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Constructing k-Slant Curves in Three Dimensional Euclidean Spaces

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Abstract— Helices and constant procession curves are special examples of slant curves. However, there is no example of a k-slant curve for a positive integer $k \geq 2$ in three dimensional Euclidean spaces. Furthermore, the position vector of a k-slant curve for a positive integer $k \geq 2$ has not been known thus far. In this paper, we propose a method for constructing k-slant curves in three dimensional Euclidean spaces. We then show that spherical k-slant curves and N_k-constant procession curves can be derived from circles, for $k \in \mathbb{N}$, the set of all nonnegative integers. In addition, we provide a new proof of the spherical curve characterization and define a curve in the sphere called a spherical prime curve. Afterward, we apply k-slant curves to magnetic curves. Finally, we discuss the need for further research.

Keywords — *General helices, spherical curves, slant curves*

Mathematics Subject Classification (2020) 53A04, 53A05

1. Introduction

Curves are geometric sets of points or loci in spaces. In differential geometry, special curves, such as geodesics, circles, circular helices, general helices, slant helices, C-slant curves, and glad helices, have been extensively studied in various spaces. Helical structures are ubiquitous in nature, appearing in physics, kinematic motion, architectural design, and even in the double helix structure of DNA. A curve is called a general helix (or constant slope curve) if its tangent vector field makes a constant angle with a fixed straight line. The classical characterization of general helices was first proposed by Lancret in 1802 and later solved by Saint-Venant in 1845, who showed that a curve is a general helix if and only if the ratio of its curvature (κ) to torsion (τ) is constant. The curve is called a circular helix if the curvature is a non-zero constant. Additionally, straight lines and circles are considered degenerate helices.

Blum first introduced the concept of slant curves in 1966 [1], who studied curves with curvature functions given by $\kappa(s) = w\cos(as + b)$ and $\tau(s) = w\sin(as + b)$. Earlier, in 1878, Mannheim investigated curves satisfying $\kappa^2 + \tau^2 = w^2$, which Blum later connected to curves of constant precession. Izumiya and Takeuchi [2] provided a key characterization of slant helices, proving that a curve α is a slant helix if and only if the geodesic curvature of the spherical image of its principal normal indicatrix satisfies:

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$$\sigma = \frac{\kappa^2}{(\kappa^2 + \tau^2)^{\frac{3}{2}}} \left(\frac{\tau}{\kappa}\right)'$$

Kula and Yaylı [3] and Menninger [4] further explored the properties of slant curves. Camcı et al. [5] studied spherical slant curves in three dimensional Euclidean spaces. Besides the classical slant curves, such as Salkowski curves, a special type of slant helices with a constant curvature function κ, many studies have been conducted on the generalized slant curves. Ali [6] provided the general equation for the position vectors of slant curves and introduced k-slant helices, which generalize the concept of slant curves. Takahashi and Takeuchi [7] defined new special curves, such as clad (2-slant) and g-clad (3-slant) helices, extending the study of helical curves in three dimensional Euclidean spaces.

Curves of constant precession are another important class of curves, characterized by the property that they are traversed with unit speed, their centrode maintains a constant angle with a fixed axis, and they revolve at a constant speed. Scofield [8] provided a detailed study of such curves, while Uzunoğlu et al. [9] introduced a new approach using an alternative moving frame. Although Izumiya and Takeuchi mentioned the term slant curve [2], Blum did the foundational study [1] in 1966. Blum's study of curves with specific curvature and torsion functions laid the groundwork for later developments, including the connection to Mannheim curves and curves of constant precession.

Recently, the properties of slant helices have been studied not only in three dimensional Euclidean spaces but also within the framework of Semi-Riemannian Geometry. These studies have focused on the geometric characterization of slant helices in various ambient spaces, including Lorentzian and Sasakian manifolds, as well as their applications in differential geometry and physics. For more details, see [10–15].

The rest of the paper is organized as follows: Section 2 presents some basic properties to be used in the following sections. Section 3 propound a method for constructing k-slant curves in three dimensional Euclidean spaces. It demonstrates that spherical k-slant curves and N_k-constant procession curves can be derived from circles, for $k \in \mathbb{N}$. Section 4 presents a new proof of the spherical curve characterization and defines a curve in the sphere called a spherical prime curve. The most crucial point is that spherical helices oscillate in the sphere's equator. Section 5 applies k-slant curves to magnetic curves. The final section inquires whether further research should be conducted.

