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Some Curves on Three Dimensional Kenmotsu Space Forms

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Abstract

The object of the present paper is to study slant curves, C-parallel slant curves on Kenmotsu space forms. As a particular case we consider Legendre curves and integral curves of the Reeb vector fields. We show that on such manifolds Legendre curves do not exist and the slant integral curves of the Reeb vector fields are a geodesics. We also study biharmonic curves on such manifolds. An example is given.

Keywords: *Kenmotsu space forms, Legendre curves, slant curves.*
2010 Mathematics Subject Classification: *53C25*

1. INTRODUCTION

In 1972, K. Kenmotsu defined a new class of almost contact manifolds known as Kenmotsu manifold [7]. A Kenmotsu manifold of constant ϕ-sectional curvature is known as Kenmotsu space forms. The Riemannian curvature R for such a manifold is given by

$$\begin{aligned} R(X,Y)Z &= \frac{c-3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ &+ \frac{c+1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi \\ &+ g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}. \end{aligned} \tag{1.1}$$

The study of different types of curves on almost contact and contact manifold have become an important topic of research. In 1994, Baikoussis and Blair studied Legendre curves on contact three manifolds [1]. In 2008, Cho and Lee studied slant curves on Pseudo-Hermition manifolds [4]. In 2014, Özgur and Güvenc studied biharmonic Legendre curves on Sasakian space forms [6]. Recently the present authors have studied certain curves on some classes of almost contact manifolds [3]. Motivated by these works, in this paper we like to study slant curves and biharmonic curves on Kenmotsu space forms. The present paper is organized as followes:
We give the required preliminaries in Section 2. Section 3, contains the study of slant curves on Kenmotsu space forms. Section 4, is devoted to study biharmonic curves on Kenmotsu space forms. The last Section contains an example.

2. Preliminaries

Let M be a $(2n+1)$- dimensional connected differentiable manifold together with an almost contact metric structure (ϕ,ξ,η,g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is 1-form and g is Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.1}$$

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \eta\phi = 0, \qquad \eta(\xi) = 1, \qquad \eta(X) = g(X,\xi), \tag{2.2}$$

and

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \qquad g(X,\phi Y) = -g(\phi X,Y), \qquad X,Y \in T(M). \tag{2.3}$$

An almost contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is said to be a trans-Sasakian manifold if $(M^{2n+1}\times R;J;G)$ belongs to the class W_4 of the Hermitian manifolds, where J is the almost complex structure on $M^{2n+1}\times R$ defined by , $J(Z,f\frac{d}{dt}) = (\phi Z - f\xi, \eta(Z)\frac{d}{dt})$; where the

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pair $(Z, f\frac{d}{dt})$ denotes a tangent vector to $M^{2n+1} \times R$. Z and $f\frac{d}{dt}$ being tangent to M^{2n+1} and R respectively, M^{2n+1} (ϕ, ξ, η, g) are said to be normal if the structure J is integrable .The necessary and sufficient condition for (ϕ, ξ, η, g) to be normal is

$$[\phi, \phi] + 2\xi \otimes \eta = 0, \tag{2.4}$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ defined by

$$[\phi, \phi](X,Y) = [\phi X, \phi Y] + \phi^2[X,Y] - \phi[\phi X, Y] - \phi[X, \phi Y], \tag{2.5}$$

for any vector field$Z$ on$M^{2n+1}$and smooth function$f$ on $M^{2n+1} \times R$ and G is the Hermitian metric on the product $M^{2n+1} \times R$. This may be expressed by the condition

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X) \tag{2.6}$$

for some smooth functions α and β on M^{2n+1}.
An almost contact metric manifold is called Kenmotsu manifold if [7]

$$(\nabla_X \eta)Y = g(\phi X,Y)\xi - \eta(Y)\phi X, \tag{2.7}$$

where ∇ is Levi-Civita connection of g. We have also on a Kenmotsu manifold

$$\nabla_X \xi = X - \eta(X)\xi = -\phi^2 X. \tag{2.8}$$

An almost contact metric manifold is called contact metric manifold if

$$d\eta(X,Y) = \Phi(X,Y) = g(X, \phi Y), \tag{2.9}$$

where Φ is called the fundamental 2-form of M^{2n+1}.
Lemma 2.1 In a Kenmotsu space form, we have [7]

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.10}$$

$$R(\xi,Y)Z = \eta(Z)Y - g(Y,Z)\xi, \tag{2.11}$$

$$R(\xi,Y)\xi = Y - \eta(Y)\xi, \tag{2.12}$$

$$S(X,\xi) = -2n\eta(X), \tag{2.13}$$

where R is the Riemannian curvature tensor and S is the Ricci tensor of type (0,2).

