Konuralp Journal of Mathematics, 6 (1) (2018) 158-162

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

A Generalization of The Taxicab Metric And Related Isometries

Harun Barış Çolakoğlu[1]

[1] *Akdeniz University, Vocational School of Technical Sciences, Department of Computer Technologies, 07070, Pınarbaşı, Antalya, TÜRKİYE.*
[*] *Corresponding author E-mail: hbcolakoglu@akdeniz.edu.tr*

Abstract

In this paper, we define a family of distance functions in the real plane, m-generalized taxicab distance function, which includes the generalized taxicab distance and so the taxicab distance functions as special cases, and we show that the m-generalized taxicab distance function determines a metric. Then we give some properties of the m-generalized taxicab metric, and determine Euclidean isometries that preserve the m-generalized taxicab metric.

Keywords: *metric, taxicab distance, m-generalized taxicab distance, isometry.*
2010 Mathematics Subject Classification: *51K05, 51K99, 51N99.*

1. Introduction

The *taxicab metric* was given in a family of metrics of the real plane by Minkowski. Using this metric, taxicab geometry was introduced by Menger [5], and developed by Krause [4]. In the taxicab geometry, circles are squares in which each diagonal is parallel to a coordinate axis. In [8], Lawrance J. Wallen altered the taxicab metric by redefining in order to get rid of imperative symmetry, and called it *the* (*slightly*) *generalized taxicab metric*. In the generalized taxicab geometry, circles are rhombuses in which each diagonal is also parallel to a coordinate axis.

In this work, we define a new distance function in the real plane $\mathbb{R}^2$, m-generalized taxicab distance function $d_{T_g(m)}$, which includes the generalized taxicab distance and so the taxicab distance functions as special cases, and we show that the m-generalized taxicab distance function determines a metric in $\mathbb{R}^2$. We see that the m-generalized taxicab metric can have any rhombus as a circle, instead of rhombuses with diagonal parallel to a coordinate axis. Then we give some properties of the m-generalized taxicab metric, and determine Euclidean isometries of the plane which also preserve m-generalized taxicab metric. We also see that there are transformations which preserve the m-generalized taxicab distance, but not preserve the Euclidean distance for some case.

2. The m-generalized taxicab distance in $\mathbb{R}^2$

The m-generalized taxicab distance and the m-taxicab distance between two points in the Cartesian plane are defined as follows:

Definition 2.1. Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points in $\mathbb{R}^2$. For any real number m and positive real numbers a and b, the function $d_{T_g(m)} : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ defined by

$$d_{T_g(m)}(P_1,P_2)=(a|(x_1-x_2)+m(y_1-y_2)|+b|m(x_1-x_2)-(y_1-y_2)|)/(1+m^2)^{1/2} \tag{2.1}$$

is called the *m-generalized taxicab distance function* in $\mathbb{R}^2$, and the real number $d_{T_g(m)}(P_1, P_2)$ is called the *m- generalized taxicab distance* between points P_1 and P_2. As a special case, if $a = b = 1$, then the function $d_{T(m)} : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ defined by

$$d_{T(m)}(P_1,P_2)=(|(x_1-x_2)+m(y_1-y_2)|+|m(x_1-x_2)-(y_1-y_2)|)/(1+m^2)^{1/2} \tag{2.2}$$

is called the *m-taxicab distance function* in $\mathbb{R}^2$, and the real number $d_{T(m)}(P_1, P_2)$ is called the *m-taxicab distance* between points P_1 and P_2.

***Email addresses*:** **hbcolakoglu@akdeniz.edu.tr (Harun Barış Çolakoğlu)**

Remark 1. We are quite familiar to the notation of d_T for the taxicab distance function. In [2] and [3], d_{T_g} was used to state the generalized taxicab distance. Here, we choose the notation $d_{T_g(m)}$ to state the m-generalized taxicab distance function, and $d_{T(m)}$ to state the m-taxicab distance function for compatibility. Clearly, there are infinitely many different distance functions in the family of distance functions $d_{T_g(m)}$, depending on values a, b and m. One can find the definition of $d_{T_g(m)}$ not to be well-defined since the m-generalized taxicab distance between two points can change also according to values a and b. To remove this confusion, we have to use values a and b in the name of the distance, just as we use m. This can be done easily; for example by using the notation $m(a,b)$ instead of m in phrases $d_{T_g(m)}$ and m-generalized taxicab distance. But we keep on using m for the sake of shortness, supposing values a and b are initially determined and fixed, unless otherwise stated.

