Musical Isomorphisms on the Semi-Tensor Bundles

Semra Yurttaçkanmaz* and Furkan Yıldırım

1Department of Mathematics, Faculty of Sci., Atatürk University, 25240, Erzurum, Turkey
2Department of Mathematics, Faculty of Sci., Atatürk University, Narman Vocational Training School, 25530, Erzurum, Turkey

*Corresponding author E-mail: semrakaya@atauni.edu.tr

Abstract

We transfer vertical lifts and complete lifts of some tensor fields from the semi-tangent bundle to the semi-cotangent bundle using a musical isomorphism between these bundles. In this article, we also analyze complete lift of vector and affinor (tensor of type (1, 1)) fields for semi-tangent (pull-back) bundle to M. We denote by \( t^*M \) using a musical isomorphism \( \delta \) and \( B \) the musical (natural) isomorphisms of any pseudo-Riemannian metric \( g \).

1. Introduction

Let \((B_m, g)\) be a smooth pseudo-Riemannian manifold of dimension \( m \). We denote by \( t(B_m) \) and \( t^*(B_m) \) the semi-tangent [9], [10], [1] and semi-cotangent bundles [3], [4] over \( B_m \) with local coordinates \((x^a, x^\alpha, x^\beta) = (x^a, x^\alpha, y^\beta)\) and \((x^a, x^\alpha, y^\beta) = (x^a, x^\alpha, p_\beta)\), \( a, b, \ldots = 1, \ldots, n - m; \alpha, \beta, \ldots = n - m + 1, \ldots, n; \bar{\alpha}, \bar{\beta}, \ldots = n + 1, \ldots, n + m \), respectively, where \( y^\beta = y^\beta \frac{\partial}{\partial x^\beta} \in t_x(B_m) \) and \( p_\beta = p_\beta dx^\beta \in t^*_x(B_m), \forall x \in B_m \).

We know that the mappings \( g^\alpha : t(B_m) \to t^*(B_m) \) and \( g^\beta : t^*(B_m) \to t(B_m) \) between the semi-tangent and semi-cotangent bundles determine the musical (natural) isomorphisms of any pseudo-Riemannian metric \( g \).

The musical isomorphisms \( g^\alpha \) and \( g^\beta \) have respectively components

\[
g^\alpha : x^I = (x^a, x^\alpha, x^\beta) = (x^a, x^\alpha, y^\beta) \to \tilde{x}^I = (x^b, x^\beta, y^\bar{\beta}) \]

\[
= (\delta^{ab}_a, \delta^\beta_b, \alpha, p_\beta = g_{\beta\alpha} \delta^\beta_a)
\]

and

\[
g^\beta : \tilde{x}^I = (x^b, x^\beta, y^\bar{\beta}) \to x^I = (x^a, x^\alpha, x^\beta)
\]

\[
= (\delta^{ab}_b, \delta^\alpha_a, \delta^\beta_b, y^\alpha = g^{ab} p_\beta)
\]

with respect to the local coordinates, where \( \delta \) is the Kronecker delta. The Jacobian of \( g^\alpha \) and \( g^\beta \) are given by

\[
(g^\alpha) = \left( \frac{\partial x^I}{\partial \tilde{x}^J} \right) = \left( \begin{array}{ccc} \delta^b_a & 0 & 0 \\ 0 & \delta^\beta_b & 0 \\ \delta^\alpha_a & \delta^\beta_b & g_{\beta\alpha} \end{array} \right)
\]

(1.1)

and

\[
(g^\beta) = \left( \frac{\partial \tilde{x}^J}{\partial x^I} \right) = \left( \begin{array}{ccc} \delta^b_a & 0 & 0 \\ 0 & \delta^\alpha_a & 0 \\ \delta^\beta_b & \delta^\alpha_a & g^{ab} \end{array} \right)
\]

(1.2)

respectively. Where \( I = (a, \alpha, \bar{\alpha}), J = (b, \beta, \bar{\beta}) \).