2. Preliminaries

In three dimensional Euclidean spaces $\mathbb{E}^3$, let γ be a unit-speed curve with a coordinate neighborhood (I, γ), and let $\{T, N, B, \kappa, \tau\}$ represent the Serret-Frenet apparatus of the curve. The derivations of the Serret-Frenet vectors are as follows:

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

The centrode of the curve C is defined as follows [16]:

$$W(s) = \tau(s)T(s) + \kappa(s)B(s)$$

If the curve is a spherical curve, then $\gamma(s)$ is perpendicular to $T = \gamma'(s)$, for all $s \in I$. Thus, $\{\gamma(s), T(s) = \gamma'(s), Y(s) = \gamma(s) \times T(s)\}$ forms an orthonormal frame along γ. This frame is called the Sabban frame [17] along γ. The Serret-Frenet formula of a spherical curve is given by

$$\begin{pmatrix} \gamma' \\ T' \\ Y' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa_g \\ 0 & -\kappa_g & 0 \end{pmatrix} \begin{pmatrix} \gamma \\ T \\ Y \end{pmatrix}$$

where $\kappa_g = \det(\gamma, T, T')$, called the geodesic curvature along the curve in 2-sphere [17].

Recently, Ali [18] has defined a unit vector as follows:

$$\psi_{k+1}(s) = \frac{\psi_k'}{\|\psi_k'\|}$$

where $\psi_0(s) = \gamma(s)$, $\psi_1(s) = T(s)$, and $\psi_2(s) = N(s)$. Hence, it can be defined a regular curve γ_k as follows [19]:

$$\gamma_k(s) = \int_0^s \psi_{k+1}(u)du$$

Let $\{T_k, N_k, B_k, \kappa_k, \tau_k\}$ be the Serret-Frenet apparatus of the curve γ_k. Then, [18] shows that

$$T_k = \psi_{k+1}, \quad N_k = \frac{\psi_{k+1}'}{\left\|\psi_{k+1}'\right\|} = \psi_{k+2} = T_{k+1}, \quad \text{and} \quad B_k = T_k \times N_k$$

The Serret-Frenet formulas of the curve γ_k is given as follows [18]:

$$\begin{pmatrix} T_k' \\ N_k' \\ B_k' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_k & 0 \\ -\kappa_k & 0 & \tau_k \\ 0 & -\tau_k & 0 \end{pmatrix} \begin{pmatrix} T_k \\ N_k \\ B_k \end{pmatrix}$$

Furthermore, [18, 19] show that $\kappa_k = \sqrt{\kappa_{k-1}^2 + \tau_{k-1}^2}$ and $\tau_k = \sigma_{k-1}\kappa_k$ where

$$\sigma_{k-1} = \frac{\kappa_{k-1}^2}{\left(\kappa_{k-1}^2 + \tau_{k-1}^2\right)^{\frac{3}{2}}} \left(\frac{\tau_{k-1}}{\kappa_{k-1}}\right)'$$

is the geodesic curvature of the spherical image of the principal normal of γ_{k+1}. The centrode of the curve γ_k is defined by

$$W_k(s) = \tau_k(s)T_k(s) + \kappa_k(s)B_k(s), \quad s \in I$$

If there exists a constant angle between $\psi_{k+1}(s)$ and any constant vector, i.e., γ_k, a general helix, then it is said that γ is a k-slant curve [18]. Thus, the following expressions are equivalent [18, 19]:

i. γ is a k-slant curve

ii. γ_{k-1} is a slant curve (1-slant)

iii. γ_k is a general helix (0-slant)

iv. γ_{k+1} is a planar curve

3. Construction of Spherical k-Slant Curves

In three dimensional Euclidean spaces, let γ be a regular spherical curve with a coordinate neighborhood (I, γ). Hence, a curve $I(\gamma) : I \to \mathbb{R}^3$ is defined as follows:

$$I(\gamma)(t, \theta_0) = \alpha(t) = \int_0^t S_\gamma(u, \theta_0)\gamma(u)du$$

where $S_\gamma : I \longrightarrow \mathbb{R}$ is a differentiable function. Thus, the following lemma is obtained:

Lemma 3.1. The curve $I(\gamma)$ is a spherical curve if and only if

$$S_\gamma(t,\theta_0) = \|\gamma'(t)\| \cos\left(\int_0^t \frac{\det(\gamma(u),\gamma'(u),\gamma''(u))}{\|\gamma'(u)\|^2} du + \theta_0 \right) \tag{3.1}$$