Remark 2. In $\mathbb{R}^2$, the m-generalized taxicab distance function involves the generalized taxicab distance and so the taxicab distance functions as special cases: For points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, if $m = 0$ in $d_{T_g(m)}$, then

$$d_{T_g(0)}(P_1,P_2) = d_{T_g}(P_1,P_2) = a\,|x_1 - x_2| + b\,|y_1 - y_2| \tag{2.3}$$

which is the generalized taxicab distance function d_{T_g}. And if $a = b = 1$ and $m = 0$ in $d_{T_g(m)}$, or if $m = 0$ in $d_{T(m)}$, then

$$d_{T(0)}(P_1,P_2) = d_T(P_1,P_2) = |x_1 - x_2| + |y_1 - y_2| \tag{2.4}$$

which is the well-known taxicab distance function d_T. In addition, if $a = b$, then $d_{T_g(m)}$ is equal to the *m-generalized absolute value distance* d_m defined in [1]. But, if $a \neq b$, then we will see in Section 2 that the m-generalized taxicab circle is a *rhombus*, while the m-generalized absolute value circle is an *octagon*.

Henceforth, we use equations $a' = a/(1+m^2)^{1/2}$ and $b' = b/(1+m^2)^{1/2}$ to shorten phrases. The following proposition shows that the m-generalized taxicab distance function satisfies the metric properties.

Proposition 2.1. *The m-generalized taxicab distance function determines a metric in* $\mathbb{R}^2$.

Proof. Clearly, $d_{T_g(m)}(P_1,P_2) = 0$ if and only if $P_1 = P_2$, and $d_{T_g(m)}(P_1,P_2) = d_{T_g(m)}(P_2,P_1)$ for all P_1 and P_2 in $\mathbb{R}^2$. So, we only will show that $d_{T_g(m)}$ satisfies the triangle inequality, that is, $d_{T_g(m)}(P_1,P_2) \leq d_{T_g(m)}(P_1,P_3) + d_{T_g(m)}(P_3,P_2)$ for points $P_i = (x_i, y_i)$, $i = 1,2,3$ in $\mathbb{R}^2$. This fact can be proven as follows:

$$\begin{aligned}
d_{T_g(m)}(P_1,P_2) &= a'\,|(x_1 - x_2) + m(y_1 - y_2)| + b'\,|m(x_1 - x_2) - (y_1 - y_2)| \\
&= a'\,|(x_1 - x_3) + m(y_1 - y_3) + (x_3 - x_2) + m(y_3 - y_2)| + \\
&\quad\; b'\,|m(x_1 - x_3) - (y_1 - y_3) + m(x_3 - x_2) - (y_3 - y_2)| \\
&\leq a'\,|(x_1 - x_3) + m(y_1 - y_3)| + b'\,|m(x_1 - x_3) - (y_1 - y_3)| + \\
&\quad\; a'\,|(x_3 - x_2) + m(y_3 - y_2)| + b'\,|m(x_3 - x_2) - (y_3 - y_2)| \\
&= d_{T_g(m)}(P_1,P_3) + d_{T_g(m)}P_3,P_1).
\end{aligned}$$

□

3. Some properties of $d_{T_g(m)}$

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points in the Cartesian coordinate plane. Let l_{P_1} be the line through P_1 with slope m, and let l_{P_2} be the line through P_2 and perpendicular to the line l_{P_1}. Since the Euclidean distances from P_1 to l_{P_2} and from P_2 to l_{P_1} are $d_E(P_1, l_{P_2}) = |(x_1 - x_2) + m(y_1 - y_2)|\,/(1+m^2)^{1/2}$ and $d_E(P_2, l_{P_1}) = |m(x_1 - x_2) - (y_1 - y_2)|\,/(1+m^2)^{1/2}$, the m-generalized taxicab distance between points P_1 and P_2 can be given by