We denote by \( \mathcal{D}^p_Q(t(B_m)) \) and \( \mathcal{D}^p_Q(t^*(B_m)) \) the modules over \( F(t(B_m)) \) and \( F(t^*(B_m)) \) of all tensor fields of type \((p,q)\) on \( t(B_m) \) and \( t^*(B_m) \), respectively, where \( F(t(B_m)) \) and \( F(t^*(B_m)) \) denote the rings of real-valued \( C^\infty \) functions on \( t(B_m) \) and \( t^*(B_m) \), respectively. On the
other hand, if \( x' = (x', x', x') \) is another system of local adapted coordinates in the semi-tangent bundle \( t(B_m) \), then we have (see, for details [1])

\[
\begin{align*}
\dot{x}' &= \dot{x}'(\dot{x}, \dot{x}), \\
\tilde{x}' &= \tilde{x}'(\tilde{x}), \\
\Pi &= \frac{\partial x'}{\partial x}
\end{align*}
\]

(1.3)

The Jacobian of (1.3) has components [1]

\[
\tilde{A} = (A_f^i) = \begin{pmatrix}
A_{β}^α & A_{β}^α & 0 \\
0 & A_{β}^α & 0 \\
0 & A_{β}^α & A_{β}^α
\end{pmatrix}
\]

(1.4)

where

\[
A_{β}^α = \frac{\partial x'}{\partial x β}, A_{β}^α = \frac{\partial^2 x'}{\partial x β \partial x α}.
\]

Let \( \tilde{c}_X \in \mathcal{H}^1(t(B_m)) \) and \( c_F \in \mathcal{H}^1(t(B_m)) \) be complete lifts of tensor fields \( \tilde{X} \in \mathcal{H}^1(M_n) \) and \( F \in \mathcal{H}^1(M_n) \) to the semi-tangent bundle \( t(B_m) \), where \( M_n \) denotes the fiber bundle [9], [11], [1] over a manifold \( B_m \). In this paper we transfer via the differential (\( g^* \)) the complete lifts \( (\tilde{c}_X \in \mathcal{H}^1(t(B_m)), c_F \in \mathcal{H}^1(t(B_m))) \) and some tensor fields that the \( \gamma \)-operator is applied from the semi-tangent bundle \( t(B_m) \) to semi-cotangent bundle \( t^*(B_m) \). On the other hand, we know that the semi-tangent \( t(B_m) \) and semi-cotangent bundles \( t^*(B_m) \) are a pull-back (induced) bundle of \( T(B_m) \) and \( T^*(B_m) \), respectively [2], [5], [7], [4]. We note that musical isomorphism and its applications were studied in [8]. The main purpose of this paper is to study musical isomorphism between semi-tangent bundles and semi-cotangent bundles. Where \( T(B_m) = \bigcup_{t \in B_m} T_t(B_m) \) and \( T^*(B_m) = \bigcup_{t \in B_m} T^*_t(B_m) \) respectively denote the tangent and cotangent bundles over \( B_m \) [6].

2. Transfer of vertical lifts of vector fields

Let \( X \in \mathcal{H}^1(M_n) \), i.e. \( X = X^α \partial α \). On putting

\[
\gamma X_i = (\gamma X^α)_i = \begin{pmatrix} 0 & 0 & X^α \end{pmatrix},
\]

(2.1)

from (1.4), we easily see that \( (\gamma X^α)_i = \tilde{A}(\gamma X^α) \). The vector field \( \gamma X \) is called the vertical lift of \( X \) to the semi-tangent bundle \( t(B_m) \). Then, using (1.1) and (2.1)

\[
\gamma X_i = \begin{pmatrix}
0 \\
0 \\
X^α
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
\tilde{A}_β^α X^α
\end{pmatrix}
\]

(3.1)

where \( (\gamma pα)_i \) is a Liouville covector field [4] on the semi-cotangent bundle \( t^*(B_m) \).

