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Research Article

On non-Newtonian Helices in Multiplicative Euclidean Space

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Abstract: In this article, spherical indicatrices of a curve and helices are re-examined using both the algebraic structure and the geometric structure of non-Newtonian (multiplicative) Euclidean space. Indicatrices of a multiplicative curve on the multiplicative sphere in multiplicative space are obtained. In addition, multiplicative general helix, multiplicative slant helix and multiplicative clad and multiplicative g-clad helix characterizations are provided. Finally, examples and drawings are given.

Keywords: Non-Newtonian calculus, spherical indicatrices, helices, multiplicative differential geometry.

1. Introduction

Classical analysis, which is a widely used mathematical theory today, was defined by Gottfried Leibniz and Isaac Newton in the second half of the 17th century based on the concepts of derivatives and integrals. Constructed upon algebra, trigonometry, and analytic geometry, the classical analysis consists of concepts such as limits, derivatives, integrals, and series. These concepts are regarded as simple versions of addition and subtraction, leading to the designation of this analysis as summational analysis. Classical analysis finds applications in various fields, including natural sciences, computer science, statistics, engineering, economics, business, and medicine, where mathematical modeling is required, and optimal solution methods are sought. However, there are situations in some mathematical models where classical analysis falls short. Therefore, alternative analyses have been defined based on different arithmetic operations while building upon classical analysis. For instance, in 1887, Volterra developed an approach known as Volterra-type analysis or multiplicative analysis since it is founded on the multiplication operation [30]. In multiplicative analysis, the roles of addition and subtraction operations in classical analysis are assumed by the multiplication and division operations, respectively. Following the definition of Volterra analysis, Grossman and Katz conducted some new studies between 1972 and 1983. This led to the development of the non-Newtonian analysis, which also involves fundamental definitions

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and concepts [14, 15]. These analyses have been referred to as geometric analysis, bigeometric analysis, and anageometric analysis. Multiplicative analysis has emerged as an alternative approach to classical analysis and has become a significant area of research and development in the field of mathematics. These new analyses may allow for a more effective resolution of various problems by examining different mathematical structures. Furthermore, these studies contribute to the expansion of the boundaries of mathematical analysis and find applications in various disciplines.

Arithmetic is an integer field which is a subset of the real numbers. An arithmetic system is the structure obtained by algebraic operations defined in this field. In fact, this field can be considered as a different interpretation of the real number field such that a countable number of infinitely ordered objects can be formed and these structures are equivalent or isomorphic to each other. The generator function, which is used to create arithmetic systems, is a one-to-one and bijective transformation whose domain is real numbers and whose value set is a subset of positive real numbers. The unit function I and the function e^x are examples of generator functions. Just as each generator produces a single arithmetic, each arithmetic can be produced with the help of a single generator. Multiplicative analysis has its own multiplicative space. In this special space, the classical number system has turned into a multiplicative number system consisting of positive real numbers, denoted by $\mathbb{R}_*$. Likewise, the basic mathematical operations in classical analysis have also turned into their purely multiplicative versions. This is clearly shown in the table below.

Table 1. Basic multiplicative operations

$a +_* b$	$e^{\log a + \log b}$	ab
$a -_* b$	$e^{\log a - \log b}$	$\frac{a}{b}$
$a \cdot_* b$	$e^{\log a \log b}$	$a^{\log b}$
$a /_* b$	$e^{\log a / \log b}$	$a^{\frac{1}{\log b}},\ b \neq 1$

Multiplicative analysis, contrast to not a completely new topic, has recently started to be explored and discovered more in today's context. The main reason behind this lies in the successful modeling of problems that cannot be addressed using classical analysis, achieved through the application of multiplicative analysis. This characteristic has led many mathematicians to prefer multiplicative analysis for solving challenging problems that are otherwise difficult to model within their respective fields. Stanley took the lead in this regard and re-announced geometric analysis as multiplicative analysis [27]. On this subject, fractal growths of fatigue defects in materials are studied by Rybaczuk and Stoppel [24] and the physical and fractional dimension concepts are studied by Rybaczuk and Zielinski [25]. In addition, there are many studies on multiplicative analysis in the field of pure mathematics. For example, the non-Newtonian efforts in complex analysis are [6, 29], in numerical analysis [1, 7, 34], in differential equations [5, 31, 33].

Also, Bashirov et al. reconsider multiplicative analysis with some basic definitions, theorems, propositions, properties and examples [4]. The multiplicative Dirac system and multiplicative time scale are studied by Emrah et al. [13, 16].

Georgiev brought a completely different perspective to multiplicative analysis with the books titled *Multiplicative Differential Calculus*, *Multiplicative Differential Geometry* and *Multiplicative Analytic Geometry* published in 2022 [10–12]. Unlike previous studies, Georgiev used operations as purely multiplicative operations and almost reconstructed the multiplicative space. These books have been recorded as the initial studies in particular for multiplicative geometry. Georgiev's book [10] serves as a guide for researchers in this field by encompassing numerous fundamental definitions and theorems pertaining to curves, surfaces, and manifolds. The book elucidates how to associate basic geometric objects such as curves, surfaces, and manifolds with multiplicative analysis, shedding light on their properties in multiplicative spaces. Additionally, it emphasizes the connections between multiplicative geometry and other mathematical domains, making it a valuable resource for researchers working in various branches of mathematics. Afterward, Nurkan et al. tried to construct geometry with geometric calculus. In addition, Gram-Schmidt vectors are obtained [23]. On the other hand, Aydın et al. studied rectifying curves in multiplicative Euclidean space. The multiplicative rectifying curves are fully classified and visualized through multiplicative spherical curves and they studied multiplicative submanifolds and of multiplicative Euclidean space [2, 3]. Has and Yılmaz constructed multiplicative conics using multiplicative arguments [17] and in another study they investigated multiplicative magnetic curves [18]. Has , Yılmaz and Yıldırım have worked on the multiplicative Lorentz-Minkowski space [19]. Ceyhan et al. performed optical fiber examined with multiplicative quaternions [8].

A helix curve is the curve that a point follows as it rotates around a fixed axis in a three-dimensional space. The helix curve is formed as a result of this rotational movement, and the rotation time around the axis determines the stability of the curve. While the helix curve is important in terms of geometry, it is also increasing in different branches of science. For example, helix is a term used for the connections of DNA. The double helix structure of DNA is called an image helix [32]. In computer graphics and 3D applications, helix curves are used in sections of complex surfaces and their results [9]. The helix is used in blades and aerospace engineering for the design and performance analysis of propellers and rotor blades [26]. In addition, helices have been traditionally studied by many researchers with their different properties [20–22, 28, 35].

In this study, spherical indicatrices and helix curves, which are important for differential geometry, are examined in multiplicative space. Spherical indicatrices, general helix, slant helix, clad helix and g-clad helix are rearranged with reference to multiplicative operations. Moreover, in the multiplicative Euclidean space, basic concepts such as orthogonal vectors, orthogonal system, curves, Frenet frame, etc. are mentioned. In addition, it is aimed to make these basic concepts more memorable by visualizing them.

2. Multiplicative Calculus and Multiplicative Space

The definitions and theorems that will be presented in this section are taken from the works of Georgiev [10–12].

Since the multiplicative space has an exponential structure, the sets of multiplicative real numbers are we have

$$\mathbb{R}_* = \{e^x : x \in \mathbb{R}\} = \mathbb{R}^+, \quad \mathbb{R}_*^+ = \{e^x : x \in \mathbb{R}^+\} = (1, \infty) \text{ and } \mathbb{R}_*^- = \{e^x : x \in \mathbb{R}\} = (0, 1). \tag{1}$$

The basic multiplicative operations for all $m, n \in \mathbb{R}_*$, are

$$m +_* n = e^{\log m + \log n} = mn, \qquad m -_* n = e^{\log m - \log n} = m/n,$$

$$m \cdot_* n = e^{\log m \log n} = m^{\log n}, \qquad m/_* n = e^{\log m / \log n} = m^{\frac{1}{\log n}}, \ n \neq 1.$$

According to the multiplicative addition operation, the multiplicative neutral and unit element are $0_* = 1$ and $1_* = e$, respectively.