$$d_{T_g(m)}(P_1,P_2) = a\,d_E(P_1, l_{P_2}) + b\;d_E(P_2, l_{P_1}). \tag{3.1}$$

If $a \geq b$, let $\frac{b}{a} = (\sec\alpha - \tan\alpha)$ for $\alpha \in [0, \pi/2)$; and if $b \geq a$, let $\frac{a}{b} = (\sec\beta - \tan\beta)$ for $\beta \in [0, \pi/2)$. Then we get the fact that if $a = b$, then $\alpha = \beta = 0$ and the m-generalized taxicab distance between points P_1 and P_2 is constant a multiple of the Euclidean length of one of the shortest paths from P_1 to P_2 composed of line segments, each parallel to one of lines with slope m or $-1/m$; if $a \geq b$, then the m-generalized taxicab distance between points P_1 and P_2 is constant a multiple of the Euclidean length of one of the shortest paths from P_1 to P_2 composed of line segments, each parallel to one of lines with slope m, $-1/m$, $[m(a^2 - b^2) + 2ab]/[(a^2 - b^2) - 2abm]$ or $[m(a^2 - b^2) - 2ab]/[(a^2 - b^2) + 2abm]$; if $b \geq a$, then the m-generalized taxicab distance between points P_1 and P_2 is constant b multiple of the Euclidean length of one of the shortest paths from P_1 to P_2 composed of line segments, each parallel to one of lines with slope m, $-1/m$, $[2abm - (a^2 - b^2)]/[m(a^2 - b^2) + 2ab]$ or $[(a^2 - b^2) + 2abm]/[2ab - m(a^2 - b^2)]$ (see Figure 1). Although there exist, in general, infinitely many shortest paths between points P_1 and P_2, we prefer to use the ones in Figure 1, and call each of them a *basic way*. Also we call each of lines $mx - y = 0$ and $x + my = 0$ an *axis of direction*.

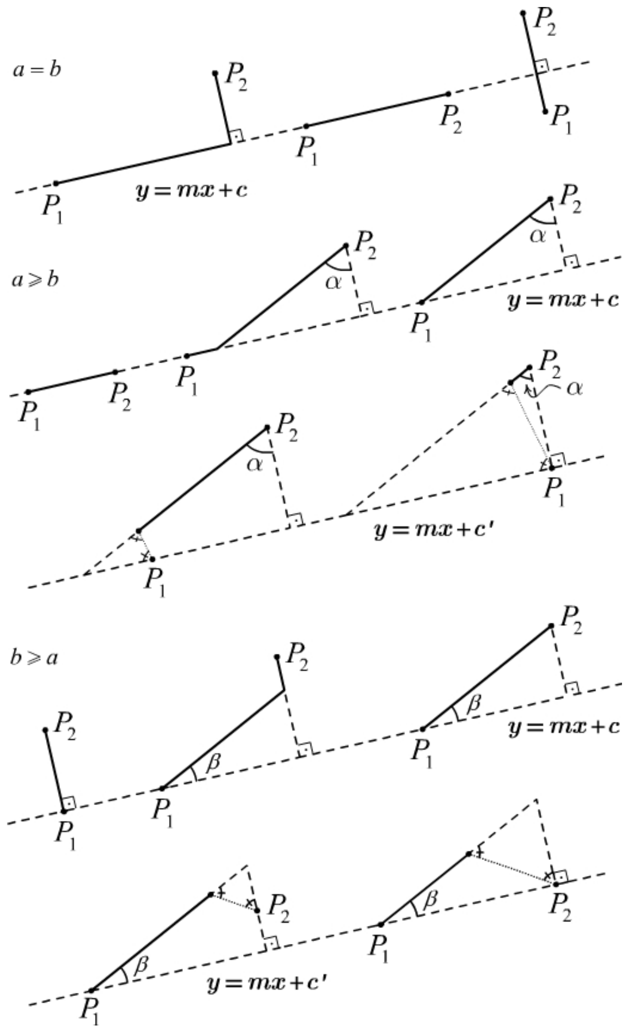

Figure 1: The basic ways between points P_1 and P_2 with respect to the m-generalized taxicab metric.