3. Transfer of complete lifts of vector fields

Let \( \tilde{X} \in \mathcal{H}^1(M_n) \) be a projectable vector field [11] with projection \( X = X^α(\partial α) \partial α \). i.e. \( \tilde{X} = \tilde{X}^α(\partial α) (\partial α) + X^α(\partial α) \partial α \). Then the complete lift \( \tilde{c}_X \) of \( \tilde{X} \) to the semi-tangent bundle \( t(B_m) \) is given by [1]

\[
\tilde{c}_X = \begin{pmatrix}
\tilde{X}^α \\
\gamma X^α
\end{pmatrix}
\]

(3.1)

with respect to the coordinates \( (x', x', x') \).

Using (1.1) and (3.1), we have

\[
\gamma X_i = \begin{pmatrix}
\tilde{X}^α \\
\gamma X^α
\end{pmatrix} = \begin{pmatrix}
\tilde{X}^α \\
\gamma X^α
\end{pmatrix} = \begin{pmatrix}
\tilde{X}^α \\
\gamma X^α
\end{pmatrix}
\]

(3.1)
\[ \begin{align*}
\left( \begin{array}{c}
\tilde{X}^b \\
\tilde{X}^\beta \\
y^\phi (L_Xg)_{\xi \beta} - p_a (\partial_\phi X^\alpha)
\end{array} \right),
\end{align*} \]

where \( L_X \) is the Lie derivation of \( g \) with respect to \( X \):

\[ (L_Xg)_{\xi \beta} = X^\alpha \partial_\alpha g_{\xi \beta} + (\partial_\phi X^\alpha)_{B\alpha} + (\partial_\phi X^\alpha)_{g\phi \alpha}. \]

In a manifold \((M, g)\), a vector field \( X \) is called a Killing vector field if \( L_Xg = 0 \). It is well known that the complete lift \( ^c\tilde{X}_r \) of \( \tilde{X} \) to the semi-cotangent bundle \( ^t(B_m) \) is given by \([4]\)

\[ ^cX_r = \left( \begin{array}{c}
\tilde{X}^a \\
\tilde{X}^\alpha \\
-p_\phi (\partial_\phi X^\xi)
\end{array} \right), \]

with respect to the coordinates \((x^a, x^\alpha, x^\phi)\).

We have from \((3.2)\)

\[ ^{g}^cX_r = ^cX_r + \gamma(L_Xg). \]

where \( \gamma(L_Xg) \) is defined by

\[ \gamma(L_Xg) = \left( \begin{array}{c}
0 \\
0 \\
y^\phi (L_Xg)_{\xi \beta}
\end{array} \right). \]

Thus, we have:

**Theorem 1.** Let \((B_m, g)\) be a pseudo-Riemannian manifold, and let \(^c\tilde{X}_r\) and \(^cX_r\) be complete lifts of a vector field \( \tilde{X} \in \mathfrak{X}(M_n) \) to the semi-tangent and semi-cotangent bundles, respectively. Then the differential (pushforward) of \(^c\tilde{X}_r\) by \( g \) coincides with \(^cX_r\), i.e. \(^{g}^cX_r = ^cX_r\) if and only if \( \tilde{X} \) is a Killing vector field.

**Theorem 2.** Let \( \tilde{X}, \tilde{Y} \in \mathfrak{X}(M_n) \). For the Lie product, we have

\[ [^c\tilde{X}_r, ^c\tilde{Y}_f] = ^c[\tilde{X}, \tilde{Y}]. \]

in the semi-tangent bundle \( t(B_m) \).

