Eq. (23), and for an element divided into n parts, it is as follows:

$$\det([K] - \eta^2[Q_r]) = 0, \qquad [K, Q_r] = \sum_{i=1}^{n}\{K, ([Q_c] + [Q_{nl}])\} \tag{25}$$

Figure 2 presents a flowchart outlining the sequence and interaction of the steps in the solution process.

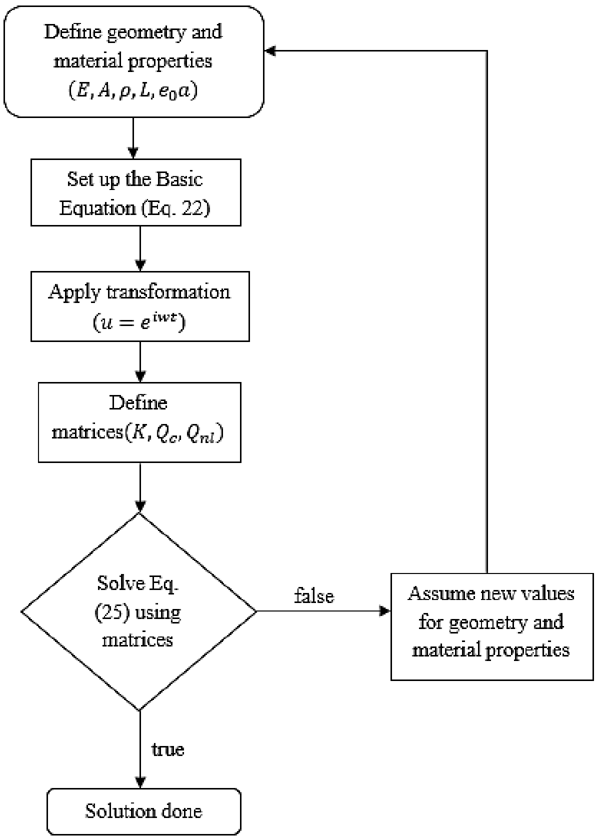

Fig. 2. Flowchart of the solution process

3. Concluding Remarks

In this paper, the derivation of the governing equation for axial vibrations of rods with ring cross-section using the theory of nonlocal elasticity and its relationship with nano dimensions is investigated. The governing equation is derived through the application of specific transformations, leading to an equation that provides the frequency and mode shapes dependent on the boundary conditions. Furthermore, by using the weighted residual method, a nonlocal finite element formulation for the bar is comprehensively developed and presented. The equations obtained clearly reveal the effect of the size effect parameter on axial vibration.

Author Contribution

Aleyna Yazıcıoglu: Conducted a literature review, Conceived and designed the analysis, Wrote the paper, Prepared the paper for publication.

Ömer Civalek: Conducted a literature review, Designed and directed the paper, Verified the theories and methods, Prepared the paper for publication.

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