Now, we examine the minimum distance set of points P_1 and P_2 in $\mathbb{R}^2_{T_g(m)}$. The *minimum distance set* of P_1 and P_2, $M(P_1,P_2)$, is defined by

$$M(P_1,P_2)=\left\{X : d_{T_g(m)}(P_1,X)+d_{T_g(m)}(X,P_2)=d_{T_g(m)}(P_1,P_2)\right\}. \tag{3.2}$$

In Euclidean plane, the minimum distance set of P_1 and P_2 is the line segment joining points P_1 and P_2. It is not difficult to see that the minimum distance set of points P_1 and P_2 with respect to the m-generalized taxicab metric is generally a rectangular region (or a line segment) with diagonal P_1P_2 bounded by the lines through P_1 and P_2 being parallel to an axis of direction, which are $mx-y-mx_i+y_i=0$ and $x+my-x_i-my_i=0$ for $i\in\{1,2\}$, as shown in Figure 2.

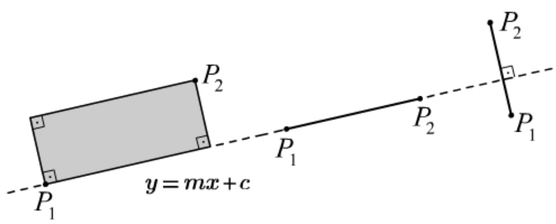

Figure 2: Minimum distance set of P_1 and P_2 with respect to the m-generalized taxicab metric.

The following two propositions follow directly from Equation (3.1) which is the geometric interpretation of the m-generalized taxicab distance.

Proposition 3.1. *Let P_1, P_2 and P_3 be three points in $\mathbb{R}^2$ such that $P_3\in M(P_1,P_2)$. Then, $d_{T_g(m)}(P_1,P_3)\leq d_{T_g(m)}(P_1,P_2)$. In addition, $d_{T_g(m)}(P_1,P_3)=d_{T_g(m)}(P_1,P_2)$ if and only if $P_3=P_2$.*

Proposition 3.2. *Let P_1, P_2, P_3 and P_4 be four points in $\mathbb{R}^2$. For $a = b$, if $M(P_1,P_2)$ and $M(P_3,P_4)$ are congruent, then $d_{T_g(m)}(P_1,P_2) = d_{T_g(m)}(P_3,P_4)$. For $a \neq b$, if $M(P_1,P_2)$ and $M(P_3,P_4)$ are congruent and equal sides of $M(P_1,P_2)$ and $M(P_3,P_4)$ are parallel, then $d_{T_g(m)}(P_1,P_2) = d_{T_g(m)}(P_3,P_4)$.*

Let us denote the real plane endowed with the m-generalized taxicab metric by $\mathbb{R}^2_{T_g(m)}$. Then m-generalized taxicab unit circle in $\mathbb{R}^2_{T_g(m)}$ is the set of points (x,y) satisfying the equation

$$(a\,|x+my|+b\,|mx-y|)/(1+m^2)^{1/2} = 1. \tag{3.3}$$

One can see by calculation that m-generalized taxicab unit circle is a rhombus with a diagonal having slope of m, and with vertices $A_1 = \left(\frac{1}{ak},\frac{m}{ak}\right)$, $A_2 = \left(\frac{-m}{bk},\frac{1}{bk}\right)$, $A_3 = \left(\frac{-1}{ak},\frac{-m}{ak}\right)$ and $A_4 = \left(\frac{m}{bk},\frac{-1}{bk}\right)$, where $k = (1+m^2)^{1/2}$ (see Figure 3). If $a = b$, then m-generalized taxicab unit circle is a square with vertices A_1,A_2,A_3 and A_4. In $\mathbb{R}^2_{T_g(m)}$, it is easy to see that the ratio of the circumference of an m-generalized taxicab circle to its diameter is $\pi_{T_g(m)} = 4$.

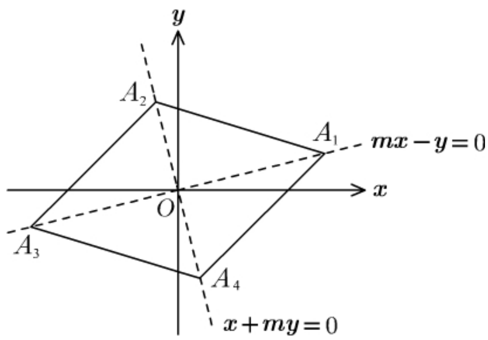

Figure 3: The m-generalized taxicab unit circle for $m = 1/4$, $a = 1/5$, $b = 1/3$.