**Proof.** If \( \tilde{X}, \tilde{Y} \in \mathfrak{X}(M_n) \) and \( \left[ \begin{array}{c}
[^c\tilde{X}_r, ^c\tilde{Y}_f]^b \\
[^c\tilde{X}_r, ^c\tilde{Y}_f]^\beta \\
[^c\tilde{X}_r, ^c\tilde{Y}_f]^\phi
\end{array} \right] \) are components of \([^c\tilde{X}_r, ^c\tilde{Y}_f]\) with respect to the coordinates \((x^b, x^\beta, x^\phi)\) on \( t(M_n) \), then we have

\[ [^c\tilde{X}_r, ^c\tilde{Y}_f] = (^c\tilde{X}_r)^b \partial_b (^c\tilde{Y}_f)^b - (^c\tilde{Y}_f)^b \partial_b (^c\tilde{X}_r)^b. \]

Firstly, if \( J = b \), we have

\[ [^c\tilde{X}_r, ^c\tilde{Y}_f]^b = (^c\tilde{X}_r)^b \partial_b (^c\tilde{Y}_f)^b - (^c\tilde{Y}_f)^b \partial_b (^c\tilde{X}_r)^b = (^c\tilde{X}_r)^a \partial_a (^c\tilde{Y}_f)^b + (^c\tilde{X}_r)^a \partial_a (^c\tilde{Y}_f)^b + (^c\tilde{X}_r)^\phi \partial_\phi (^c\tilde{Y}_f)^b - (^c\tilde{Y}_f)^a \partial_a (^c\tilde{X}_r)^b - (^c\tilde{Y}_f)^\phi \partial_\phi (^c\tilde{X}_r)^b = (^c\tilde{X}_r)^a \partial_a (^c\tilde{Y}_f)^b - (^c\tilde{Y}_f)^a \partial_a (^c\tilde{X}_r)^b = X^a \partial_a (^c\tilde{Y}_f)^b - Y^a \partial_a (^c\tilde{X}_r)^b = X^a \partial_a \tilde{Y}^b - Y^a \partial_a \tilde{X}^b = [\tilde{X}, \tilde{Y}]. \]
by virtue of (3.1). Thirdly, if \( J = \overline{\beta} \), then we have
\[
\left[\overline{cc}X, \overline{cc}Y\right] = \left(\overline{cc}X\right)^i \partial_i \left(\overline{cc}Y\right)^\beta - \left(\overline{cc}Y\right)^i \partial_i \left(\overline{cc}X\right)^\beta
\]
\[
= \left(\overline{cc}X\right)^i \partial_i \left(\overline{cc}Y\right)^\beta + \left(\overline{cc}Y\right)^i \partial_i \left(\overline{cc}X\right)^\beta + \left(\overline{cc}X\right)^i \partial_i \left(\overline{cc}Y\right)^\beta - \left(\overline{cc}Y\right)^i \partial_i \left(\overline{cc}X\right)^\beta
\]
\[
= X^\alpha \partial_\alpha \left(\overline{cc}Y\right)^\beta + Y^\alpha \partial_\alpha \left(\overline{cc}X\right)^\beta - Y^\alpha \partial_\alpha \left(\overline{cc}Y\right)^\beta - X^\alpha \partial_\alpha \left(\overline{cc}X\right)^\beta
\]
\[
= \gamma^\alpha \partial_\alpha \left(\overline{cc}Y\right)^\beta - \gamma^\alpha \partial_\alpha \left(\overline{cc}X\right)^\beta
\]
by virtue of (3.1). On the other hand, we know that \( \overline{cc}[X,Y] \) have components
\[
\overline{cc}[X,Y] = \left(\begin{array}{c}
\left[X,Y\right]^\beta \\
\left[X,Y\right]^\beta \\
\gamma^\alpha \partial_\alpha \left[X,Y\right]^\beta
\end{array}\right)
\]
with respect to the coordinates \((\beta, \alpha, \gamma)\) on \( t(M_n) \).

Thus, we have \([\overline{cc}X, \overline{cc}Y] = \overline{cc}[X,Y] \) in \( t(B_m) \).

