Konuralp Journal of Mathematics, 13 (1) (2025) 108-116

Konuralp Journal of Mathematics

Research Paper
Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

Representation of Solutions to a Two-Dimensional System of Difference Equations

Mehmet Gümüş[1], R. Abo-Zeid[2] and Ana C. Carapito[3*]

[1]*Department of Mathematics, Faculty of Science, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Türkiye*
[2]*Department of Basic Science, The Higher Institute for Engineering & Technology, Al-Obour, Cairo, Egypt*
[2]*Center of Mathematics and Applications of University of Beira Interior (CMA-UBI), Department of Mathematics, University of Beira Interior, Portugal*
[*]*Corresponding author*

Abstract

In this paper, we conduct a comprehensive investigation to introduce a representation to the well-defined solutions of the following system of higher-order difference equations

$$x_{n+1} = \frac{y_{n-1}(a_1 y_{n-1} + b_1 x_{n-3})}{c_1 x_{n-3} + d_1 y_{n-1}}, \quad y_{n+1} = \frac{x_{n-1}(a_2 x_{n-1} + b_2 y_{n-3})}{c_2 y_{n-3} + d_2 x_{n-1}}, \quad n = 0, 1, \ldots,$$

where a_i, b_i, c_i, d_i, $i = 1, 2$, and the initial values $x_{-3}, \ldots, x_0, y_{-3}, \ldots, y_0$ are real numbers such that $|b_1| + |b_2| + |c_1| + |c_2| \neq 0$. Finally, the theoretical findings of the study are supported by some numerical examples.

Keywords: *Well-defined solutions, forbidden sets, system of difference equations, numerical simulations*
2010 Mathematics Subject Classification: *39A05, 39A20*

1. Introduction

One of the fundamental structures of applied mathematics is the difference equations that are used to understand the behavior of models defined in discrete time periods, to reveal their properties and to analyze them. Difference equations are also used to build real-life mathematical models. Difference equations help to understand many areas from biology to medicine, from economics to physics (see [1]). In other words, they are a part of discrete mathematics that describes changes in small time periods.
The history of difference equations dates back to ancient times. Although they are seen as discrete structures of differential equations, they have a much older history. Difference equations first appeared in 1202 in the rabbit problem in the work "Liber Abaci (Abacus)" by the famous Italian mathematician Fibonacci. In addition, in the 17th century, many mathematicians used difference equations to understand different dynamic systems. However, the foundation of the theory used today was laid by the important research of the French mathematician Abraham de Moivre in the 18th century. De Moivre investigated the repetitive sequences that are important in probability and conducted research to understand how systems behave when repeated events occur (see [2, 3]).
Difference equations are used in many scientific fields because they provide unique ways to model and analyze real-life problems (see [4]). In biology and ecology, they are the most effective tools for studying population change, how species interact, and how diseases spread over time (see [5]). By modeling populations with these factors, such as birth rates, death rates, and migration, difference equations help predict population growth, extinction risks, and disease spread (see [6]) Therefore, it enables better decisions to be made in taking measures to continue human life and protect. In medical sciences, problems related to physiological processes and disease progression, especially the procedure of drug behavior in the diseased body and how diseases progress, are interpreted for the necessary control using difference equations. Mathematical models built in this field analyze treatments and interventions, allowing the improvement of many factors such as health plans, drug dosage, and patient treatment. With these models, the spread of epidemic diseases can be predicted and analyzed. In addition, the response of patients to treatment and how health resources are beneficial to the patient can be evaluated. Many phenomena in engineering sciences are calculated using difference equations. Mathematical models are used in complex structures such as electrical circuits, machines, and communication networks. Engineers can simulate how the systems they deal with work under different inputs through

Email addresses: **m.gumus@beun.edu.tr (M. Gümüş), abuzead73@yahoo.com (R. Abo-Zeid), ana.carapito@ubi.pt, (Ana C. Carapito)**

these mathematical models. In economics, difference equations have a significant place in the procedure of estimating financial trends and analyzing policies for economic development (see [7]). Likewise, in physics, especially in quantum mechanics and computational physics, difference equations are used to model many dynamic systems. Here, its use is to convert continuous mathematical models into discrete forms and thus to perform simulations of these models. In other words, in many branches of science, difference equations are used as a very powerful mathematical tool to study the behavior of dynamic systems. For this and many other reasons, many scientists have taken up difference equations as a field of study and have produced very good studies. In [8], the authors studied the difference equation

$$x_{n+1}=\frac{x_n(ax_{n-1}+bx_n)}{cx_{n-1}+dx_n},\ n=0,1,...,$$

where the parameters and the initial conditions are real numbers. In [9], Stevic et al. studied the system of difference equations

$$x_n=\frac{x_{n-k}(ay_{n-l}+by_{n-k-l})}{cy_{n-l}+dy_{n-k-l}},\quad y_n=\frac{y_{n-k}(\alpha x_{n-l}+\beta x_{n-k-l})}{\gamma x_{n-l}+\delta x_{n-k-l}},\quad n=0,1,\ldots,$$

where l and k are natural numbers, $a,b,c,d,\ \alpha,\beta,\gamma,\delta$ and the initial conditions are real numbers. In [10], Elsayed et al. introduced the form of the solutions of the systems of difference equations

$$w_{n+1}=\frac{s_n(w_{n-3}+s_{n-4})}{s_{n-4}+w_{n-3}-s_n},\quad s_{n+1}=\frac{w_{n-2}(w_{n-2}+s_{n-3})}{2w_{n-2}+s_{n-3}},\quad n=0,1,\ldots,$$

and

$$w_{n+1}=\frac{s_n(s_{n-4}-w_{n-3})}{s_{n-4}-w_{n-3}+s_n},\quad s_{n+1}=\frac{w_{n-2}(s_{n-3}-w_{n-2})}{s_{n-3}},\quad n=0,1,\ldots.$$