Thus, the m-generalized taxicab metric has rhombuses with diagonal having slope of m, as circles, instead of rhombuses with diagonal parallel to a coordinate axis, and the m-taxicab metric has squares with diagonal having slope of m, as circles, instead of squares with diagonal parallel to a coordinate axis.

The following proposition gives an equation which relates the Euclidean distance to the m-generalized taxicab distance between two points in the Cartesian coordinate plane:

Proposition 3.3. *For any two points P_1 and P_2 in $\mathbb{R}^2$ that do not lie on a vertical line, if n is the slope of the line through P_1 and P_2, then*

$$d_E(P_1,P_2) = \mu(n)d_{T_g(m)}(P_1,P_2) \tag{3.4}$$

where $\mu(n) = (1+n^2)^{1/2}/(a'\,|1+mn|+b'\,|m-n|)$. If P_1 and P_2 lie on a vertical line, then

$$d_E(P_1,P_2) = [1/(a'\,|m|+b')]d_{T_g(m)}(P_1,P_2). \tag{3.5}$$

Proof. Let $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$ such that $x_1 \neq x_2$. Then $n = (y_2-y_1)/(x_2-x_1)$. Clearly, $d_{T_g(m)}(P_1,P_2) = |x_1-x_2|\,(a'\,|1+mn|+b'\,|m-n|)$ and $d_E(P_1,P_2) = |x_1-x_2|(1+n^2)^{1/2}$, thus we have $d_E(P_1,P_2) = \mu(n)d_{T_g(m)}(P_1,P_2)$, where $\mu(n) = (1+n^2)^{1/2}/(a'\,|1+mn|+b'\,|m-n|)$. If $x_1 = x_2$, then $d_{T_g(m)}(P_1,P_2) = |x_1-x_2|\,(a'\,|m|+b')$ and we have Equation (3.5). □

The following two corollaries follow directly from Proposition 3.2 or Proposition 3.3:

Corollary 3.1. *Let P_1, P_2, P_3 and P_4 be four points in $\mathbb{R}^2$. If lines P_1P_2 and P_3P_4 are coincident or parallel or symmetric about a line parallel to an axis of direction; in addition P_1P_2 and P_3P_4 are perpendicular to each other for $a = b$, then*

$$d_{T_g(m)}(P_1,P_2) = d_{T_g(m)}(P_3,P_4) \text{ if and only if } d_E(P_1,P_2) = d_E(P_3,P_4). \tag{3.6}$$

Corollary 3.2. *If P_1, P_2 and X are three distinct collinear points in $\mathbb{R}^2$, then*

$$d_{T_g(m)}(P_1,X)/d_{T_g(m)}(X,P_2) = d_E(P_1,X)/d_E(X,P_2). \tag{3.7}$$

As a consequence of Corollary 3.2, it is clear that Thales, Menelaus, and Ceva theorems are true in $\mathbb{R}^2_{T_g(m)}$.

4. Isometries of $\mathbb{R}^2_{T_g(m)}$

Notice that by Proposition 3.2 and Corollary 3.1, the m-generalized taxicab distance between two points is invariant under all translations. In addition, the m-generalized taxicab distance between two points is invariant under rotation of π radian around a point for $a \neq b$; and rotations of $\pi/2$, π and $3\pi/2$ radians around a point for $a = b$. Besides, if $a \neq b$, then the m-generalized taxicab distance between two points is invariant under the reflections in lines parallel to $mx - y = 0$ or $x + my = 0$; if $a = b$, then the m-generalized taxicab distance between two points is invariant under reflections in lines parallel to $mx - y = 0$, $x + my = 0$, $(1+m)x - (1-m)y = 0$, or $(1-m)x + (1+m)y = 0$. On the other hand, if we denote reflection in line l by σ_l, and rotation about a point C through angle θ by $\rho_{C,\theta}$, using coordinate definition of the m -generalized taxicab distance for any points P_1 and P_2, one can get that $d_{T_g(m)}(P_1,P_2) = d_{T_g(m)}(\sigma_l(P_1),\sigma_l(P_2))$ for $l \in \{mx-y=0, x+my=0\}$ and $d_{T_g(m)}(P_1,P_2) = d_{T_g(m)}(\rho_{O,\theta}(P_1),\rho_{O,\theta}(P_2))$ for $\theta \in \{0,\pi\}$; in addition if $a = b$ then the equations holds also for $l \in \{(1+m)x-(1-m)y=0, (1-m)x+(1+m)y=0\}$ and $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$.