The inverse elements of multiplicative addition and multiplicative multiplication operations for all $m \in \mathbb{R}_*$ are as follows, respectively:

$$-_* m = 1/m, \quad m^{-1_*} = e^{\frac{1}{\log m}}.$$

Absolute value function in multiplicative space, we have

$$|m|_* = \begin{cases} m, & m \geq 0_* \\ -_* m, & m < 0_*. \end{cases}$$

With the help of multiplicative arguments, the multiplicative power function can be given as for all $m \in \mathbb{R}_*$ and $k \in \mathbb{N}$

$$m^{k_*} = e^{(\log m)^k}, \quad m^{\frac{1}{2}_*} = \sqrt[*]{m} = e^{\sqrt{\log m}}.$$

A vector whose components are elements of the space $\mathbb{R}_*$ is called a multiplicative vector and satisfies the following properties $\vec{\mathbf{r}} = (r_1, r_2, \cdots, r_n), \vec{\mathbf{s}} = (s_1, s_2, \cdots, s_n) \in \mathbb{R}_*^n$ multiplicative vectors and $\lambda \in \mathbb{R}_*$, as follows

$$\begin{aligned} \vec{\mathbf{r}} +_* \vec{\mathbf{s}} &= (r_1 +_* s_1, \cdots, r_n +_* s_n) = (r_1 s_1, \cdots, r_n s_n), \\ \lambda \cdot_* \vec{\mathbf{r}} &= (\lambda \cdot_* r_1, \cdots, \lambda \cdot_* r_n) = (r_1^{\log \lambda}, \cdots, r_n^{\log \lambda}) = e^{\log \vec{\mathbf{r}} \log \lambda}, \end{aligned}$$

where $\log \overrightarrow{\mathbf{r}} = (\log r_1, \log r_2, \cdots, \log r_n)$. Let $\overrightarrow{\mathbf{r}} = (r_1, r_2, \cdots, r_n)$ and $\overrightarrow{\mathbf{s}} = (s_1, s_2, \cdots, s_n) \in \mathbb{R}_*^n$ be two multiplicative vectors in the multiplicative vector space $\mathbb{R}_*^n$. Thus the multiplicative inner product of two multiplicative vectors is follow

$$\langle \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{s}} \rangle_* = r_1 \cdot_* s_1 +_* \cdots +_* r_n \cdot_* s_n = e^{\langle \log \overrightarrow{\mathbf{r}}, \log \overrightarrow{\mathbf{s}} \rangle}.$$

If the multiplicative vectors $\overrightarrow{\mathbf{r}}$ and $\overrightarrow{\mathbf{s}}$ are multiplicative orthogonal to each other, they are denoted by $\overrightarrow{\mathbf{r}} \perp_* \overrightarrow{\mathbf{s}}$ and this relation is as follows

$$\langle \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{s}} \rangle_* = 0_*.$$

In Figure 1, we present the graph of the multiplicative orthogonal vectors.

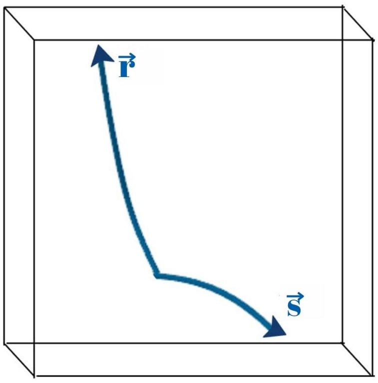

Figure 1: Multiplicative orthogonal vectors $\overrightarrow{\mathbf{r}} = (e^{\frac{1}{2}}, e^{-\frac{3}{4}}, e^{\frac{3}{2}})$ and $\overrightarrow{\mathbf{s}} = (e^{\frac{3}{4}}, e, e^{\frac{1}{4}})$

The multiplicative norm of the multiplicative vector $\overrightarrow{\mathbf{r}} \in \mathbb{R}_*^n$ is given by the multiplicative inner product is defined as follows:

$$\| \overrightarrow{\mathbf{r}} \|_* = e^{\langle \log \overrightarrow{\mathbf{r}}, \log \overrightarrow{\mathbf{r}} \rangle^{\frac{1}{2}}}.$$

Let $\overrightarrow{\mathbf{r}} = (r_1, r_2, r_3)$ and $\overrightarrow{\mathbf{s}} = (s_1, s_2, s_3)$ be 3D multiplicative vectors, and the multiplicative cross products of $\overrightarrow{\mathbf{r}}$ and $\overrightarrow{\mathbf{s}}$, we have

$$\overrightarrow{\mathbf{r}} \times_* \overrightarrow{\mathbf{s}} = (e^{\log r_2 \log s_3 - \log r_3 \log s_2}, e^{\log r_3 \log s_1 - \log r_1 \log s_3}, e^{\log r_1 \log s_2 - \log r_2 \log s_1}).$$

Multiplicative cross product preserves the properties of traditional cross product with its arguments. For example, cross products of multiplicative vectors $\overrightarrow{\mathbf{r}}$ and $\overrightarrow{\mathbf{s}}$ are multiplicative orthogonal to both $\overrightarrow{\mathbf{r}}$ and $\overrightarrow{\mathbf{s}}$. We give this visually in Figure 2 The multiplicative angle between the multiplicative unit direction vectors $\overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{s}} \in \mathbb{R}_*^n$ is given by

$$\phi = \arccos_* (e^{\langle \log \overrightarrow{\mathbf{r}}, \log \overrightarrow{\mathbf{s}} \rangle}).$$

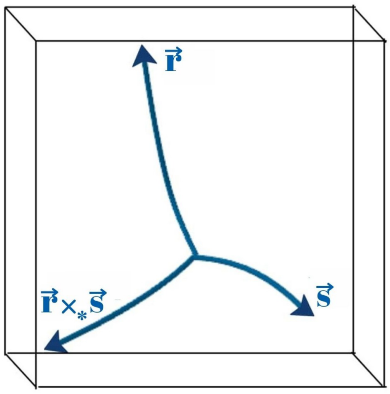

Figure 2: Multiplicative orthogonal system $\overrightarrow{\mathbf{r}} = (e^{\frac{1}{2}}, e^{-\frac{3}{4}}, e^{\frac{3}{2}})$, $\overrightarrow{\mathbf{s}} = (e^{\frac{3}{4}}, e, e^{\frac{1}{4}})$ and $\overrightarrow{\mathbf{r}} \times_* \overrightarrow{\mathbf{s}} = (e^{-\frac{27}{16}}, e, e^{\frac{17}{16}})$

Multiplicative trigonometric functions with the help of multiplicative angles

$$\sin_* \phi = e^{\sin \log \phi}, \quad \cos_* \phi = e^{\cos \log \phi},$$

$$\tan_* \phi = e^{\tan \log \phi}, \quad \cot_* \phi = e^{\cot \log \phi}.$$

Multiplicative trigonometric functions provide the same algebraic properties as traditional trigonometric functions, but with their own arguments. For example, there is the equality $\sin_*^{2_*} \theta +_* \cos_*^{2_*} \theta = 1_*$. For other relations, see [11].

The multiplicative derivative of the multiplicative function $f(t) \subset \mathbb{R}_*$ for $t \in I \subset \mathbb{R}_*$ is as follows

$$\begin{aligned} f^*(t) &= \lim_{h \to 0_*} \left(\left(f(t +_* h) -_* f(t) \right) /_* h \right) \\ &= \lim_{h \to 1} \exp \left[\frac{\log f(th) - \log f(t)}{\log(h)} \right] \\ &= \lim_{h \to 1} \exp \left[\frac{thf'(th)}{f(th)} \right] \\ &= e^{t \frac{f'(t)}{f(t)}}. \end{aligned}$$

Multiplicative differentiation realizes many properties provided in classical differentiation, such as linearity, Leibniz rule, chain rules, etc., based on multiplicative arguments. For examples $(f(x) \cdot_* g(x))^* = f^*(x) \cdot_* g(x) +_* g^*(x) \cdot_* f(x)$. It can also be stated as $f^*(x) = d_* f /_* d_* x$. For other relations, see [11].