In [11], Kara et al. studied the global behavior of the system of difference equations

$$u_{n+1}=f^{-1}\left(g(v_{n-1})\frac{A_1f(u_{n-2})+B_1g(v_{n-4})}{C_1f(u_{n-2}+D_1g(v_{n-4})}\right),$$
$$v_{n+1}=g^{-1}\left(f(u_{n-1})\frac{A_2g(v_{n-2})+B_2f(u_{n-4})}{C_2g(v_{n-2})+D_2f(u_{n-4})}\right),$$

where $n\geq 0$, ${A_r}^2+{B_r}^2\neq 0$, ${C_r}^2+{D_r}^2\neq 0$, $r\in\{1,2\}$, f and g are continuous and strictly monotone functions, $f(R)=R$, $g(R)=R$, $f(0)=0$, $g(0)=0$.
For more related difference equations and systems of difference equations, (see [14]-[29] and the references therein).
Inspired by the above-mentioned works, this paper aims to introduce a representation of the well-defined solutions of the following difference equation system

$$x_{n+1}=\frac{y_{n-1}(a_1y_{n-1}+b_1x_{n-3})}{c_1x_{n-3}+d_1y_{n-1}},\quad y_{n+1}=\frac{x_{n-1}(a_2x_{n-1}+b_2y_{n-3})}{c_2y_{n-3}+d_2x_{n-1}},\quad n=0,\ldots,\tag{1.1}$$

where a_i,b_i,c_i,d_i, $i=1,2$, and the initial values $x_{-3},...,x_0,y_{-3},...,y_0$ are real numbers such that $|b_1|+|b_2|+|c_1|+|c_2|\neq 0$.
We remark here that system (1.1) can be written as

$$x_{n+1}=\alpha_1y_{n-1}+\frac{\beta_1y_{n-1}x_{n-3}}{\gamma_1x_{n-3}+\delta_1y_{n-1}},\quad y_{n+1}=\alpha_2x_{n-1}+\frac{\beta_2x_{n-1}y_{n-3}}{\gamma_2y_{n-3}+\delta_2x_{n-1}},\quad n=0,\ldots,\tag{1.2}$$

where $\alpha_1=\frac{a_1}{d_1}$, $\beta_1=b_1-\frac{a_1c_1}{d_1}$, $\gamma_1=c_1$, $\delta_1=d_1$, $\alpha_2=\frac{a_2}{d_2}$, $\beta_2=b_2-\frac{a_2c_2}{d_2}$, $\gamma_2=c_2$ and $\delta_2=d_2$. We mention here [12] in which the form (1.2) was used.
Our main results in this article are as follows:

- Deriving a representation of the well-defined solutions of the system (1.1)
- Deriving the forbidden set for system (1.1)
- Providing some special cases of the system (1.1)
- Validating some of the theoretical results numerically.

2. Main Results

In this section, we introduce the solutions of the system (1.1).

If we set

$$u_n=\frac{x_n}{y_{n-2}},\ v_n=\frac{y_n}{x_{n-2}},\ n\geq -1,\tag{2.1}$$

then system (1.1) becomes

$$u_{n+1}=\frac{a_1v_{n-1}+b_1}{d_1v_{n-1}+c_1},\ v_{n+1}=\frac{a_2u_{n-1}+b_2}{d_2u_{n-1}+c_2},$$

and so

$$u_{n+1}=\frac{A_1u_{n-3}+B_1}{C_1u_{n-3}+D_1},\ v_{n+1}=\frac{A_2v_{n-3}+B_2}{C_2v_{n-3}+D_2},\ n\geq 2,\tag{2.2}$$

where

$$A_1=a_1a_2+b_1d_2,\ B_1=a_1b_2+b_1c_2,\ C_1=d_1a_2+c_1d_2,\ D_1=d_1b_2+c_1c_2,$$
$$A_2=a_1a_2+d_1b_2,\ B_2=b_1a_2+c_1b_2,\ C_2=a_1d_2+d_1c_2,\ D_2=b_1d_2+c_1c_2.$$

This implies that

$$u_{4n+i}=\frac{A_1u_{4(n-1)+i}+B_1}{C_1u_{4(n-1)+i}+D_1},\ i=\overline{-1,2},\ n\geq 1, \tag{2.3}$$

and

$$v_{4n+i}=\frac{A_2v_{4(n-1)+i}+B_2}{C_2v_{4(n-1)+i}+D_2}, i=\overline{-1,2},\ n\geq 1. \tag{2.4}$$

In [13], the author represented the solutions of the equation

$$x_{n+1}=\frac{ax_n+b}{cx_n+d},\ n\geq 0.$$

So, when ${A_1}^2+{D_1}^2>2A_1D_1-4B_1C_1$, Equations (2.3) and (2.4) can be represented as

$$u_{4n+i}=\frac{-\mu u_is_{n-1}+(A_1u_i+B_1)s_n}{(C_1u_i-A_1)s_n+s_{n+1}},\ i=\overline{-1,2},\ n\geq 1, \tag{2.5}$$

and

$$v_{4n+i}=\frac{-\mu v_is_{n-1}+(A_2v_i+B_2)s_n}{(C_2v_i-A_2)s_n+s_{n+1}},\ i=\overline{-1,2},\ n\geq 1. \tag{2.6}$$

where $\mu=A_1D_1-B_1C_1$, and the sequence $\{s_n\}_{n=0}^{\infty}$ is the solution of

$$s_{n+1}-(A_1+D_1)s_n+\mu s_{n-1}=0, \tag{2.7}$$
$$s_0=0 \text{ and } s_1=1.$$

Theorem 2.1. *Let $\{x_n,y_n\}_{n=-3}^{\infty}$ be a solution of system (1.1). The solution can be represented in the form*

$$x_{4n+1}=x_{-3}\left[\prod_{l=0}^{n}\left(\frac{-\mu\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}s_{l-1}+\left(A_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}+B_1\right)s_l}{\left(C_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}-A_1\right)s_l+s_{l+1}}\right)\left(\frac{-\mu\frac{A_2y_{-1}+B_2x_{-1}}{C_2y_{-1}+D_2x_{-3}}s_{l-2}+\left(A_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}+B_2\right)s_{l-1}}{\left(C_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}-A_2\right)s_{l-1}+s_l}\right)\right],$$