Now, we state following three propositions as results of our observations above:

Proposition 4.1. *Every translation is an isometry of* $\mathbb{R}^2_{T_g(m)}$.

Proposition 4.2. *In* $\mathbb{R}^2_{T_g(m)}$, *the set of isometric reflections in lines through origin is*

$$S_{T_g(m)} = \begin{cases} \sigma_{mx-y=0},\ \sigma_{x+my=0} & , \textit{if } a \neq b \\ \sigma_{mx-y=0},\ \sigma_{x+my=0},\ \sigma_{(1+m)x-(1-m)y=0},\ \sigma_{(1-m)x+(1+m)y=0} & , \textit{if } a = b \end{cases}.$$

Proposition 4.3. *In* $\mathbb{R}^2_{T_g(m)}$, *the set of isometric rotations about origin for* $\theta \in [0,2\pi)$ *is*

$$R_{T_g(m)} = \begin{cases} \rho_{O,0},\ \rho_{O,\pi} & , \textit{if } a \neq b \\ \rho_{O,0},\ \rho_{O,\frac{\pi}{2}},\ \rho_{O,\pi},\ \rho_{O,\frac{3\pi}{2}} & , \textit{if } a = b \end{cases}.$$

The following proposition states that there is no other Euclidean isometry preserving the m-generalized taxicab distance:

Proposition 4.4. *Let* $\phi : \mathbb{R}^2_{T_g(m)} \to \mathbb{R}^2_{T_g(m)}$ *be a Euclidean isometry preserving the m-generalized taxicab distance such that* $\phi(O) = O$. *Then* $\phi \in R_{T_g(m)}$ *or* $\phi \in S_{T_g(m)}$.

Proof. Given a Euclidean isometry $\phi : \mathbb{R}^2_{T_g} \to \mathbb{R}^2_{T_g}$ such that $\phi(O) = O$. Assume that $\phi(A_1) \in (A_1,A_2)$, then $\phi(A_2) \in [A_3,A_4]$ since $d_{T_g}(\phi(A_1),\phi(A_2)) = 2$. Then $\phi(A_3) \in [A_1,A_2]$, and we have $d_{T_g}(\phi(A_1),\phi(A_3)) < 2$, which is a contradiction since $d_{T_g}(A_1,A_3) = 2$. Thus, $\phi(A_1) \notin (A_1,A_2)$. Similarly, one can see that $\phi(A_i) \notin (A_j,A_{j+1})$ for $i,j \in \{1,2,3,4\}$ and (assume that $A_5 = A_1$). Now, it is clear that $\phi(A_i) \in \{A_1,A_2,A_3,A_4\}$ for $i \in \{1,2,3,4\}$. For $a \neq b$, if $\phi(A_1) = A_i$, then $\phi(A_2) = A_{i+1}$ for $i \in \{1,3\}$, and ϕ is a rotation with the angle $\theta = \frac{(1-i)\pi}{2}$. If $\phi(A_1) = A_{i+1}$, then $\phi(A_2) = A_i$ for $i \in \{2,4\}$, and ϕ is a reflection in line $mx - y = 0$ or $x + my = 0$. For $a = b$, if $\phi(A_1) = A_i$, then $\phi(A_2) = A_{i+1}$ for $i \in \{1,2,3,4\}$, and ϕ is a rotation with the angle $\theta = \frac{(1-i)\pi}{2}$. If $\phi(A_1) = A_{i+1}$, then $\phi(A_2) = A_i$ for $i \in \{2,4\}$, and ϕ is a reflection in line $mx - y = 0$, $x + my = 0$, $(1+m)x - (1-m)y = 0$ or $(1-m)x + (1+m)y = 0$. $\square$