The multiplicative integral of the multiplicative function $f(t) \subset \mathbb{R}_*$ is as follows for $t \in I \subset \mathbb{R}_*$

$$\int_* f(x) \cdot_* d_* x = e^{\int \frac{1}{x} \log f(x) dx}, \quad x \in \mathbb{R}_*.$$

The geometric location of points with equal multiplicative distances from a point in multiplicative space is called a multiplicative sphere. The equation of the sphere with centered at $C(a,b,c)$ and radius r is

$$\|P -_* C\|_* = r,$$

where $P = (x,y,z)$ is the representation point of the multiplicative sphere, so

$$e^{(\log x - \log a)^2 + (\log y - \log b)^2 + (\log z - \log c)^2} = e^{(\log r)^2}.$$

In Figure 3 we show the multiplicative sphere with centered at multiplicative origin $O(0_*,0_*,0_*)$ and radius 1_*

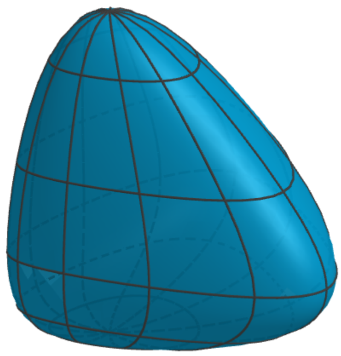

Figure 3: A multiplicative sphere with centered at multiplicative origin $O(0_*,0_*,0_*)$ and radius 1_*

3. Differential Geometry of Curves in Multiplicative Space

A multiplicative parametrization of class C_*^k $(k \geq 1_*)$ for a curve $\mathbf{x}$ in $\mathbb{R}_*^3$ (i.e., the component-functions of $\mathbf{x}$ are k-times continuously multiplicative differentiable), is a multiplicative vector valued function $\mathbf{x} : I \subset \mathbb{R}_* \to \mathbb{E}_*^3$, where s is mapped to $\mathbf{x}(s) = (x_1(s), x_2(s), x_3(s))$. In particular, a parametric multiplicative curve $\mathbf{x}$ is regular if and only if $\|\mathbf{x}^*(s)\|_* \neq 0_*$ for any $s \in I$. Looking at it dynamically, the multiplicative vector $\mathbf{x}^*(s)$ represents the multiplicative velocity of the multiplicative curve at time s. For a multiplicative curve $\mathbf{x}$ to have multiplicative naturally parameters, the necessary and sufficient condition is that the curve is from the class C_*^k and $\|\mathbf{x}^*(s)\|_* = 1_*$ for each $s \in I$.

Given $s_0 \in I$, the multiplicative arc length of a multiplicative regular parameterized curve $\mathbf{x}(s)$ from the point s_0, is by definition

$$h(s) = \int_{*s_0}^{s} \|\mathbf{x}^*(t)\|_* \cdot_* d_*t. \tag{2}$$

As an example, the multiplicative circle curve in multiplicative plane with center $(0_*, 0_*, 0_*)$ and radius $r = e^{-2}$ is given by the equation $\mathbf{x}(s) = e^{-2} \cdot_* (e^{\frac{1}{2}} \cos_* 2s, e^{\frac{1}{2}} \cdot_* \sin_* 2s, e^{\sqrt{3}})$ in $\mathbb{R}^3_*$. It can be plotted as in Figure 4.

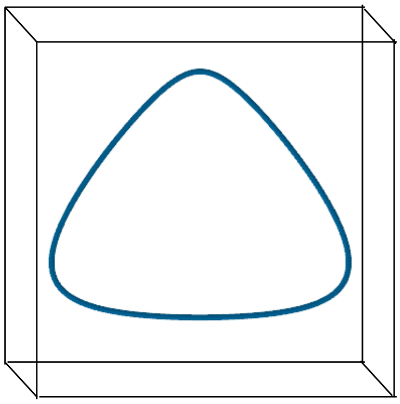

Figure 4: A multiplicative circle in the plane $z = e^{\frac{\sqrt{3}}{2}}$ with centered at $(0_*, 0_*, 0_*)$, radius $r = 1/e^2$ and $0_* < s < e^{2\pi}$

The multiplicative Frenet trihedron of a naturally parameterized multiplicative curve $\mathbf{x}(s)$ are

$$\mathbf{t}(s) = \mathbf{x}^*(s), \qquad \mathbf{n}(s) = \mathbf{x}^{**}(s) /_* \|\mathbf{x}^{**}(s)\|_*, \qquad \mathbf{b}(s) = \mathbf{t}(s) \times_* \mathbf{n}(s).$$

The vector field $\mathbf{t}(s)$ (resp. $\mathbf{n}(s)$ and $\mathbf{b}(s)$) along $\mathbf{x}(s)$ is said to be multiplicative tangent (resp. multiplicative principal normal and multiplicative binormal). It is direct to prove that $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ is mutually multiplicative orthogonal and $\mathbf{n}(s) \times_* \mathbf{b}(s) = \mathbf{t}(s)$ and $\mathbf{b}(s) \times_* \mathbf{t}(s) = \mathbf{n}(s)$. We also point out that the arc length parameter and multiplicative Frenet frame are independent from the choice of multiplicative parametrization [10].

To give an example, the multiplicative Frenet vectors of the multiplicative curve

$$\mathbf{x}(s) = \left((e^3 /_* e^5) \cdot_* \cos_* s, (e^3 /_* e^5) \cdot_* \sin_* s, e^4 /_* e^5 \cdot_* e^s\right)$$

are

$$\begin{aligned}
\mathbf{t}(s) &= (-_*(e^3 /_* e^5) \cdot_* \sin_* s, (e^3 /_* e^5) \cdot_* \cos_* s, e^4 /_* e^5), \\
\mathbf{n}(s) &= (-_* \cos_* s, -_* \sin_* s, 0_*), \\
\mathbf{b}(s) &= ((e^4 /_* e^5) \cdot_* \sin_* s, -_*(e^4 /_* e^5) \cdot_* \cos_* s, e^3 /_* e^5).
\end{aligned}$$

In Figure 5, we present the graph of the multiplicative Frenet frame on the multiplicative curve $\mathbf{x}(s)$.

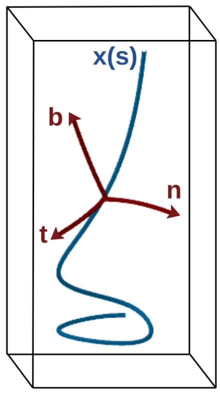

Figure 5: Multiplicative curve and its multiplicative Frenet frame

The multiplicative Frenet formulae of $\mathbf{x}$ are given by

$$\begin{aligned} \mathbf{t}^* &= \kappa \cdot_* \mathbf{n}, \\ \mathbf{n}^* &= -_* \kappa \cdot_* \mathbf{t} +_* \tau \cdot_* \mathbf{b}, \\ \mathbf{b}^* &= -_* \tau \cdot_* \mathbf{n}, \end{aligned}$$

where $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are the curvature and the torsion functions of $\mathbf{x}$, calculated by

$$\kappa(s) = \|\mathbf{x}^{**}(s)\|_* = e^{\langle \log \mathbf{x}^{**}, \log \mathbf{x}^{**} \rangle^{\frac{1}{2}}}, \tag{3}$$

$$\tau(s) = \langle \mathbf{n}^*(s), \mathbf{b}(s) \rangle_* = e^{\langle \log \mathbf{n}^*(s), \log \mathbf{b}(s) \rangle}. \tag{4}$$

4. Main Results

4.1. Multiplicative Spherical Indicatries

Consider a multiplicative curve $\mathbf{x}(s) \in \mathbb{R}^3_*$. The multiplicative Frenet vectors of $\mathbf{x}$ also evolve along the curve as a multiplicative vector field. The thing to note here is that since the multiplicative Frenet vectors of the multiplicative curve $\mathbf{x}$ are multiplicative unit vectors, they form a curve on the multiplicative sphere. In this section, such curves will be examined.