Let \( \overline{X} \) and \( \overline{Y} \) be a Killing vector fields on \( M_n \). Then we have
\[
L_{[\overline{X},\overline{Y}]}g = [L_{\overline{X}}L_{\overline{Y}}]g = L_{\overline{X}} \circ L_{\overline{Y}}g - L_{\overline{Y}} \circ L_{\overline{X}}g = 0,
\]
i.e. \([\overline{X},\overline{Y}]) is a Killing vector field. Since \( \overline{cc}[\overline{X},\overline{Y}] = [\overline{cc}X, \overline{cc}Y] = \gamma^\alpha \partial_\alpha \left[\overline{cc}X, \overline{cc}Y\right] \) (see [4]), from Theorem 1. and Theorem 2. we have

Theorem 3. If \( \overline{X} \) and \( \overline{Y} \) be a Killing vector fields on \( M_n \), then
\[
g^\alpha_{\gamma}[\overline{cc}X, \overline{cc}Y] = [\overline{cc}X, \overline{cc}Y],
\]
where \( g^\alpha_{\gamma} \) is a differential (pushforward) of musical isomorphism \( g^\alpha \).

4. Transfer of \((\gamma F)_i\) and \((\gamma T)_i\)

For any \( F \in \mathfrak{F}(B_m) \), if we take account of (1.4), we can prove that \((\gamma F)_i = \overline{\alpha}(\gamma F)_i\), where \((\gamma F)_i\) is a vector field on the semi-tangent bundle \( t(B_m) \) defined by
\[
(\gamma F)_i = \left(\gamma F^\alpha\right)_i = \left(\begin{array}{c}
0 \\
0 \\
\gamma^\alpha F^\alpha_i
\end{array}\right)
\]
with respect to the coordinates \((\alpha, \alpha, \beta)\) . On the other hand, vector field \((\gamma F)_\gamma\) on the semi-cotangent bundle \( t^*(B_m) \) is defined by [4]:
\[
(\gamma F)_\gamma = \left(\gamma F^\alpha\right)_\gamma = \left(\begin{array}{c}
0 \\
0 \\
p^{\alpha} F^\alpha_{\gamma}
\end{array}\right)
\]
Let \( T \in \mathfrak{S}(B_m) \). On putting
\[
(\gamma T)_i = \left(\gamma T^\alpha\right)_i = \left(\begin{array}{c}
0 \\
0 \\
\gamma^\alpha T^\alpha_{\beta}
\end{array}\right),
\]
from (1.4), we easily see that \( \left(\gamma T^\alpha\right)_i = A^\beta_{\gamma} A^\alpha_{\beta} \left(\gamma T\right)_\gamma \), where \( (\overline{A})^{-1} = (A^\alpha_{\beta}) \) is the inverse matrix of \( \overline{A} \).

Theorem 4. If \( F \in \mathfrak{F}(B_m) \) and \( T \in \mathfrak{S}(B_m) \), then
\[
(i) \quad g^\alpha_{\gamma}\left(\gamma F\right)_i = (\gamma F)_\gamma,
(ii) \quad g^\alpha_{\gamma}\left(\gamma T\right)_i = (\gamma T)_\gamma.
\]
Proof. (i) From (1.1) and (4.1), we have:
\[
g^i_j (\gamma F)_t = \begin{pmatrix}
\delta^i_a & 0 & 0 \\
0 & \delta^i_b & 0 \\
y^i \frac{\partial x_\beta}{\partial x^i} & s_{\beta \alpha} & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
y^i F^\alpha_j
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
y^i s_{\beta \alpha} F^\alpha_j
\end{pmatrix} = (\gamma F)_t.
\]

It is well known that \((\gamma F)_t\) have components [4]:
\[
(\gamma F)_t = \begin{pmatrix}
0 \\
0 \\
p a F^\alpha_j
\end{pmatrix}
\]
with respect to the coordinates \((x^\alpha, x^\alpha, x^\alpha)\) on the semi-cotangent bundle \(t^*(B_m)\). Thus, we have (i) of Theorem 4.