Let M be a 3-dimensional Riemannian manifold. Let $\gamma : I \to M$, I being an interval, be a curve in M which is parameterized by arc length, and let $\nabla_{\dot{\gamma}}$ denotes the covariant differentiation along γ with respect to the Levi-Civita connection on M. It is said to that γ is a Frenet curve if one of the following three cases holds:
(a) γ is of osculating order 1, if, $\nabla_T T = 0$ (geodesic), $T = \dot{\gamma}$. Here, . denotes differentiation with respect to the arc length.
(b) γ is of osculating order 2, if,there exist two orthonormal vector fields $T(= \dot{\gamma})$, N and a non-negative function κ (curvature) along γ such that $\nabla_T T = \kappa N$, $\nabla_T N = -\kappa T$.
(c) γ is of osculating order 3, if, there exist three orthonormal vectors $T = (\dot{\gamma}), N, B$ and two non-negative function κ(curvature) and τ(torsion) along γ such that

$$\nabla_T T = \kappa N, \tag{2.14}$$

$$\nabla_T N = -\kappa T + \tau B, \tag{2.15}$$

$$\nabla_T B = -\tau N, \tag{2.16}$$

Where $T = \dot{\gamma}$ and $\{T, N, B\}$is the Frenet frame κ and τ are the curvature and torsion of the curve. With respect to Levi-Civita connection, a Frenet curve of osculating order 3 is called a Geodesic if $\kappa = 0$. It is called a circle if κ is a positive constant and $\tau = 0$. The curve is called a helix in M if κ and τ both are positive constants and the curve is called a generalized Helix if $\frac{\kappa}{\tau} =$ constant.
A Frenet curve γ in an almost contact metric manifold is said to be a Legendre curve or almost contact curve if it is an integral curve of the contact distribution $D = \ker\eta$. Formally, it is said that a Frenet curve γ in an almost contact metric manifold is a Legendre curve if and only if $\eta(\dot{\gamma}) = 0$ and $g(\dot{\gamma}, \dot{\gamma}) = 1$. For more details we refer [1], [4].

3. SLANT CURVE IN KENMOTSU SPACE FORM:

Definition 3.1. A unit speed curve γ in an almost contact metric manifold $M(\phi,\xi,\eta,g)$ is said to be slant if its tangent vector field makes constant angle θ with ξ i.e., $\eta(\dot{\gamma}) = \cos\theta$ is constant alonge γ.
By definition, slant curves with constant angle 0 are trajectories of ξ, slant curves with constant angle $\frac{\pi}{2}$ are called almost Legendre curves or almost contact curves.
The example of slant curves in almost contact metric manifold is given in the paper [5],(example 4, page 98).
Definition 3.2. A slant curve γ is called C-parallel if it is satisfies

$$\nabla_T H = \lambda\xi, \tag{3.1}$$

where λ is a (non-zero for proper case) differentiable function and $H = \nabla_T T$ is the mean curvature vector field of the manifold.
Consider a slant curve γ on a Kenmotsu manifold, then we get by definition

$$g(T,\xi) = \cos\theta,$$

where θ is a constant. Differentiating both side with respect to T we get

$$\nabla_T g(T,\xi) - g(\nabla_T T,\xi) - g(T,\nabla_T\xi) = 0. \tag{3.2}$$

Using (2.8) in the above equation we get,

$$-\kappa\eta(N) - 1 + \cos^2\theta = 0, \tag{3.3}$$

where $\{T,N,B\}$ is a Frenet frame with $T = \dot{\gamma}$. From above we get

$$\kappa\eta(N) = \cos^2\theta - 1. \tag{3.4}$$

In particular, let $\theta = \frac{\pi}{2}$, and we consider, $\dot{\gamma} = T$ and $\xi = B$. Then we can take $\{T,N,B\}$ as Frenet frame. So, κ is undefined. Therefore, we can conclude the following:
Theorem 3.1 In a Kenmotsu space form Legendre curves do not exist.
In particular, let $\theta = 0$, then $\kappa = 0$. Thus we obtain the following:
Theorem 3.2 The Reeb flow line on a Kenmotsu space form is geodesic.
We also obtain the following:
Theorem 3.3 The curvature of a curve other than the Legendre curve and Reeb flow line on a three-dimensional Kenmotsu space form is negative.