The multiplicative curve $\mathbf{x}(s)$ is associated with multiplicative Frenet vectors $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$. Now, let us consider the unit multiplicative tangent vectors along $\mathbf{x}(s)$. These vectors collectively form another curve, denoted by $\mathbf{x}_t = \mathbf{t}$. This new curve resides on the surface of a multiplicative sphere with a radius of 1_* and centered at the multiplicative origin $O = (0_*, 0_*, 0_*)$. The multiplicative curve $\mathbf{x}_t$ is often referred to as the multiplicative spherical indicatrix associated with the unit multiplicative tangent vector $\mathbf{t}$. We will call this curve multiplicative tangent indicatrix of the original multiplicative curve $\mathbf{x}$, in line with the more conventional notation. With similar thought, we will call multiplicative curves $\mathbf{x}_n = \mathbf{n}$ and $\mathbf{x}_b = \mathbf{b}$ as multiplicative normal indicatrix and multiplicative binormal indicatrix of $\mathbf{x}$, respectively.

Proposition 4.1 *Let $\mathbf{x}_t$ be the multiplicative tangent indicatrix of a multiplicative naturally parameterized curve $\mathbf{x}$. The multiplicative naturally parameter s_t of $\mathbf{x}_t$ is given by*

$$s_t = e^{\int^s \frac{1}{u}\kappa(u)du},$$

where s is multiplicative naturally parameter of $\mathbf{x}$ and $\kappa(s)$ is multiplicative curvature of $\mathbf{x}$.

Proof Let $\mathbf{x}(s)$ be the multiplicative naturally parameterized curve. Also, let $\mathbf{x}_t(s) = \mathbf{t}(s)$ be the multiplicative tangent indicatrix of $\mathbf{x}(s)$. Considering (2), we get the multiplicative naturally parameter of $\mathbf{x}_t$ as follows

$$s_t = \int_*^s \|\mathbf{t}^*(u)\|_* \cdot_* d_* u.$$

Then from the definition of multiplicative curvature, we get

$$s_t = \int_*^s \kappa(u) \cdot_* d_* u$$

or equivalently

$$s_t = e^{\int^s \frac{1}{u}\kappa(u)du}.$$

□

Theorem 4.2 *Let $\mathbf{x}_t$ be the multiplicative tangent indicatrix of a multiplicative naturally parameterized curve $\mathbf{x}$ with $\kappa \neq 0_*$ on I. The multiplicative Frenet vectors $\{T_t, N_t, B_t\}$ of $\mathbf{x}_t$ satisfy*

$$\begin{aligned} T_t &= \mathbf{n}, \\ N_t &= (-_*\mathbf{t} +_* f \cdot_* \mathbf{b})/_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}, \\ B_t &= (f \cdot_* \mathbf{t} +_* \mathbf{b})/_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}, \end{aligned}$$

where $f = f(s)$ and $f = \tau/_*\kappa$.

Proof Since $\mathbf{x}_t$ is the multiplicative tangent indicatrix of $\mathbf{x}$, we have

$$\mathbf{x}_t = \mathbf{t}.$$

Taking the multiplicative derivative of both sides of the above equation with respect to s,

$$(d_*\mathbf{x}_t/_* d_* s_t) \cdot_* (d_* s_t/_* d_* s) = \kappa \cdot_* \mathbf{n}.$$

Using Proposition 4.1 and putting $T_t = d_* \mathbf{x}_t /_* d_* s_t$, we get

$$T_t = \mathbf{n}. \tag{5}$$

If we take the multiplicative derivative of (5) with respect to s and apply multiplicative Frenet formulas, we have

$$\kappa \cdot_* (d_* T_t /_* d_* s_t) = -_* \kappa \cdot_* \mathbf{t} +_* \tau \cdot_* \mathbf{b}$$

and

$$d_* T_t /_* d_* s_t = -_* \mathbf{t} +_* (\tau /_* \kappa) \cdot_* \mathbf{b}.$$

Considering the multiplicative norm, the following equation is obtained:

$$\begin{aligned} \| d_* T_t /_* d_* s_t \|_* &= e^{(\langle -\log \mathbf{t}, -\log \mathbf{t} \rangle + (\frac{\log \tau}{\log \kappa})^2 \langle \log \mathbf{b}, \log \mathbf{b} \rangle)^{\frac{1}{2}}} \\ &= e^{\sqrt{(1+(\frac{\log \tau}{\log \kappa})^2)}}. \end{aligned} \tag{6}$$

In that case, we can see that

$$N_t = (d_* T_t /_* d_* s_t) /_* \| d_* T_t /_* d_* s_t \|_* = (-_* \mathbf{t} +_* (\tau /_* \kappa) \cdot_* \mathbf{b}) /_* e^{(1+(\frac{\log \tau}{\log \kappa})^2)^{\frac{1}{2}}}.$$

Setting $\tau /_* \kappa = f$,

$$N_t = (-_* \mathbf{t} +_* f \cdot_* \mathbf{b}) /_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}. \tag{7}$$

On the other hand, if we take into account (5) and (7) along with the multiplicative Frenet formulas, we obtain the final Frenet vector as

$$B_t = [\mathbf{n} \times_* (-_* \mathbf{t} +_* f \cdot_* \mathbf{b})] /_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}.$$

When we organize the multiplicative operations, we obtain

$$B_t = (f \cdot_* \mathbf{t} +_* \mathbf{b}) /_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}. \tag{8}$$

□

Proposition 4.3 *Let $\mathbf{x}_t$ be the multiplicative tangent indicatrix of a multiplicative naturally parameterized curve $\mathbf{x}$ with $\kappa \neq 0_*$ on I. The multiplicative curvatures of $\mathbf{x}_t$ are*

$$\kappa_t = e^{(1+(\log f(s))^2)^{\frac{1}{2}}} \quad \textit{and} \quad \tau_t = \sigma \cdot_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}},$$

where $\sigma = f^* /_* (e^{\log \kappa (1+(\log f(s))^2)^{\frac{3}{2}}})$.

Proof The first equality follows by (6),

$$\kappa_t = e^{\langle \log \mathbf{x}_t^{**}, \log \mathbf{x}_t^{**} \rangle^{\frac{1}{2}}} = e^{(1+(\log f(s))^2)^{\frac{1}{2}}}.$$

Next considering calculate the multiplicative derivative of N_t with respect to s_t, considering (7). Also, is chosen $e^{(1+(\log f(s))^2)^{\frac{1}{2}}} = \lambda$ in (7), so

$$(d_* N_t /_* d_* s_t) \cdot_* (d_* s_t /_* d_* s) = [(-_* \kappa \cdot_* \mathbf{n} +_* f^* \cdot_* \mathbf{b} -_* f \cdot_* \tau \cdot_* \mathbf{n}) \cdot_* \lambda -_* \lambda \cdot_* (-_* \mathbf{t} +_* f \cdot \mathbf{b})] /_* \lambda^{2*}.$$

After this we can write

$$N_t^* \cdot_* \kappa = [\lambda^* \cdot_* \mathbf{t} -_* \lambda \cdot_* (\kappa +_* f \cdot_* \tau) \cdot_* \mathbf{n} +_* (\lambda \cdot_* f^* -_* \lambda^* \cdot_* f) \cdot_* \mathbf{b}] /_* \lambda^{2*}$$

and so

$$N_t^* = [\lambda^* \cdot_* \mathbf{t} -_* \lambda \cdot_* (\kappa +_* f \cdot_* \tau) \cdot_* \mathbf{n} +_* (\lambda \cdot_* f^* -_* \lambda^* \cdot_* f) \cdot_* \mathbf{b}] /_* \lambda^{2*} \cdot_* \kappa. \tag{9}$$

Then from (8) and (9), we obtain

$$\begin{aligned} \tau_t = \langle \log N_t^*, \log B_t \rangle_* &= (f \cdot_* \lambda^*) /_* \lambda^{3*} \cdot_* \kappa +_* (\lambda \cdot_* f^* -_* \lambda^* \cdot_* f) /_* \lambda^{3*} \cdot_* \kappa \\ &= (f^* \cdot_* \lambda) /_* \lambda^{3*} \cdot_* \kappa. \end{aligned}$$

Here again let's consider the choice $e^{(1+(\log f(s))^2)^{\frac{1}{2}}} = \lambda$, so we get

$$[f^* /_* (e^{(1+(\log f(s))^2)^{\frac{3}{2}}} \cdot_* \kappa)] \cdot_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}.$$

Finally, if a choice is made in the form $\sigma = f^* /_* (e^{\log \kappa (1+(\log f(s))^2)^{\frac{3}{2}}} \cdot_* \kappa)$, the above-mentioned equation becomes

$$\tau_t = \sigma \cdot_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}.$$

□

Using similar arguments, we may have the following results.