$$y_{4n+1}=y_{-3}\left[\prod_{l=0}^{n}\left(\frac{-\mu\frac{a_2x_{-1}+b_2y_{-3}}{c_2y_{-3}+d_2x_{-1}}s_{l-1}+\left(A_2\frac{a_2x_{-1}+b_2y_{-3}}{c_2y_{-3}+d_2x_{-1}}+B_2\right)s_l}{\left(C_2\frac{a_2x_{-1}+b_2y_{-3}}{c_2y_{-3}+d_2x_{-1}}-A_2\right)s_l+s_{l+1}}\right)\left(\frac{-\mu\frac{A_1x_{-1}+B_1y_{-3}}{C_1x_{-1}+D_1y_{-3}}s_{l-2}+\left(A_1\frac{A_1x_{-1}+B_1y_{-3}}{C_1x_{-1}+D_1y_{-3}}+B_1\right)s_{l-1}}{\left(C_1\frac{A_1x_{-1}+B_1y_{-3}}{C_1x_{-1}+D_1y_{-3}}-A_1\right)s_{l-1}+s_l}\right)\right],$$

$$x_{4n+2}=x_{-2}\left[\prod_{l=0}^{n}\left(\frac{-\mu\frac{a_1y_0+b_1x_{-2}}{c_1x_{-2}+d_1y_0}s_{l-1}+\left(A_1\frac{a_1y_0+b_1x_{-2}}{c_1x_{-2}+d_1y_0}+B_1\right)s_l}{\left(C_1\frac{a_1y_0+b_1x_{-2}}{c_1x_{-2}+d_1y_0}-A_1\right)s_l+s_{l+1}}\right)\left(\frac{-\mu\frac{A_2y_0+B_2x_{-2}}{C_2y_0+D_2x_{-2}}s_{l-2}+\left(A_2\frac{A_2y_0+B_2x_{-2}}{C_2y_0+D_2x_{-2}}+B_2\right)s_{l-1}}{\left(C_2\frac{A_2y_0+B_2x_{-2}}{C_2y_0+D_2x_{-2}}-A_2\right)s_{l-1}+s_l}\right)\right],$$

$$y_{4n+2}=y_{-2}\left[\prod_{l=0}^{n}\left(\frac{-\mu\frac{a_2x_0+b_2y_{-2}}{c_2y_{-2}+d_2x_0}s_{l-1}+\left(A_2\frac{a_2x_0+b_2y_{-2}}{c_2y_{-2}+d_2x_0}+B_2\right)s_l}{\left(C_2\frac{a_2x_0+b_2y_{-2}}{c_2y_{-2}+d_2x_0}-A_2\right)s_l+s_{l+1}}\right)\left(\frac{-\mu\frac{A_1x_0+B_1y_{-2}}{C_1x_0+D_1y_{-2}}s_{l-2}+\left(A_1\frac{A_1x_0+B_1y_{-2}}{C_1x_0+D_1y_{-2}}+B_1\right)s_{l-1}}{\left(C_1\frac{A_1x_0+B_1y_{-2}}{C_1x_0+D_1y_{-2}}-A_1\right)s_{l-1}+s_l}\right)\right],$$

$$x_{4n+3}=x_{-1}\left[\prod_{l=0}^{n}\left(\frac{-\mu\frac{A_1x_{-1}+B_1y_{-3}}{C_1x_{-1}+D_1y_{-3}}s_{l-1}+\left(A_1\frac{A_1x_{-1}+B_1y_{-3}}{C_1x_{-1}+D_1y_{-3}}+B_1\right)s_l}{\left(C_1\frac{A_1x_{-1}+B_1y_{-3}}{C_1x_{-1}+D_1y_{-3}}-A_1\right)s_l+s_{l+1}}\right)\left(\frac{-\mu\frac{a_2x_{-1}+b_2y_{-3}}{c_2y_{-3}+d_2x_{-1}}s_{l-1}+\left(A_2\frac{a_2x_{-1}+b_2y_{-3}}{c_2y_{-3}+d_2x_{-1}}+B_2\right)s_l}{\left(C_2\frac{a_2x_{-1}+b_2y_{-3}}{c_2y_{-3}+d_2x_{-1}}-D_2\right)s_l+s_{l+1}}\right)\right],$$

$$y_{4n+3}=y_{-1}\left[\prod_{l=0}^{n}\left(\frac{-\mu\frac{A_2y_{1-k}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-1}}s_{l-1}+\left(A_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}+B_2\right)s_l}{\left(C_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}-A_2\right)s_l+s_{l+1}}\right)\left(\frac{-\mu\frac{ay_{1-k}+bx_{1-2k}}{cx_{1-2k}+dy_{1-k}}s_{l-1}+\left(A\frac{ay_{1-k}+bx_{1-2k}}{cx_{1-2k}+dy_{1-k}}+B\right)s_l}{\left(C\frac{ay_{1-k}+bx_{1-2k}}{cx_{1-2k}+dy_{1-k}}-A\right)s_l+s_{l+1}}\right)\right],$$

$$\vdots$$

$$x_{4n+4}=x_0\left[\prod_{l=0}^{n}\left(\frac{-\mu\frac{A_1x_0+B_1y_{-2}}{C_1x_0+D_1y_{-2}}s_{l-1}+\left(A_1\frac{A_1x_0+B_1y_{-2}}{C_1x_0+D_1y_{-2}}+B_1\right)s_l}{\left(C_1\frac{A_1x_0+B_1y_{-2}}{C_1x_0+D_1y_{-2}}-A_1\right)s_l+s_{l+1}}\right)\left(\frac{-\mu\frac{a_2x_0+b_2y_{-2}}{c_2y_{-2}+d_2x_0}s_{l-1}+\left(A_2\frac{a_2x_0+b_2y_{-2}}{c_2y_{-2}+d_2x_0}+B_2\right)s_l}{\left(C_2\frac{a_2x_0+b_2y_{-2}}{c_2y_{-2}+d_2x_0}-A_2\right)s_l+s_{l+1}}\right)\right],$$

$$y_{4n+4}=y_0\left[\prod_{l=0}^{n}\left(\frac{-\mu\frac{A_2y_0+B_2x_{-k}}{C_2y_0+D_2x_{-k}}s_{l-1}+\left(E\frac{A_2y_0+Fx_{-k}}{C_2y_0+D_2x_{-k}}+B_2\right)s_l}{\left(G\frac{A_2y_0+B_2x_{-k}}{C_2y_0+D_2x_{-k}}-A_2\right)s_l+s_{l+1}}\right)\left(\frac{-\mu\frac{a_1y_0+b_1x_{-2}}{c_1x_{-2}+d_1y_0}s_{l-1}+\left(A_1\frac{a_1y_0+b_1x_{-2}}{c_2x_{-2}+d_2y_0}+B_1\right)s_l}{\left(C_1\frac{a_1y_0+b_1x_{-2}}{c_1x_{-2}+d_1y_0}-A_1\right)s_l+s_{l+1}}\right)\right],$$