Consider the slant curve is C-parallel, then by definition we get

$$\nabla_T H = \lambda\xi, \tag{3.5}$$

Putting $H = \nabla_T T$ and if $\{T,N,B\}$ is a Frenet frame then (3.5) implies

$$-\kappa^2 T + \kappa' N + \kappa\tau B = \lambda\xi. \tag{3.6}$$

Taking inner product of the above equation with T,N,B respectively we get

$$\eta(T) = -\frac{1}{\lambda}\kappa^2, \tag{3.7}$$

$$\eta(N) = \frac{1}{\lambda}\kappa', \tag{3.8}$$

$$\eta(B) = \frac{1}{\lambda}\kappa\tau. \tag{3.9}$$

But for a slant curves with constant slant angle θ, $\eta(T) = \cos\theta$, hence from (3.7) we get

$$\kappa^2 = -\lambda\cos\theta. \tag{3.10}$$

But by Theorem 3.2 for slant curve $\kappa = 0$, hence we get, $\lambda = 0$, which contradicts Definition 3.2 for proper case. Thus we are in a position to state the following:
Theorem 3.4: There exists no proper C-parallel slant curves on Kenmotsu space form.

And for $0 < \theta < \frac{\pi}{2}$, $\lambda \neq 0$, we can state the following:
Theorem 3.5 There exists proper C-parallel slant curve on Kenmotsu space form for $0 < \theta < \frac{\pi}{2}$.

4. BIHARMONIC LEGENDRE CURVES ON KENMOTSU SPACE FORM

Definition 4.1. The contact angle of γ is the function $\theta : I \longrightarrow [0,2\pi)$ given by

$$\cos\theta(s) = g(T(s),\xi). \tag{4.1}$$

γ is a slant curve (or more precisely θ-slant curve) if θ is a constant function. In the particular case of $\theta = \frac{\pi}{2}$ (or $\frac{3\pi}{2}$), γ is called Legendre curve [1].

Definition 4.2 A Legendre curve is called biharmonic [4] if it satisfies

$$\nabla_T^3 T - \kappa R(N,T)T = 0, \tag{4.2}$$

where $T = \dot{\gamma}$.

Since for a Legendre curve T is orthogonal to ξ, we can take $T = \dot{\gamma}$, and $\{T,N,B\}$ as Frenet frame. Let us consider a biharmonic Legendre curve on Kenmotsu space form. Then

$$\nabla_T^3 T - \kappa R(N,T) = 0. \tag{4.3}$$

By Serret-Frenet formula we get

$$\nabla_T^3 T = -3\kappa\kappa' T + (\kappa'' - \kappa^3 - \kappa\tau^2)N + (2\tau\kappa' + \kappa\tau')B. \tag{4.4}$$

For Legendre curve $\eta(T) = 0$, $\eta(N) = 0$, because we have considered the Frenet frame $T, N = \phi T, B = \xi$. Using these facts in (2.11) we get, after simplification

$$R(\xi,T)T = -\xi. \tag{4.5}$$

Now for $N = \xi$, and in the view of (4.3) and (4.4), it follows that

$$\begin{aligned} \nabla_T^3 T - \kappa R(\xi,T)T &= -3\kappa\kappa' T + (\kappa'' - \kappa^3 - \kappa\tau^2 + \kappa)\xi \\ &+ (2\tau\kappa' + \kappa\tau')B \\ &= 0. \end{aligned} \tag{4.6}$$

From the first component, we get κ =constant. And from third component, we get τ=constant. Hence the curve is a helix.

From second component, we get

$$\kappa^2 + \tau^2 = 1. \tag{4.7}$$

Hence we are in position to state the following:

Theorem 4.1. The curvature and torsion of a biharmonic curve in a Kenmotsu space form are constant and they are related by $\kappa^2 + \tau^2 = 1$.

5. Example

In this section we state the well known example of a three-dimensional β-Kenmotsu manifold and then Legendre curves on it.