(ii) For simplicity, we take \(g^i_j (\gamma T)_t = (\gamma T^f)_t\). In fact,
\[
(\gamma T^f)_t = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
p c T^\beta_\alpha & 0
\end{pmatrix}
\]
with respect to the coordinates \((x^\alpha, x^\alpha, x^\alpha)\). Thus, we have \(g^i_j (\gamma T)_t = (\gamma T)_t\).

5. Complete lift of affinor fields

Let \(\bar{F} \in \mathfrak{X} (M)\) be a projectable affinor field [10] with projection \(F = F^\alpha_\beta (x^\alpha) \partial_\alpha \otimes dx^\beta\), i.e. \(\bar{F}\) has components
\[
\bar{F} = \begin{pmatrix}
\bar{F}^\alpha_\beta \\
0
\end{pmatrix}
\]
which respects the coordinates \((x^\alpha, x^\alpha)\). On putting
\[
(\overset{\circ}{\bar{F}})_t = \begin{pmatrix}
\overset{\circ}{F}^\alpha_\beta \\
0
\end{pmatrix}
\]
we easily see that \(\overset{\circ}{\bar{F}}_t = A^\alpha_f A^\beta_f (\overset{\circ}{\gamma} F)_t\).
We call \(\overset{\circ}{\gamma} \bar{F}_t\) the complete lift of the tensor field \(\bar{F}\) of type (1,1) to the semi-tangent bundle \(t(B_m)\).

Proof. For simplicity, put \(l = \bar{\alpha}, j = \beta\) in \(\overset{\circ}{\gamma} F^\alpha_f\) and take account of (1.4) and (5.1), we obtain
\[
(\overset{\circ}{\gamma} F^\alpha_\beta)_t = A^\alpha_\alpha A^\beta_\beta F^\alpha_\beta + A^\alpha_\sigma A^\beta_\sigma F^\alpha_\beta + A^\alpha_\beta \bar{F}^\alpha_\beta + A^\alpha_\beta \overset{\circ}{\gamma} A^\beta_\sigma F^\alpha_\sigma + A^\alpha_\beta \overset{\circ}{\gamma} A^\beta_\sigma F^\alpha_\sigma + A^\alpha_\beta \overset{\circ}{\gamma} F^\alpha_\beta
\]

Similarly, we can easily find another components of \(\overset{\circ}{\gamma} F^\alpha_\beta\).

\[\square\]
6. Transfer of complete lifts of affinor fields

Let \( \tilde{F} \) be projectable affinor fields [10] on \( M_a \) with projection \( F \) on \( B_m \). Using (1.1), (1.2) and (5.1), we have

\[
\dot{g}^i_{\alpha}(\varepsilon_{\dot{F}})_{\dot{I}J} = \dot{A}^i_{\mu} \dot{A}^j_{\nu} \dot{F}^K_{\dot{L}} \tag{6.1}
\]

where \( \dot{g}^i_{\alpha} = (\dot{g}_{\alpha\beta}) \) and \( \dot{g}^{-1} = (\dot{g}^{\alpha\beta}) \) are pure tensor fields with respect to \( F \), we find

\[
\dot{g}_{\beta\alpha} \theta^\beta = \dot{g}_{\beta\alpha} \theta^\alpha \tag{6.2}
\]