where the sequence $\{s_n\}_{n=0}^{\infty}$ *is the solution to the equation*

$$s_{n+1}-(A_1+D_1)s_n+\mu s_{n-1}=0,\ s_0=0 \text{ and } s_1=1, n\in\mathbb{N}_0,$$

such that

$$\mu=A_1D_1-B_1C_1.$$

Proof. We prove by induction on n. When $n=0$, if $i=1$, then

$$\begin{aligned}
x_1 &= x_{-3}\left(\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}\right)\left(\frac{(A_2+D_2)\left(\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}\right)-\left(A_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}+B_2\right)}{-C_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}+A_2}\right)\\
&= x_{-3}\left(\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}\right)\times\frac{D_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}-B_2}{-C_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}+A_2}\\
&= x_{-3}\left(\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}\right)\times\frac{y_{-1}}{x_{-3}}=y_{-1}\left(\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}\right).
\end{aligned}$$

Suppose that the representation is true for a certain $n\geq 1$. We first compute

$$A_1x_{4n+1}+B_1y_{4n-1}.$$

For

$$\begin{aligned}
&A_1x_{-3}\prod_{l=0}^{n}\left(\frac{-\mu\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}s_{l-1}+\left(A_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}+B_1\right)s_l}{\left(C_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}-A_1\right)s_l+s_{l+1}}\right)\left(\frac{-\mu\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}s_{l-2}+\left(A_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}+B_2\right)s_{l-1}}{\left(C_2\frac{A_2y_{-1}+Fx_{-3}}{C_2y_{-1}+D_2x_{-3}}-A_2\right)s_{l-1}+s_l}\right)\\
&+B_1y_{-1}\prod_{l=0}^{n}\left(\frac{-\mu\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}s_{l-1}+\left(A_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}+B_2\right)s_l}{\left(C_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}-A_2\right)s_l+s_{l+1}}\right)\left(\frac{-\mu\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}s_{l-1}+\left(A_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}+B_1\right)s_l}{\left(C_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}-A_1\right)s_l+s_{l+1}}\right)\\
&=\left(\prod_{l=0}^{n-1}\left(\frac{-\mu\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}s_{l-1}+\left(A_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}+B_1\right)s_l}{\left(C_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}-A_1\right)s_l+s_{l+1}}\right)\right)\left(\prod_{l=1}^{n}\left(\frac{-\mu\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}s_{l-2}+\left(A_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}+B_2\right)s_{l-1}}{\left(C_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}-A_2\right)s_{l-1}+s_l}\right)\right)\times\\
&\left(A_1x_{-3}\left(\frac{-\mu\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}s_{n-1}+\left(A_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}+B_1\right)s_n}{\left(C_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}-A_1\right)s_n+s_{n+1}}\right)\left(\frac{-\mu\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}s_{-2}+\left(A_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}+B_2\right)s_{-1}}{\left(C_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}-A_2\right)s_{-1}+s_0}\right)+By_{-1}\right).
\end{aligned}$$

Then

$$\begin{aligned}
A_1x_{4n+1}+B_1y_{4n-1}=y_{-1}&\left(\prod_{l=0}^{n-1}\left(\frac{-\mu\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}s_{l-1}+\left(A_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}+B_1\right)s_l}{\left(C_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}-A_1\right)s_l+s_{l+1}}\right)\right)\left(\prod_{l=1}^{n}\left(\frac{-\mu\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}s_{l-2}+\left(A_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}+B_2\right)s_{l-1}}{\left(C_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}-A_2\right)s_{l-1}+s_l}\right)\right)\\
&\times\left(A_1\left(\frac{-\mu\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}s_{n-1}+\left(A_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}+B_1\right)s_n}{\left(C_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}-A_1\right)s_n+s_{n+1}}\right)+B_1\right).
\end{aligned}$$

In the same way, we get

$$\begin{aligned}
C_1x_{4n+1}+D_1y_{4n-1}=y_{-1}&\left(\prod_{l=0}^{n-1}\left(\frac{-\mu\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}s_{l-1}+\left(A_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}+B_1\right)s_l}{\left(C_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}-A_1\right)s_l+s_{l+1}}\right)\right)\left(\prod_{l=1}^{n}\left(\frac{-\mu\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}s_{l-2}+\left(A_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}+B_2\right)s_{l-1}}{\left(C_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}-A_2\right)s_{l-1}+s_l}\right)\right)\\
&\times\left(C\left(\frac{-\mu\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}s_{n-1}+\left(A\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}+B_1\right)s_n}{\left(C_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}-A_1\right)s_n+s_{n+1}}\right)+D_1\right).
\end{aligned}$$

This implies that

$$\frac{A_1x_{4n+1}+B_1y_{4n-1}}{C_1x_{4n+1}+D_1y_{4n-1}}=\frac{A_1\left(\frac{-\mu\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}s_{n-1}+\left(A_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}+B_1\right)s_n}{\left(C_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}-A_1\right)s_n+s_{n+1}}\right)+B_1}{C_1\left(\frac{-\mu\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}s_{n-1}+\left(A_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}+B_1\right)s_n}{\left(C_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}-A_1\right)s_n+s_{n+1}}\right)+D_1}$$

$$=\frac{-\mu A_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}s_{n-1}+\left({A_1}^2+B_1C_1\right)\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}s_n+B_1s_{n+1}}{-\mu C_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}s_{n-1}+\left(\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}\left(C_1A_1+D_1C_1\right)+B_1C_1-D_1A_1\right)s_n+D_1s_{n+1}}$$

$$=\frac{-\mu\left(\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}\right)s_n+\left(A_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}+B_1\right)s_{n+1}}{\left(C_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}-A_1\right)s_{n+1}+s_{n+2}}.$$