Let us consider a 3-dimensional manifold $M = \{(x,y,z) \in R^3 : z \neq 0\}$, where (x,y,z) are the standard coordinates in R^3.

Let $e_1 = z\frac{\partial}{\partial x}$, $e_2 = z\frac{\partial}{\partial y}$ and $e_3 = z\frac{\partial}{\partial z}$, which are linearly independent vector fields at each point of M. Let g be a Riemannian metric define by

$g(e_1,e_2) = g(e_2,e_3) = g(e_1,e_3) = 0$, $g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1$.

Let η be 1-form defined by $\eta(Z) = g(Z,e_3)$, for any $Z \in TM$ and ϕ be the tensor field of type $(1,1)$ defined by $\phi e_1 = -e_2$, $\phi e_2 = e_1$, $\phi e_3 = 0$. Then by applying linearity of ϕ and g, we have

$\eta(e_3) = 1$, $\phi^2 Z = -Z + \eta(Z)e_3$, $g(\phi Z,\phi U) = g(Z,U) - \eta(Z)\eta(U)$, for any $Z,U \in TM$. Hence for $e_3 = \xi$, (ϕ,ξ,η,g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to g and R be the curvature tensor of type $(1,3)$, then we have

$[e_1,e_2] = 0$, $[e_1,e_3] = -e_1$, $[e_2,e_3] = -e_2$.

By using Koszul formula for Levi-Civita connection with respect to g, we obtain

$$\begin{array}{lll} \nabla_{e_1} e_3 = -e_1, & \nabla_{e_2} e_3 = -e_2, & \nabla_{e_3} e_3 = 0, \\ \nabla_{e_1} e_2 = 0, & \nabla_{e_2} e_2 = e_3, & \nabla_{e_3} e_2 = 0, \\ \nabla_{e_1} e_1 = e_3, & \nabla_{e_2} e_1 = 0, & \nabla_{e_3} e_1 = 0. \end{array}$$

From the above we see that the manifold satisfies $\nabla_X \xi = \beta(X - \eta(X)\xi)$, for $\xi = e_3$ and $\beta = -1$. Consequently $M(\phi,\xi,\eta,g,\varepsilon)$ is a three-dimensional β-Kenmotsu manifold.

With the help of the above results it can be verified that

$$\begin{array}{lll} R(e_1,e_2)e_3 = 0, & R(e_2,e_3)e_3 = -e_2, & R(e_1,e_3)e_3 = -e_1, \\ R(e_1,e_2)e_2 = -e_1, & R(e_2,e_3)e_2 = e_3, & R(e_1,e_3)e_2 = 0, \\ R(e_1,e_2)e_1 = e_2, & R(e_2,e_3)e_1 = 0, & R(e_1,e_3)e_1 = e_3. \end{array}$$

Also from the above expressions if we choose c=-1, the the curvature tensor satisfies

$$\begin{aligned} R(X,Y)Z &= \frac{c-3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ &+ \frac{c+1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi \\ &+ g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \end{aligned} \tag{5.1}$$

Hence the manifold is a Kenmotsu space form. Now we give an example of unit speed curves on the manifold.
Example 5.1. Consider a curve $\gamma : I \longrightarrow M$ defined by $\gamma(s) = (0,0,s)$. Hence $\dot{\gamma}_1 = 0$, $\dot{\gamma}_2 = 0$ and $\dot{\gamma}_3 = 1$,
$\eta(\dot{\gamma}) = g(\dot{\gamma}, e_3) = g(\dot{\gamma}_1 e_1 + \dot{\gamma}_2 e_2 + \dot{\gamma}_3 e_3, e_3) = 1.$
$g(\dot{\gamma}, \dot{\gamma}) = g(\dot{\gamma}, e_3) = g(\dot{\gamma}_1 e_1 + \dot{\gamma}_2 e_2 + \dot{\gamma}_3 e_3, \dot{\gamma}_1 e_1 + \dot{\gamma}_2 e_2 + \dot{\gamma}_3 e_3)$
. $= \dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \dot{\gamma}_3^2$
. $= \dot{\gamma}_3^2$
. $= 1.$
Hence the curve is unit speed and it is the flow line of the Reeb vector field ξ. For this curve $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. Hence the Reeb flow line is geodesic.

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