Proposition 4.4 *Let* $\mathbf{x}_n$ *be the multiplicative normal indicatrix of a multiplicative naturally parameterized curve* $\mathbf{x}$. *The multiplicative arc parameter* s_n *of the multiplicative curve* $\mathbf{x}_n$ *provides*

$$s_n = \int_* \kappa(s) \cdot_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}} \cdot_* d_* s, \tag{10}$$

where $f = f(s)$ *and* $f = \tau /_* \kappa$.

Theorem 4.5 *Let* $\mathbf{x}_n$ *be the multiplicative normal indicatrix of a multiplicative naturally parameterized curve* $\mathbf{x}$*. The multiplicative Frenet vectors* $\{T_n, N_n, B_n\}$ *of* $\mathbf{x}_n$ *as follows*

$$\begin{aligned} T_n &= (-_*\mathbf{t} +_* f \cdot_* \mathbf{b})/_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}, \\ N_n &= (\sigma/_* e^{(1+(\log \sigma(s))^2)^{\frac{1}{2}}}) \cdot_* [((f \cdot_* \mathbf{t} +_* \mathbf{b})/_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}) -_* \mathbf{n}/_* \sigma], \\ B_n &= (e/_* e^{(1+(\log \sigma(s))^2)^{\frac{1}{2}}}) \cdot_* [((f \cdot_* \mathbf{t} +_* \mathbf{b})/_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}} -_* \mathbf{n} \cdot_* \sigma], \end{aligned}$$

where $\sigma = f^* /_* (e^{\log \kappa (1+(\log f(s))^2)^{\frac{3}{2}}})$.

Proposition 4.6 *Let* $\mathbf{x}_n$ *be the multiplicative normal indicatrix of the multiplicative curve* $\mathbf{x}$*. The multiplicative curvatures of the normal indicatrix* $\mathbf{x}_n$ *are described as follows*

$$\kappa_n = e^{(1+(\log \sigma(s))^2)} \quad and \quad \tau_n = \Gamma \cdot_* e^{(1+(\log f(s))^2)}, \tag{11}$$

where $\Gamma = \sigma^* /_* (e^{\log \kappa (1+(\log f(s))^2)(1+(\log \sigma(s))^2)^{\frac{3}{2}}}$.

Proposition 4.7 *Let* $\mathbf{x}_b$ *be the multiplicative binormal indicatrix of a multiplicative naturally parameterized curve* $\mathbf{x}$*. The multiplicative arc parameter* s_b *of the multiplicative curve* $\mathbf{x}_b$ *provides*

$$s_b = \int_* \tau(s) d_* s.$$

Theorem 4.8 *Let* $\mathbf{x}_b$ *be the multiplicative binormal indicatrix of a multiplicative naturally parameterized curve* $\mathbf{x}$*. The multiplicative Frenet vectors* $\{T_b, N_b, B_b\}$ *of* $\mathbf{x}_b$ *satisfy*

$$\begin{aligned} T_b &= -_*\mathbf{n}, \\ N_b &= (\mathbf{t} -_* f \cdot_* \mathbf{b})/_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}, \\ B_b &= (f \cdot_* \mathbf{t} +_* \mathbf{b})/_* e^{(1+(\log f(s))^2)^{\frac{1}{2}}}, \end{aligned}$$

where $f = f(s)$ *and* $f = \tau /_* \kappa$.

Proposition 4.9 *Let the multiplicative curve, denoted as* $\mathbf{x}_b$*, be the binormal indicatrix of the multiplicative curve* $\mathbf{x}$*. Then, the multiplicative curvatures of the* $\mathbf{x}_b$ *are described as follows*

$$\kappa_b = e^{(1+(\log f(s))^2)^{\frac{1}{2}}} /_* f \quad and \quad \tau_b = (-_* \sigma \cdot_* e^{(1+(\log \sigma(s))^2)^{\frac{1}{2}}}) /_* f,$$

where $\sigma = f^* /_* (\kappa \cdot_* (e^{1+(\log f(s))^2})^{\frac{3}{2}*})$.

Example 4.10 *Let* $\mathbf{x} : I \subset \mathbb{R}_* \to \mathbb{E}^3_*$ *be multiplicative naturally parametrized curve in* $\mathbb{R}^3_*$ *parameterized by*

$$\mathbf{x}(s) = \left(e^s, e^{\frac{e^2}{2}}, e^{\frac{e^3}{6}}\right).$$

In Figure 6, we present the graph of the multiplicative spherical indicatrices of $\mathbf{x}$.

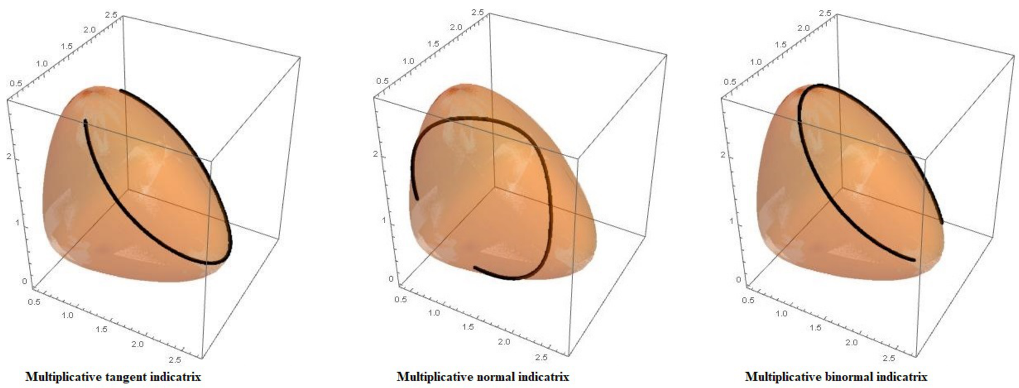

Figure 6: Multiplicative spherical indicatrices

4.2. Multiplicative Helices

Definition 4.11 *Consider the multiplicative curve* $\mathbf{x} : I \subset \mathbb{R}_* \to \mathbb{E}^3_*$ *with* $\kappa \neq 0_*$. *If the multiplicative tangent vector field of the curve* $\mathbf{x}$ *makes a constant multiplicative angle with a constant multiplicative vector, then the curve* $\mathbf{x}$ *is referred to as a multiplicative general helix [10].*

Theorem 4.12 *Let* $\mathbf{x} : I \subset \mathbb{R}_* \to \mathbb{E}^3_*$ *be a multiplicative curve with* $\kappa \neq 0_*$. *The multiplicative space curve* $\mathbf{x}$ *is a multiplicative general helix if and only if the multiplicative ratio of multiplicative torsion and multiplicative curvature is constant. In other words, it is*

$$\kappa /_* \tau = c, \quad c \in \mathbb{R}_*.$$

Proof The proof of the theorem is explained by Georgiev (see [10]). □

Example 4.13 *Let* $\mathbf{x} : I \subset \mathbb{R}_* \to \mathbb{E}^3_*$ *be multiplicative naturally parametrized general helix curve in* $\mathbb{R}^3_*$ *parameterized by*

$$\mathbf{x}(s) = \left(e^3 /_* e^5 \cdot_* \cos_* s, e^3 /_* e^5 \cdot_* \sin_* s, e^4 /_* e^5 \cdot_* e^s\right).$$

With the help of multiplicative curvature formulas from (4), we give

$$\kappa(s) = e^3 /_* e^5 \quad \text{and} \quad \tau = e^4 /_* e^5.$$

Since

$$\tau /_* \kappa = e^{(\log e^4 / \log e^5)/(\log e^3 / \log e^5)} = e^4 /_* e^3$$

is a multiplicative constant, $\mathbf{x}$ *is a multiplicative helix. In Figure 7, we present the graph of the multiplicative general helix.*