Consequently, we have the orthogonal group $O_{T_g(m)}(2) = R_{T_g(m)} \cup S_{T_g(m)}$. Note that there are four more isometries of $\mathbb{R}^2_{T_g(m)}$ such that $\Psi_1(A_1,A_2,A_3,A_4) = (A_2,A_1,A_4,A_3)$, $\Psi'_1(A_1,A_2,A_3,A_4) = (A_4,A_3,A_2,A_1)$, $\Psi_2(A_1,A_2,A_3,A_4) = p(A_2,A_3,A_4,A_1)$, $\Psi'_2(A_1,A_2,A_3,A_4) = (A_4,A_1,A_2,A_3)$ and $\Psi_i(O) = \Psi'_i(O) = O$: $\Psi_i(x,y) = (U_i,V_i)$, $\Psi'_i(x,y) = (-U_i,-V_i)$, where $U_1 = \frac{(b^2-a^2m^2)y-m(a^2+b^2)x}{ab(1+m^2)}$, $V_1 = \frac{m(a^2+b^2)y+(a^2-b^2m^2)x}{ab(1+m^2)}$, $U_2 = \frac{(-b^2-a^2m^2)y-m(a^2-b^2)x}{ab(1+m^2)}$, $V_2 = \frac{m(a^2-b^2)y+(a^2+b^2m^2)x}{ab(1+m^2)}$. One can check that these transformations do not preserve the Euclidean distance for $a \neq b$, while they preserve the m-generalized taxicab distance! If $a = b$, then $\Psi_1 = \sigma_{(1+m)x-(1-m)y=0}$, $\Psi'_1 = \sigma_{(1-m)x+(1+m)y=0}$, $\Psi_2 = \rho_{O,\frac{\pi}{2}}$ and $\Psi'_2 = \rho_{O,\frac{3\pi}{2}}$.

Theorem 4.1. *Let* $f : \mathbb{R}^2_{T_g(m)} \to \mathbb{R}^2_{T_g(m)}$ *be a Euclidean isometry preserving the m-generalized taxicab distance. Then there exists a unique* $T_a \in T(2)$ *and* $\phi \in O_{T_g(m)}(2)$ *such that* $f = T_a \circ \phi$.

Proof. Suppose that $f(O) = A$ where $a = (a_1,a_2)$. Define $\phi = T_{-a} \circ f$. It is clear that ϕ is an isometry and $\phi(O) = O$. Thus, we get $\phi \in O_{T_g(m)}(2)$ by Proposition 4.4, and $f = T_a \circ \phi$. The proof of uniqueness is trivial. $\square$

Finally, by Theorem 4.1 we determine the group of Euclidean isometries preserving the m-generalized taxicab distance is semidirect product of the translation group $T(2)$ consisting of all translations and the symmetry group of the m-generalized taxicab unit circle $O_{T_g(m)}(2)$.

References

[1] Çolakoğlu, H.B. and Kaya, R. *A generalization of some well-known distances and related isometries*, Math. Commun. Vol:16 (2011), 21-35.
[2] Ekmekçi, S., Bayar, A. and Altıntaş, A.K. *On the group of isometries of the generalized taxicab plane*, International Journal of Contemporary Mathematical Sciences Vol:10, No.4 (2015), 159-166.
[3] Ekmekçi, S., Akça, Z. and Altıntaş, A.K. *On trigonometric functions and norm in the generalized taxicab metric*, Mathematical Sciences And Applications E-Notes Vol:3, No.2 (2015), 27-33.
[4] Krause, E.F. *Taxicab Geometry*, Addison-Wesley, Menlo Park, California, 1975.
[5] Menger, K. *You Will Like Geometry*, Guidebook of Illinois Institute of Technology Geometry Exhibit, Museum of Science and Industry, Chicago, Illinois, 1952.
[6] Richard, S.M. and George, D.P. *Geometry, A Metric Approach with Models*, Springer-Verlag, New York, 1981.
[7] Schattschneider, D.J. *The taxicab group*, American Mathematical Monthly Vol:91, No.7 (1984), 423-428.
[8] Wallen, L.J. *Kepler, the taxicab metric, and beyond: An isoperimetric primer*, The College Mathematics Journal Vol:26, No.3 (1995), 178-190.