Therefore,

$$\begin{aligned}
x_{4(n+1)+1}&=y_{4n+3}\frac{A_1x_{4n+1}+B_1y_{4n-1}}{C_1x_{4n+1}+D_1y_{4n-1}}\\
&=y_{-1}[\prod_{l=0}^{n}(\frac{-\mu\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}s_{l-1}+(A_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}+B_2)s_l}{(C_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}-A_2)s_l+s_{l+1}})(\frac{-\mu\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}s_{l-1}+(A_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-3}}+B_1)s_l}{(C_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}-A_1)s_l+s_{l+1}})]\\
&\times\left(\frac{-\mu(\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}})s_n+(A_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}+B_1)s_{n+1}}{(C_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}-A_1)s_{n+1}+s_{n+2}}\right)\\
&=y_{-1}[\prod_{l=0}^{n+1}(\frac{-\mu\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}s_{l-1}+(A_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}+B_1)s_l}{(C_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}-A_1)s_l+s_{l+1}})\prod_{l=1}^{n+1}(\frac{-\mu\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}s_{l-2}+(A_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}+B_2)s_{l-1}}{(C_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}-A_2)s_{l-1}+s_l})\\
&=x_{-3}[\prod_{l=0}^{n+1}(\frac{-\mu\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}s_{l-1}+(A_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}+B_1)s_l}{(C_1\frac{a_1y_{-1}+b_1x_{-3}}{c_1x_{-3}+d_1y_{-1}}-A_1)s_l+s_{l+1}})\prod_{l=0}^{n+1}(\frac{-\mu\frac{A_2y_{-1}+B_2x_{-1}}{C_2y_{-1}+D_2x_{-3}}s_{l-2}+(A_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}+B_2)s_{l-1}}{(C_2\frac{A_2y_{-1}+B_2x_{-3}}{C_2y_{-1}+D_2x_{-3}}-A_2)s_{l-1}+s_l}).
\end{aligned}$$

For $i=2,3,4$, the proof is similar. Also, for y_{4m+i}, $i=\overline{1,4}$, the proof is similar.
This completes the proof. □

For simplicity, we can write the representation of the system (1.1) when

$${A_1}^2+{D_1}^2>2A_1D_1-4B_1C_1$$

in the form

$$\begin{aligned}
x_{4n+i}=&x_{-4+i}\prod_{j=0}^{n}\left(\frac{-\mu u_is_{n-1}+(A_1u_i+B_1)s_n}{(C_1u_i-A_1)s_n+s_{n+1}}\right)\\
&\times\left(\frac{-\mu v_{-2+i}s_{n-1}+(A_2v_{-2+i}+B_2)s_n}{(C_2v_{-2+i}-A_2)s_n+s_{n+1}}\right),\\
y_{4n+i}=&y_{-4+i}\prod_{j=0}^{n}\left(\frac{-\mu v_is_{n-1}+(A_2v_i+B_2)s_n}{(C_2v_i-A_2)s_n+s_{n+1}}\right)\\
&\times\left(\frac{-\mu u_{-2+i}s_{n-1}+(A_1u_{-2+i}+B_1)s_n}{(C_1u_{-2+i}-A_1)s_n+s_{n+1}}\right),
\end{aligned}\tag{2.8}$$

where $n\geq 0$ and $i=\overline{1,4}$. When

$${A_1}^2+{D_1}^2=2A_1D_1-4B_1C_1,$$

we can write the representation of the system (1.1) in the form

$$\begin{aligned}
x_{4n+i}=x_{-4+i}&\prod_{j=0}^{n}\left(\frac{D_1+A_1}{2C_1}\frac{j(C_1u_i+\frac{D_1-A_1}{2})+C_1u_i+D_1}{j(C_1u_i+\frac{D_1-A_1}{2})+\frac{D_1+A_1}{2C_1}}-\frac{D_1}{C_1}\right)\\
&\times\left(\frac{D_2+A_2}{2C_2}\frac{j(C_2v_{-2+i}+\frac{D_2-A_2}{2})+C_2v_{-2+i}+D_2}{j(C_2v_{-2+i}+\frac{D_2-A_2}{2})+\frac{D_2+A_2}{2C_2}}-\frac{D_2}{C_2}\right)\\
y_{4n+i}=y_{-4+i}&\prod_{j=0}^{n}\left(\frac{D_2+A_2}{2C_2}\frac{j(C_2v_i+\frac{D_2-A_2}{2})+C_2v_i+D_2}{j(C_2v_i+\frac{D_2-A_2}{2})+\frac{D_2+A_2}{2C_2}}-\frac{D_2}{C_2}\right)\\
&\times\left(\frac{D_1+A_1}{2C_1}\frac{j(C_1u_{-2+i}+\frac{D_1-A_1}{2})+C_1u_{-2+i}+D_1}{j(C_1u_i+\frac{D_1-A_1}{2})+\frac{D_1+A_1}{2C_1}}-\frac{D_1}{C_1}\right),
\end{aligned}\tag{2.9}$$

where $n \geq 0$ and $i = \overline{1,4}$. Finally, when

$$A_1{}^2 + D_1{}^2 < 2A_1D_1 - 4B_1C_1,$$

we can write the representation of the system (1.1) in the form

$$\begin{aligned} x_{4n+i} &= x_{-4+i} \prod_{j=0}^{n} \left(\frac{\sqrt{\mu}}{C_1} \frac{(C_1u_i + D_1)\sin(j+1)\theta - \sqrt{\mu}\sin j\theta}{(C_1u_i + D_1)\sin j\theta - \sqrt{\mu}\sin(j-1)\theta} \right) \\ &\times \left(\frac{\sqrt{\mu}}{C_2} \frac{(C_2v_{-2+i} + D_2)\sin(j+1)\theta - \sqrt{\mu}\sin j\theta}{(C_2v_{-2+i} + D_2)\sin j\theta - \sqrt{\mu}\sin(j-1)\theta} \right) \\ y_{4n+i} &= y_{-4+i} \prod_{j=0}^{n} \left(\frac{\sqrt{\mu}}{C_2} \frac{(C_2v_i + D_2)\sin(j+1)\theta - \sqrt{\mu}\sin j\theta}{(C_2v_i + D_2)\sin j\theta - \sqrt{\mu}\sin(j-1)\theta} \right) \\ &\times \left(\frac{\sqrt{\mu}}{C_1} \frac{(C_1u_{-2+i} + D_1)\sin(j+1)\theta - \sqrt{\mu}\sin j\theta}{(C_1u_{-2+i} + D_2)\sin j\theta - \sqrt{\mu}\sin(j-1)\theta} \right), \end{aligned} \tag{2.10}$$

such that $\theta = tan^{-1} \frac{\sqrt{2A_1D_1 - 4B_1C_1 - A_1{}^2 - D_1{}^2}}{A_1 + D_1}$, where $n \geq 0$ and $i = \overline{1,4}$.