and

\[
\begin{align*}
\dot{g}^i_{\alpha}(\varepsilon_{\dot{F}})_{\dot{I}J} & = \dot{F}^a_{\dot{I}} \dot{F}^{\dot{J}}_{\dot{B}} + \dot{F}^a_{\dot{I}} \dot{F}^{\dot{J}}_{\dot{B}} + \dot{F}^a_{\dot{I}} \dot{F}^{\dot{J}}_{\dot{B}} \\
& = \dot{g}^i_{\alpha}(\varepsilon_{\dot{F}})_{\dot{I}J} + \dot{g}^i_{\alpha}(\varepsilon_{\dot{F}})_{\dot{I}J} + \dot{g}^i_{\alpha}(\varepsilon_{\dot{F}})_{\dot{I}J} \\
& = \dot{g}^i_{\alpha}(\varepsilon_{\dot{F}})_{\dot{I}J} + \dot{g}^i_{\alpha}(\varepsilon_{\dot{F}})_{\dot{I}J} + \dot{g}^i_{\alpha}(\varepsilon_{\dot{F}})_{\dot{I}J}. \tag{6.3}
\end{align*}
\]

Where \( \dot{I} = (a, \alpha, \dot{\alpha}), \dot{J} = (b, \beta, \dot{\beta}), \dot{K} = (c, \theta, \dot{\theta}), \dot{L} = (d, \sigma, \dot{\sigma}) \). Also, the component \( (\varepsilon_{\dot{F}})_{\dot{I}J} \) of \( (\varepsilon_{\dot{F}})_{\dot{I}J} \) is defined as Tachibana operator \( \phi_{\theta \gamma} \) of \( F \), i.e.,

\[
\dot{g}^i_{\alpha}(\varepsilon_{\dot{F}})_{\dot{I}J} = \dot{g}^i_{\alpha}(\varepsilon_{\dot{F}})_{\dot{I}J} + \dot{g}^i_{\alpha}(\varepsilon_{\dot{F}})_{\dot{I}J} + \dot{g}^i_{\alpha}(\varepsilon_{\dot{F}})_{\dot{I}J}. \tag{6.4}
\]

It is well known that the complete lift \( (\varepsilon_{\dot{F}})_{\dot{I}J} \) of \( \tilde{F} \in \mathfrak{S}^1(M_a) \) to the semi-tangent bundle \( \mathfrak{t}^*(B_m) \) is given by [4]

\[
(\varepsilon_{\dot{F}})_{\dot{I}J} = \left( \begin{array}{ccc}
\dot{F}^a_{\dot{I}} & \dot{F}^a_{\dot{J}} & 0 \\
0 & \dot{F}^a_{\dot{I}} & \dot{F}^a_{\dot{J}} \\
0 & 0 & \dot{F}^{\dot{I}}_{\dot{J}}
\end{array} \right) \tag{6.5}
\]

with respect to the coordinates \((\varepsilon^a, \varepsilon^\alpha, \varepsilon^{\dot{\alpha}})\) on \( \mathfrak{t}^*(B_m) \). From (6.4) and (6.5), we easily obtain

\[
\dot{g}^i_{\alpha}(\varepsilon_{\dot{F}})_{\dot{I}J} = (\varepsilon_{\dot{F}})_{\dot{I}J} + \gamma(\phi_{\theta \gamma}).
\]

where

\[
\gamma(\phi_{\theta \gamma}) = \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array} \right) \tag{6.6}
\]

Finally, we can prove

**Theorem 5.** Let \((\varepsilon_{\dot{F}})_{\dot{I}J}\) and \((\varepsilon_{\dot{F}})_{\dot{I}J}\) be complete lifts of \( \tilde{F} \in \mathfrak{S}^1(M_a) \) to the semi-tangent and semi-cotangent bundles, respectively. Then the differential of \((\varepsilon_{\dot{F}})_{\dot{I}J}\) by \( \dot{g}^i_{\alpha} \) coincides with \((\varepsilon_{\dot{F}})_{\dot{I}J}\), i.e. \( \dot{g}^i_{\alpha}(\varepsilon_{\dot{F}})_{\dot{I}J} = (\varepsilon_{\dot{F}})_{\dot{I}J} \) if and only if \( \phi_{\theta \gamma} = 0 \).
References