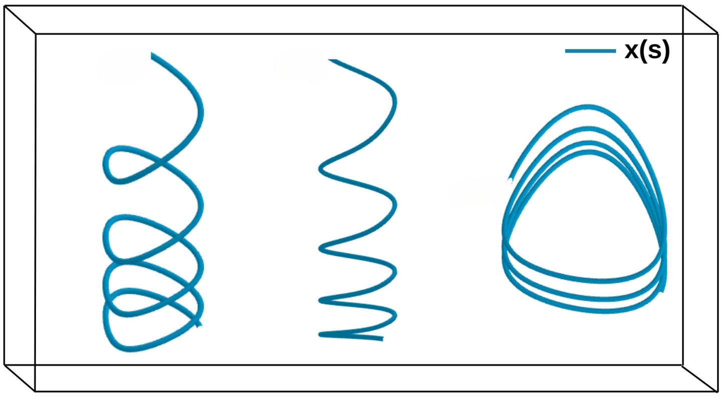

Figure 7: Multiplicative general helix

Definition 4.14 *Let* $\mathbf{x} : I \subset \mathbb{R}_* \to \mathbb{E}^3_*$ *be the multiplicative curve with* $\kappa \neq 0_*$. *If the multiplicative normal vector field of the curve* $\mathbf{x}$ *makes a constant multiplicative angle with a constant multiplicative vector, then the curve* $\mathbf{x}$ *is referred to as a multiplicative slant helix.*

Theorem 4.15 *Let* $\mathbf{x} : I \subset \mathbb{R}_* \to \mathbb{E}^3_*$ *be multiplicative curve with* $\kappa \neq 0_*$. *The multiplicative curve* $\mathbf{x}$ *is a multiplicative slant helix if and only if the following equality is a multiplicative constant function*

$$\sigma(s) = [\kappa^{2*}(s) /_* (\kappa^{2*}(s) +_* \tau^{2*}(s))^{\frac{3}{2}*}] \cdot_* (\tau(s) /_* \kappa(s))^*. \tag{12}$$

Proof Suppose that multiplicative naturally parametrized curve $s \mapsto \mathbf{x}(s)$ is a multiplicative slant helix. Since the multiplicative normal vector field of the multiplicative curve $\mathbf{x}$ makes a constant multiplicative angle with $\mathbf{v}$, which is a constant multiplicative vector, we have

$$\langle \mathbf{n}, \mathbf{v} \rangle_* = \cos_* \theta, \tag{13}$$

where θ constant multiplicative angle. Taking a multiplicative derivative of (13), we get

$$\langle \mathbf{n}^*, \mathbf{v} \rangle_* = 0_* \tag{14}$$

and

$$-_* \kappa \cdot_* \langle \mathbf{t}, \mathbf{v} \rangle_* +_* \tau \cdot_* \langle \mathbf{b}, \mathbf{v} \rangle_* = 0_*.$$

As can be seen from the elements of multiplicative Frenet frame and (13), there is a constant angle between $\mathbf{n}$ and fixed direction $\mathbf{v}$ and there is also a constant angle between $\mathbf{b}$ and fixed direction $\mathbf{v}$. Then the following equations are provided,

$$\langle \mathbf{t}, \mathbf{v} \rangle_* = (c \cdot_* \tau)/_* \kappa, \tag{15}$$

$$\langle \mathbf{b}, \mathbf{v} \rangle_* = c, \ c \in \mathbb{R}_*. \tag{16}$$

In terms of the multiplicative Frenet frame, we can write the decomposition for $\mathbf{v}$ as

$$\begin{aligned} \mathbf{v} &= e^{\langle \log \mathbf{t}, \log \mathbf{v} \rangle \log \mathbf{t} + \langle \log \mathbf{n}, \log \mathbf{v} \rangle \log \mathbf{n} + \langle \log \mathbf{b}, \log \mathbf{v} \rangle \log \mathbf{b}} \\ &= \langle \mathbf{t}, \mathbf{v} \rangle_* \cdot_* \mathbf{t} +_* \langle \mathbf{n}, \mathbf{v} \rangle_* \cdot_* \mathbf{n} +_* \langle \mathbf{b}, \mathbf{v} \rangle_* \cdot_* \mathbf{b}. \end{aligned}$$

The constant direction $\mathbf{v}$ from (13), (15) and (16) is obtained as follows

$$\mathbf{v} = (c \cdot_* \tau)/_* \kappa \cdot_* \mathbf{t} +_* \cos_* \theta \cdot_* \mathbf{n} +_* c \cdot_* \mathbf{b}. \tag{17}$$

Since $\mathbf{v}$ is the multiplicative unit vector, taking the multiplicative norm of both sides of the above equation, we get

$$\begin{aligned} e^{\langle \log \mathbf{v}, \log \mathbf{v} \rangle^{\frac{1}{2}}} &= ((c \cdot_* \tau)/_* \kappa)^{2*} \cdot_* e^{\langle \log \mathbf{t}, \log \mathbf{t} \rangle^{\frac{1}{2}}} +_* \cos_*^{2*} \theta \cdot_* e^{\langle \log \mathbf{n}, \log \mathbf{n} \rangle^{\frac{1}{2}}} \\ +_* & \ c^{2*} \cdot_* e^{\langle \log \mathbf{b}, \log \mathbf{b} \rangle^{\frac{1}{2}}} \end{aligned}$$

or

$$c^{2*} \cdot_* (\tau^{2*}/_* \kappa^{2*} +_* e) = \sin_*^{2*} \theta.$$

If the necessary algebraic operations are performed here, we obtain

$$c = (\kappa/_* (\kappa^{2*} +_* \tau^{2*})^{\frac{1}{2}*} \cdot_* \sin_* \theta.$$

Therefore, we can easily write $\mathbf{v}$ as

$$\mathbf{v} = \tau/_* (\kappa^{2*} +_* \tau^{2*})^{\frac{1}{2}*} \cdot_* sin_* \theta \cdot_* \mathbf{t} +_* \cos_* \theta \cdot_* \mathbf{n} +_* \kappa/_* (\kappa^{2*} +_* \tau^{2*})^{\frac{1}{2}*} \cdot_* sin_* \theta \cdot_* \mathbf{b}. \tag{18}$$

Take the multiplicative derivative of (14), we get

$$\langle \mathbf{n}^{**}, \mathbf{v} \rangle_* = 0_*. \tag{19}$$

From multiplicative Frenet frame and (18) and (19), we have

$$\begin{aligned} &\langle -_* \kappa^* \cdot_* \mathbf{t} -_* (\kappa^{2*} +_* \tau^{2*}) \cdot_* \mathbf{n} +_* \tau^* \mathbf{b}, \tau/_* (\kappa^{2*} +_* \tau^{2*})^{\frac{1}{2}*} \cdot_* sin_* \theta \cdot_* \mathbf{t} \\ &+_* \cos_* \theta \cdot_* \mathbf{n} +_* \kappa/_* (\kappa^{2*} +_* \tau^{2*})^{\frac{1}{2}*} \cdot_* sin_* \theta \cdot_* \mathbf{b} \rangle_* = 0_*. \end{aligned}$$

Here the following equation exists

$$(\kappa \cdot_* \tau^* - \tau \cdot_* \kappa^*)/_*(\kappa^{2*} +_* \tau^{2*})^{\frac{3}{2}*} \cdot_* \tan_* \theta + e = 0_*$$

and finally, we get

$$\tan_* \theta = (\kappa \cdot_* \tau^* - \tau \cdot_* \kappa^*)/_*(\kappa^{2*} +_* \tau^{2*})^{\frac{3}{2}*}.$$

Since the multiplicative angle θ is constant, after the necessary adjustments, we obtain that

$$\kappa^{2*}/_*(\kappa^{2*} +_* \tau^{2*})^{\frac{3}{2}*} \cdot_* (\tau/_*\kappa)^* = c, \quad c \in \mathbb{R}_*.$$

□

Example 4.16 *Let* $\mathbf{x} : I \subset \mathbb{R}_* \to \mathbb{E}_*^3$ *be multiplicative naturally parametrized slant helix curve in* $\mathbb{R}_*^3$ *as*

$$\mathbf{x}(s) = (x_1(s), x_2(s), x_3(s)),$$

where

$$\begin{aligned} x_1(s) &= e^9/_*e^{400} \cdot_* e^{\sin\log 25s} +_* e^{25}/_*e^{144} \cdot_* e^{\sin\log 9s}, \\ x_2(s) &= -_*e^9/_*e^{400} \cdot_* e^{\cos\log 25s} +_* e^{25}/_*e^{144} \cdot_* e^{\cos\log 9s}, \\ x_3(s) &= e^{15}/_*e^{136} \cdot_* e^{\sin\log 17s}. \end{aligned}$$

In Figure 8, we present the graph of the multiplicative slant helix.