Theorem 2.2. *The forbidden set for system* (1.1) *is*

$$\begin{aligned} F = &\bigcup_{i=0}^{1} \{(x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0) \in \mathbb{R}^8 : x_{-i} = 0\} \cup \\ &\bigcup_{i=0}^{1} \{(x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0) \in \mathbb{R}^8 : y_{-i} = 0\} \cup \\ &\bigcup_{i=0}^{1} \bigcup_{n=0}^{\infty} \{(x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0) \in \mathbb{R}^8 : y_{-1+i} = x_{-3+i}(f^{-1} \circ g^{-1})^{n+1}(-\frac{c_1}{d_1})\} \cup \\ &\bigcup_{i=0}^{1} \bigcup_{n=0}^{\infty} \{(x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0) \in \mathbb{R}^8 : x_{-1+i} = y_{-3+i}(g^{-1} \circ f^{-1})^{n+1}(-\frac{c_2}{d_2})\} \cup \\ &\bigcup_{i=0}^{1} \bigcup_{n=0}^{\infty} \{(x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0) \in \mathbb{R}^8 : x_{-1+i} = y_{-3+i}(g^{-1} \circ f^{-1})^{n}(-\frac{D_1}{C_1})\} \cup \\ &\bigcup_{i=0}^{1} \bigcup_{n=0}^{\infty} \{(x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0) \in \mathbb{R}^8 : y_{-1+i} = x_{-3+i}(f^{-1} \circ g^{-1})^{n}(-\frac{D_2}{C_2})\}, \end{aligned}$$

where $f(t) := \frac{a_1t + b_1}{d_1t + c_1}$ *and* $g(t) := \frac{a_2t + b_2}{d_2t + c_2}$.

Proof. Let $\{(x_n, y_n)\}_{n=-3}^{\infty}$ be a solution to system (1.1).
Using (2.2), we can write for each $n \in \mathbb{N}_0$

$$\begin{aligned} u_{4n+i} &= \underbrace{f \circ g \circ f \circ g \circ \ldots \circ f \circ g}_{2n}(u_i), \ i = 1, 2, \\ u_{4n+i} &= \underbrace{f \circ g \circ f \circ g \circ \ldots \circ f \circ g}_{2n+2}(u_{-4+i}), \ i = 3, 4, \end{aligned} \tag{2.11}$$

The solution $\{(x_n, y_n)\}_{n=-2}^{\infty}$ of system (1.1) is not defined if for a certain $n \in \mathbb{N}_{-1}$, $u_n = -\frac{c_2}{d_2}$ or $v_n = -\frac{c_1}{d_1}$.
Using Equation (2.11) we get

$$\begin{aligned} u_{-i} &= \underbrace{g^{-1} \circ f^{-1} \circ g^{-1} \circ f^{-1} \circ \ldots \circ g^{-1} \circ f^{-1}}_{2n+2}(-\frac{c_2}{d_2}), \ i = 0, 1, \\ v_{-i} &= \underbrace{f^{-1} \circ g^{-1} \circ f^{-1} \circ g^{-1} \circ \ldots \circ f^{-1} \circ g^{-1}}_{2n}(-\frac{D_2}{C_2}), \ i = 0, 1. \end{aligned} \tag{2.12}$$

In the same way, we can get

$$\begin{aligned} u_{-i} &= \underbrace{g^{-1} \circ f^{-1} \circ g^{-1} \circ f^{-1} \circ \ldots \circ g^{-1} \circ f^{-1}}_{2n}(-\frac{D_1}{C_1}), \ i = 0, 1, \\ v_{-1} &= \underbrace{f^{-1} \circ g^{-1} \circ f^{-1} \circ g^{-1} \circ \ldots \circ f^{-1} \circ g^{-1}}_{2n+2}(-\frac{c_1}{d_1}), \ i = 0, 1. \end{aligned} \tag{2.13}$$

The proof is completed by taking into account the case $\prod_{i=0}^{1} x_{-i}y_{-i} = 0$. □

3. Special case and illustrative examples

When $\mu = 0$, the solution of system (1.1) is reduced to a simpler form. In fact, using (2.3), we have for each $i = \overline{1,4}$ and $n \geq 1$

$$\begin{aligned} u_{4n+i} =& \frac{A_1 u_{4(n-1)+i} + B_1}{C_1 u_{4(n-1)+i} + D_1} \\ =& \frac{A_1 u_{4(n-1)+i} + \frac{A_1 D_1}{C_1}}{C_1 u_{4(n-1)+i} + D_1} \\ =& \frac{A_1}{C_1}. \end{aligned}$$

Similarly,

$$v_{4n+i} = \frac{A_2}{C_2},\ i = \overline{1,4},\ n \geq 0.$$

This implies that the solution of system (1.1) in this case is of the form

$$\begin{aligned} x_{4n+i} &= x_i \left(\frac{A_1 A_2}{C_1 C_2}\right)^n \\ y_{4n+i} &= y_i \left(\frac{A_1 A_2}{C_1 C_2}\right)^n \end{aligned}$$

where $n \geq 1$ and $i = \overline{1,4}$.
The following result is true when $\mu = 0$, and its proof is omitted for simplicity.

Theorem 3.1. *Let $\{x_n, y_n\}_{n=-3}^{\infty}$ be a solution of system (1.1). If $\mu = 0$, then the solution $\{x_n, y_n\}_{n=-3}^{\infty}$ satisfies*

1. *If $|A_1 A_2| < |C_1 C_2|$, then $x_n \to 0$ and $y_n \to 0$.*
2. *If $|A_1 A_2| > |C_1 C_2|$, then the solution is unbounded in both of its arguments.*
3. *If $|A_1 A_2| = |C_1 C_2|$, then we have the following:*
   - *If $A_1 A_2 = C_1 C_2$, then $x_{4n+i} = x_i$ and $y_{4n+i} = y_i$, $i = \overline{1,4}$ for each $n \geq 1$.*
   - *If $A_1 A_2 = -C_1 C_2$, then $x_{8n+4+i} = -x_i$, $y_{8n+4+i} = -y_i$, and $x_{8n+8+i} = x_i$, $y_{8n+8+i} = y_i$ $i = \overline{1,4}$ for each $n \geq 0$.*

Example (1) Figure 1 shows that a solution $\{(x_n, y_n)\}_{n=-3}^{\infty}$ of system (1.1) with $a_1 = 0.25$, $b_1 = -1.1$, $c_1 = 1$, $d_1 = -0.5$, $a_2 = 0.2$, $b_2 = 0.5$, $c_2 = 1.2$, $d_2 = 0.5$ and with initial values $x_{-3} = 2$, $x_{-2} = 1$, $x_{-1} = -1$, $x_0 = 3$, $y_{-3} = 0.3$, $y_{-2} = -1$, $y_{-1} = 1.4$ and $y_0 = -1$ converges to zero (case (1) in Theorem (3.1)).

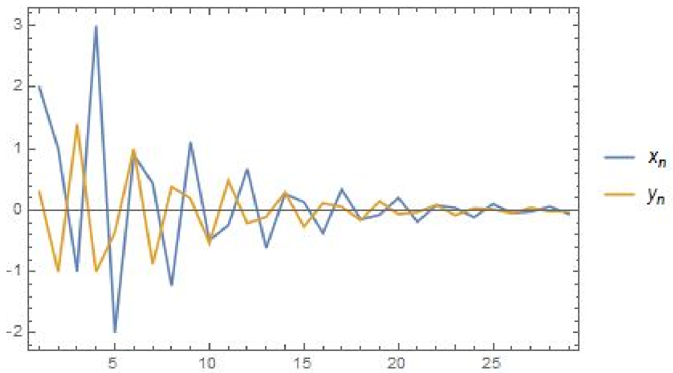

Figure 3.1: $x_{n+1} = y_{n-1}\frac{-0.25y_{n-1}+x_{n-3}}{-2x_{n-3}+0.5y_{n-1}}, y_{n+1} = x_{n-1}\frac{x_{n-1}+2y_{n-3}}{y_{n-3}+0.5x_{n-1}}$

Example (2) Figure 2 shows that a solution $\{(x_n, y_n)\}_{n=-3}^{\infty}$ of system (1.1) with $a_1 = -0.25$, $b_1 = 1$, $c_1 = -2$, $d_1 = 0.50$, $a_2 = 1$, $b_2 = 2$, $c_2 = 1$, $d_2 = 0.50$ and with initial values $x_{-3} = 2$, $x_{-2} = -1$, $x_{-1} = 1$,$x_0 = -1.5$, $y_{-3} = -1.3$,$y_{-2} = -1.7$, $y_{-1} = 0.4$ and $y_0 = 0.9$ satisfies $x_{4n+i} = x_i$ and $y_{4n+i} = y_i$, $i = \overline{1,4}$, $n \geq 1$ (case (3) in Theorem (3.1)).

Example (3) Figure 3 shows that a solution $\{(x_n, y_n)\}_{n=-3}^{\infty}$ of system (1.1) with $a_1 = 2$, $b_1 = 1$, $c_1 = 0.5$, $d_1 = 1$, $a_2 = 2$, $b_2 = 3$, $c_2 = 0.5$, $d_2 = 0.25$ and with initial values $x_{-3} = -2$, $x_{-2} = 1$, $x_{-1} = 1$, $x_0 = 2.5$, $y_{-3} = 0.3$, $y_{-2} = -1.7$, $y_{-1} = 0.4$ and $y_0 = 3.9$ satisfies is unbounded (case (2) in Theorem (3.1)).

4. Conclusion

This work is an investigation that explicitly reveals a representation of solutions to a system of higher-order difference equations by extending somehow the work of McGrath [8] in two dimensions. The main result of this work is the representation of the solutions to the system of higher-order difference equations and deriving the forbidden set. Finally, the theoretical results are supported by numerical simulations that validate our findings. These simulations demonstrate a variety of dynamical behaviors, including stability, eventual periodicity and unboundedness linking theory and practice.

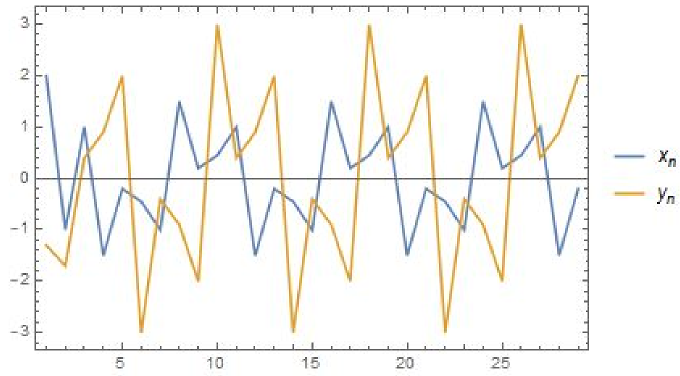

Figure 3.2: $x_{n+1} = y_{n-1}\frac{-0.25y_{n-1}+x_{n-3}}{-2x_{n-3}+0.50y_{n-1}}, y_{n+1} = x_{n-1}\frac{x_{n-1}+2y_{n-3}}{y_{n-3}+0.50x_{n-1}}$

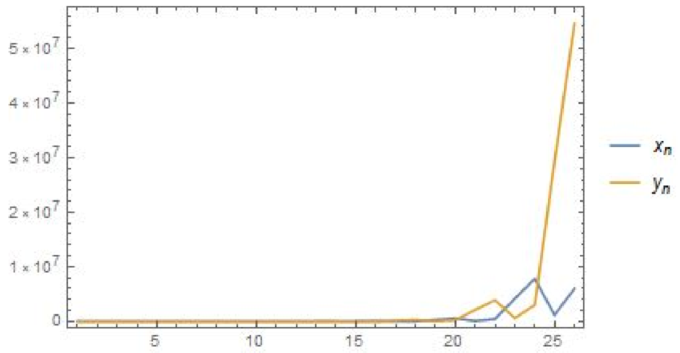

Figure 3.3: $x_{n+1} = y_{n-1}\frac{2y_{n-1}+x_{n-3}}{0.5x_{n-3}+1y_{n-1}}, y_{n+1} = x_{n-1}\frac{2x_{n-1}+3y_{n-3}}{0.5y_{n-3}+0.25x_{n-1}}$