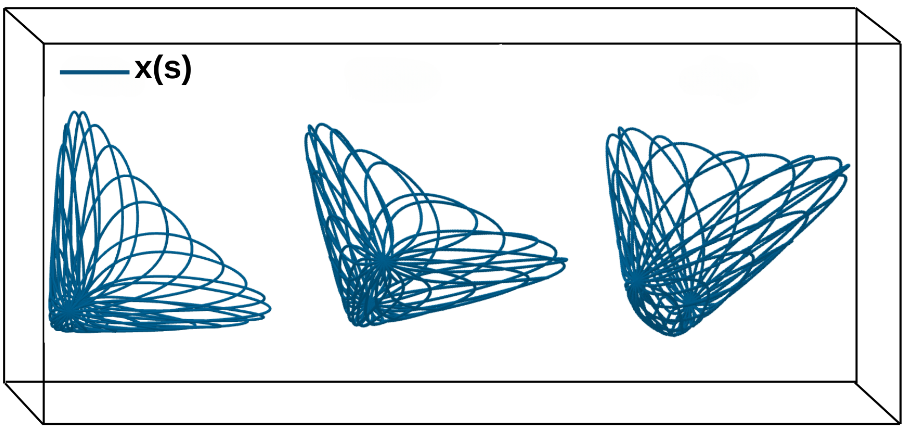

Figure 8: Multiplicative slant helix

Definition 4.17 *Let a regular multiplicative curve* $\mathbf{x}$ *be given in the multiplicative space with* $\kappa \neq 0_*$. *The multiplicative curve* $\mathbf{x}$ *is called the multiplicative clad helix if the multiplicative spherical image of the multiplicative principal normal vector* $\mathbf{n} : I \to \mathbb{S}^2_*$ *(*$\mathbb{S}^2_*$ *denotes the multiplicative sphere) of the curve* $\mathbf{x}$ *is part of the multiplicative cylindrical helix in* $\mathbb{S}^2_*$.

Therefore we remark that a multiplicative slant helix is a multiplicative clad helix. We have the following characterization of clad helices.

Theorem 4.18 *Let* $\mathbf{x}$ *be a multiplicative naturally parametrized curve with* $\kappa \neq 0_*$. *Then* $\mathbf{x}$ *is a multiplicative clad helix if and only if*

$$\Gamma = \sigma^* /_* [\kappa \cdot_* (e +_* f^{2*}) \cdot_* (e +_* \sigma^{2*})^{\frac{3}{2}*}]$$

is a constant function. Here, $f = \tau /_* \kappa$ *and* $\sigma = f^* /_* (\kappa \cdot_* (e +_* f^{2*})^{\frac{3}{2}*})$.

Proof With the multiplicative normal indicatrix of the multiplicative curve $\mathbf{x}$ being $\mathbf{x}_n$, we know from (11) that the multiplicative curvatures of $\mathbf{x}_n$ are as follows

$$\kappa_n = (e +_* \sigma^{2*})^{\frac{1}{2}*},$$

$$\tau_n = \Gamma \cdot_* (e +_* \sigma^{2*})^{\frac{1}{2}*}.$$

It follows that $\Gamma = \tau_n /_* \kappa_n$. For a part of $\mathbf{x}_n$ to be a multiplicative cylindrical helix, $\tau_n /_* \kappa_n$ must be a multiplicative constant. This means that Γ is a multiplicative constant. □

Definition 4.19 *Let a regular multiplicative curve* $\mathbf{x}$ *be given in the multiplicative space with* $\kappa \neq 0_*$. *The multiplicative curve* $\mathbf{x}$ *is called the multiplicative g-clad helix if the multiplicative spherical image of the multiplicative principal normal vector* $\mathbf{n} : I \to \mathbb{S}^2_*$ *of the curve* $\mathbf{x}$ *is part of the multiplicative slant helix in* $\mathbb{S}^2_*$.

We have the following characterization of g-clad helices.

Theorem 4.20 *Let* $\mathbf{x}$ *be a multiplicative naturally parametrized curve with* $\kappa \neq 0_*$. *Then* $\mathbf{x}$ *is a multiplicative g-clad helix if and only if*

$$\psi(s) = \Gamma^*(s) /_* \left((\kappa^{2*}(s) +_* \tau^{2*}(s))^{\frac{1}{2}*} \cdot_* (e +_* \sigma^{2*}(s))^{\frac{1}{2}*} \cdot_* (e +_* \Gamma^{2*}(s))^{\frac{3}{2}*} \right)$$

is a constant function.

Proof With the multiplicative normal indicatrix of the multiplicative curve $\mathbf{x}$ being $\mathbf{x}_n$, from (11) the multiplicative curvatures of $\mathbf{x}_n$ are as follows

$$\begin{aligned} \kappa_n &= (e +_* \sigma^{2*})^{\frac{1}{2}*}, \\ \tau_n &= \Gamma \cdot_* (e +_* \sigma^{2*})^{\frac{1}{2}*}. \end{aligned}$$

If the necessary algebraic operations are performed here, we get

$$\kappa_n^{2*} +_* \tau_n^{2*} = (e +_* \sigma^{2*}) \cdot_* (e +_* \Gamma^{2*}).$$

From (12), we know that

$$\left(\kappa_n^{2*} /_* (\kappa_n^{2*} +_* \tau_n^{2*})^{\frac{3}{2}*}\right) \cdot_* (\tau_n /_* \kappa_n)^* = c, c \in \mathbb{R}_*.$$

So, we can easily see that

$$\Gamma^* /_* \left((\kappa^{2*} +_* \tau^{2*})^{\frac{1}{2}*} \cdot_* (e +_* \sigma^{2*})^{\frac{1}{2}*} \cdot_* (e +_* \Gamma^{2*})^{\frac{3}{2}*}\right). \tag{20}$$

If the normal indicatrix of the multiplicative curve $\mathbf{x}$ is a slant helix, (20) is a constant function. This completes the proof. □

Proposition 4.21 *Considering Theorems 4.18 and 4.20 that a multiplicative slant helix is a multiplicative helix with the condition $\sigma \equiv 0_*$, a multiplicative clad helix is a multiplicative slant helix with the condition $\Gamma \equiv 0_*$ and a multiplicative g-clad helix is a multiplicative clad helix with the condition $\psi \equiv 0_*$. Hence, we have the following relation*

$$\left\{ \begin{array}{l} \textit{the family of} \\ \textit{multiplicative} \\ \textit{helices} \end{array} \right\} \subset \left\{ \begin{array}{l} \textit{the family of} \\ \textit{multiplicative} \\ \textit{slant helices} \end{array} \right\} \subset \left\{ \begin{array}{l} \textit{the family of} \\ \textit{multiplicative} \\ \textit{clad helices} \end{array} \right\} \subset \left\{ \begin{array}{l} \textit{the family of} \\ \textit{multiplicative} \\ \textit{g-clad helices} \end{array} \right\}.$$

Example 4.22 *Let $\mathbf{x} : I \subset \mathbb{R}_* \to \mathbb{E}_*^3$ be multiplicative naturally parametrized clad helix curve in $\mathbb{R}_*^3$ parameterized by*

$$\mathbf{x}(s) = (x_1(s), x_2(s), x_3(s)),$$

where

$$\begin{aligned} x_1(s) &= e^{18} \cdot_* \cos_* 3s \cdot_* \cos_*(e^6 \cdot_* \cos_* 3s), \\ x_2(s) &= e^{-18} \cdot_* \cos_* 3s \cdot_* \sin_*(e^6 \cdot_* \cos_* 3s), \\ x_3(s) &= \sin_* 2s. \end{aligned}$$