Article Information

Acknowledgements: This work was partially supported by the Portuguese Foundation for Science and Technology (FCT- Fundação para a Ciência e a Tecnologia), through the Center of Mathematics and Applications of University of Beira Interior (CMA-UBI), within Project UIDB/00212/2020.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable

References

[1] S. N. Elaydi, *Discrete chaos: with applications in science and engineering*, Chapman and Hall/CRC, 2007.
[2] A. de Moivre, *De Fractionibus Algebraicis Radicalitate immunibus ad Fractiones Simpliciores reducendis, deque summandis Terminis quarumdam Serierum aequali Intervallo a se distantibus*, Philos. Trans. 32 (1722), 162–178 (in Latin).
[3] A. de Moivre, *Miscellanea analytica de seriebus et quadraturis*, 1730 (in Latin).
[4] R. M. May, *Simple mathematical models with very complicated dynamics*, Nature, 261 (1976), no. 5560, 459–467.
[5] M. Gümüş and K. Türk, *Dynamical behavior of a hepatitis B epidemic model and its NSFD scheme*, J. Appl. Math. Comput. 70 (2024), 3767–3788 . https://doi.org/10.1007/s12190-024-02103-6.
[6] A. Q. Khan, A. Maqbool, M. J. Uddin, and S. M. S. Rana, *Dynamical analysis of a two-dimensional discrete predator–prey model*. Journal of Computational and Applied Mathematics, 440 (2024), 115578.
[7] M. F. Ansori and F. H. Gümüş, *A Difference Equation of Banking Loan with Nonlinear Deposit Interest Rate*. Journal of Mathematical Sciences and Modelling, 7 (2024), no. 1, 14–19.
[8] L.C. Mcgrath and C. Teixeira, *Existence and behavior of solutions of the rational equation* $x_{n+1} = \frac{x_n(ax_{n-1}+bx_n)}{cx_{n-1}+dx_n}$, Rocky Mountain J. Math., 36 (2006), no. 2, 649–674.
[9] S. Stevic and D. Tollu, *On a two-dimensional nonlinear system of difference equations close to the bilinear system*, AIMS Mathematics, 8(9) (2023), 20561–20575.
[10] E. Elsayed, J. Al-Jauid, and H. Malaikah, *On the Solutions of Systems of Rational Difference Equations*, J. Prog. Rese.Math., 19 (2022), no. 2, 49-59.

[11] M. Kara and Y. Yazlik, *On a Class of Difference Equations System of Fifth-Order*, Fund. J. Math. Appl., 7 (2024), no. 3, 186–202.
[12] M. Kara, Y. Yazlik, and D. Tollu, *Solvability of a system of higher order nonlinear difference equations*, Hacet. J. Math. Stat., 49 (2020), no. 5, 1566–1593. DOI : 10.15672/hujms.474649
[13] S. Stevic, *Representation of solutions of bilinear difference equations in terms of generalized Fibonacci sequences*, Electronic J. Qualit. Theory Diff. Equat., 67 (2014), 1–15.
[14] M. Kara, *On a General Non-Linear Difference Equation of Third-Order*, Turk. J. Math. Comput. Sci., 16 (2024), no. 1, 126–136.
[15] S. Stevic, B. Iricanin, W. Kosmala, and S. Zdenek, *On some classes of solvable systems of difference equations* Adv Differ Equ 39 (2019). https://doi.org/10.1186/s13662-019-1959-x
[16] S. Stevic, B. Iricanin, and S. Zdenek, *On the system of difference equations* $x_n = \frac{x_{n-1}y_{n-2}}{ay_{n-2}+by_{n-1}}$, $x_n = \frac{x_{n-1}y_{n-2}}{ay_{n-2}+by_{n-1}}$, Appl. Math. Comput., 270 (2015), 688–704.
[17] S. Stevic, *On a two-dimensional solvable system of difference equations*, Electron. J. Qual. Theory Differ. Equ., 2018, No. 104, 1–18.
[18] S. Stevic, B. Iricanin, W. Kosmala, and S. Zdenek, *On a solvable system of rational difference equations*, J. Difference Equ. Appl., 20 (2014), no.5-6, 811–825. http://dx.doi.org/10.1080/10236198.2013.817573
[19] R. Abo-Zeid, *On the solutions of a higher order difference equation*, Georgian Math. J., 27 (2020), no. 2, 165-175.
[20] R. Abo-Zeid, *Global behavior of two third order rational difference equations with quadratic terms*, Math. Slovaca, 69 (2019), no. 2, 147-158. https://doi.org/10.1515/gmj-2018-0008
[21] R. Abo-Zeid, *Forbidden sets and stability in some rational difference equations*, J. Difference Equ. Appl., 24 (2018), no. 2, 220–239.
[22] R. Abo-Zeid, *Global asymptotic stability of a second order rational difference equation*, J. Appl. Math. & Inf., 28 (2010), no. 4-3, 797–804.
[23] M. Gümüş and Ş. I. Eğilmez, *The Qualitative Analysis of Some Difference Equations Using Homogeneous Functions*, Fundam. J. Math. Appl., 6 (2023), no. 4, 218 – 231.
[24] M. Aloqeili and A. Shareef, *Neimark-Sacker Bifurcation of a Third Order Difference Equation*, Fundam. J. Math. Appl., 2 (2019), 40–49.
[25] F.H. Gümüş and R. Abo-Zeid, *Global Behavior of a Nonlinear System of Difference Equations*, Fundam. J. Math. Appl.,8(1) (2025), 43 - 54.
[26] M. Kara and Y. Yazlik, *On a Class of Difference Equations System of Fifth-Order*, Fundam. J. Math. Appl., 7(3) (2024), 186–202.
[27] H. Althagafi, A. Ghezal, Global Stability of a System of Fuzzy Difference Equations of Higher-Order. J. Appl. Math. Comput. (2024). https://doi.org/10.1007/s12190-024-02302-1
[28] A. Ghezal and I. Zemmour, *Solution forms for generalized hyperbolic cotangent type systems of p-difference equations*, Bol. Soc. Paran. Mat., 42 (2024), 1-14. https://doi.org/10.5269/bspm.66516
[29] A. Ghezal, M. Balegh and I. Zemmouri, *Solutions and local stability of the Jacobsthal system of difference equations*, AIMS Math., 9(2) (2024), 3576–3591.