In Figure 9, we present the graph of the multiplicative clad helix

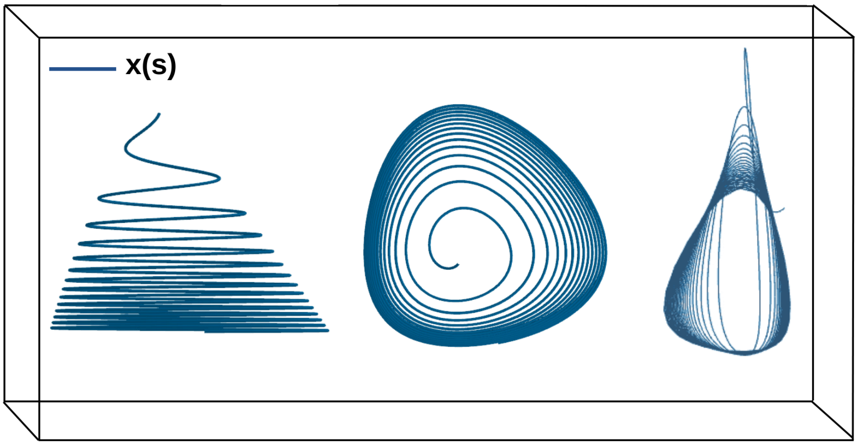

Figure 9: Multiplicative clad helix.

5. Conclusion

In this article, helices were examined using multiplicative arguments. The key point of this study is that the concept of metric, which is very important for geometry, is different from the traditional Euclidean metric. The metric here is a metric of multiplicative space based on proportional difference. Thanks to the multiplicative metric, helices, which are an important field of study in differential geometry, have been re-characterized and this change has been supported with examples. In this way, some applications of multiplicative space in differential geometry have been introduced and will be an example for other researchers.

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Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Aykut Has]: Collected the data, contributed to research method or evaluation of data, wrote the manuscript (%50).

Author [Beyhan Yılmaz]: Thought and designed the research/problem, contributed to research method or evaluation of data, wrote the manuscript (%50).

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] Aniszewska D., *Multiplicative Runge-Kutta methods*, Nonlinear Dynamics, 50, 262-272, 2007.

[2] Aydın M.E., Has A., Yılmaz B., *A non-Newtonian approach in differential geometry of curves: Multiplicative rectifying curves*, Bulletin of the Korean Mathematical Society, 61(3), 849-866, 2024.

[3] Aydın M.E., Has A., Yılmaz B., *Multiplicative rectifying submanifolds of multiplicative Euclidean space*, Mathematical Methods in the Applied Sciences, 48(1), 329-339, 2025.

[4] Bashirov A.E., Kurpınar E.M., Özyapıcı A., *Multiplicative calculus and its applications*, Journal of Mathematical Analysis and Applications, 337, 36-48, 2008.

[5] Bashirov A.E., Mısırlı E., Tandoğdu Y., Özyapıcı A., *On modeling with multiplicative differential equations*, Applied Mathematics-A Journal of Chinese Universities, 26(4), 425-438, 2011.

[6] Bashirov A.E., Rıza M., *On complex multiplicative differentiation*, TWMS Journal of Applied and Engineering Mathematics, 1(1), 75-85, 2011.

[7] Boruah K., Hazarika B., *Some basic properties of bigeometric calculus and its applications in numerical analysis*, Afrika Matematica, 32, 211-227, 2021.

[8] Ceyhan H., Özdemir Z., Nurkan S.K., Gürgil I., *A non-Newtonian approach to geometric phase through optic fiber via multiplicative quaternions*, Revista Mexicana de Física, 70(6), 061301, 2024.

[9] Foley J.D., Dam A.V., Feiner S.K., Hughes J.F., Computer Graphics: Principles and Practice, Addison-Wesley Professional Publishing, 2013.

[10] Georgiev S.G., Multiplicative Differential Geometry, Chapman and Hall/CRC, 2022.

[11] Georgiev S.G., Zennir K., Multiplicative Differential Calculus, Chapman and Hall/CRC., 2022.

[12] Georgiev S.G., Zennir K., Boukarou A., Multiplicative Analytic Geometry, Chapman and Hall/CRC, 2022.

[13] Göktaş S., Kemaloglu H., Yılmaz E., *Multiplicative conformable fractional Dirac system*, Turkish Journal of Mathematics, 46, 973–990, 2022.

[14] Grossman M., Bigeometric Calculus: A System with a Scale-Free Derivative, Archimedes Foundation, 1983.

[15] Grossman M., Katz R., Non-Newtonian Calculus, Lee Press, 1972.

[16] Gülsen T., Yılmaz E., Göktaş S., *Multiplicative Dirac system*, Kuwait Journal of Science, 49(3), 1-11, 2022.

[17] Has A., Yılmaz B., *A non-Newtonian conics in multiplicative analytic geometry*, Turkish Journal of Mathematics, 48(5), 976-994, 2024.

[18] Has A., Yılmaz B., *A non-Newtonian magnetic curves in multiplicative Riemann manifolds*, Physica Scripta, 99(4), 045239, 2024.

[19] Has A., Yılmaz B., Yıldırım H., *A non-Newtonian perspective on multiplicative Lorentz–Minkowski space* $\mathbb{L}_*^3$, Mathematical Methods in the Applied Sciences, 47(18), 13875-13888, 2024.

[20] Izumuya S., Takeuchi N., *New special curves and developable surfaces*, Turkish Journal of Mathematics, 28, 153-163, 2024.

[21] Kaya S., Ateş O., Gök I., Yaylı Y., *Timelike clad helices and developable surfaces in Minkowski 3-space*, Rendiconti del Circolo Matematico di Palermo Series 2, 68, 259–273, 2019.

[22] Mak M., *Framed clad helices in Euclidean 3-space*, Filomat, 37(28), 9627-9640, 2023.

[23] Nurkan S.K., Gürgil I., Karacan M.K., *Vector properties of geometric calculus*, Mathematical Methods in the Applied Sciences, 46(17), 17672-17691, 2023.

[24] Rybaczuk M., Stoppel P., *The fractal growth of fatigue defects in materials*, International Journal of Fracture, 103, 71-94, 2000.

[25] Rybaczuk M., Zielinski W., *The concept of physical and fractal dimension I. The projective dimensions*, Chaos, Solitons and Fractals, 12(13), 2517-2535, 2001.

[26] Sadraey M.H., Aircraft Design: A Systems Engineering Approach, Wiley, 2013.

[27] Stanley D., *A multiplicative calculus*, Primus, 9(4), 310-326, 1999.

[28] Takahashi T., Takeuchi N., *Clad helices and developable surface*, Tokyo Gakugei University Bulletin; Natural Science, 66, 1-9, 2014.

[29] Uzer A., *Multiplicative type complex calculus as an alternative to the classical calculus*, Computers and Mathematics with Applications, 60, 2725-2737, 2010.

[30] Volterra V., Hostinsky B., Operations Infinitesimales Lineares, Herman, 1938.

[31] Waseem M., Noor N.A., Shah F.A., Noor K.I., *An efficient technique to solve nonlinear equations using multiplicative calculus*, Turkish Journal of Mathematics, 42, 679-691, 2018.

[32] Watson J.D., Baker T., Stephen P.B., Gann A., Michael L., et al., Molecular Biology of the Gene, Pearson Publishing, 2013.

[33] Yalçın N., Çelik E., *Solution of multiplicative homogeneous linear differential equations with constant exponentials*, New Trends in Mathematical Sciences, 6(2), 58-67, 2018.

[34] Yazici M., Selvitopi H., *Numerical methods for the multiplicative partial differential equations*, Open Mathematics, 15, 1344-1350, 2017.

[35] Yılmaz B., Has A., *New approach to slant helix*, International Electronic Journal of Geometry, 12, 111-115, 2019.