Proof. Without loss of generality, suppose that the sphere's center is the origin. If $I(\gamma) = \alpha$ is a regular spherical curve, then

$$\|\alpha(t)\| = \left\| \int_0^t S_\gamma(u)\gamma(u) du \right\| = 1$$

and $\alpha(t)$ is perpendicular to $\gamma(t)$, for all $t \in I$. Hence, there exist functions f and g such that

$$\alpha(t) = \int_0^t S_\gamma(u)\gamma(u) du = f(t)\gamma'(t) + g(t)Y(t) \tag{3.2}$$

where $Y(t) = \gamma(t) \times \gamma'(t)$. From (3.2),

$$(f(t))^2 + (g(t))^2 = \frac{1}{\|\gamma'(t)\|^2}$$

or

$$f(t) = -\frac{1}{\|\gamma'(t)\|}\cos\theta(t) \quad \text{and} \quad g(t) = \frac{1}{\|\gamma'(t)\|}\sin\theta(t) \tag{3.3}$$

where $\theta : I \longrightarrow \mathbb{R}$ is a function. If we derivate (3.2), then

$$S_\gamma(t)\gamma(t) = f'(t)\gamma'(t) + f(t)\gamma''(t) + g'(t)(\gamma(t)\times\gamma'(t)) + g(t)(\gamma(t)\times\gamma''(t)) \tag{3.4}$$

From (3.3) and (3.4),

$$S_\gamma(t) = \|\gamma'(t)\| \cos\theta(t) \tag{3.5}$$

Moreover, using (3.3),

$$f'(t) = \frac{\langle \gamma'(t), \gamma''(t)\rangle}{\|\gamma'(t)\|^3}\cos\theta(t) + \frac{\theta'(t)}{\|\gamma'(t)\|}\sin\theta(t) \tag{3.6}$$

and from (3.4),

$$f'(t)\,\|\gamma'(t)\|^2 + f(t)\,\langle \gamma'(t), \gamma''(t)\rangle - g(t)\det(\gamma(t),\gamma'(t),\gamma''(t)) = 0 \tag{3.7}$$

Using (3.3), (3.6), and (3.7),

$$\theta'(t)\,\|\gamma'(t)\| - \frac{\det(\gamma(t),\gamma'(t),\gamma''(t))}{\|\gamma'(t)\|} = 0 \tag{3.8}$$

If we integrate (3.8), then

$$\theta(t) = \int_0^t \frac{\det(\gamma(u),\gamma'(u),\gamma''(u))}{\|\gamma'(u)\|^2} du + \theta_0$$

Using (3.5),

$$S_\gamma(t,\theta_0) = \|\gamma'(t)\| \cos\left(\int_0^t \frac{\det(\gamma(u),\gamma'(u),\gamma''(u))}{\|\gamma'(u)\|^2} du + \theta_0 \right)$$

□

Corollary 3.2. Let SC be a set of spherical regular curves. From Lemma 3.1, we can define a map as

$$\begin{aligned} I : SC \times [0,2\pi] &\to \quad SC \\ (\gamma,\theta_0) \quad &\to I(\gamma,\theta_0) \end{aligned}$$

where

$$S_{I^0(\gamma)}(t,\theta_0) := \|\gamma'(t)\| \cos\left(\int_0^t \frac{\det(\gamma(u),\gamma'(u),\gamma''(u))}{\|\gamma'(u)\|^2} du + \theta_0 \right)$$

$$I(\gamma,\theta_0)(t) := I(\gamma)(t,\theta_0) = \alpha(t) = \int_0^t S_{I^0(\gamma)}(u,\theta_0) I^0(\gamma)(u) du$$

and $I^0(\gamma) := \gamma$. Thus,

$$S_{I(\gamma)}(t,\widetilde{\theta}_1) = \|I(\gamma)'(t,\theta_0)\| \cos\left(\int_0^t \frac{\det(I(\gamma)(u,\theta_0), I(\gamma)'(u,\theta_0), I(\gamma)''(u,\theta_0))}{\|I(\gamma)'(u,\theta_0)\|^2} du + \theta_1 \right)$$

and

$$I(I(\gamma)(t,\theta_0),\theta_1) := I^2(\gamma)(t,\widetilde{\theta}_1) = \int_0^t S_{I(\gamma)}(u,\widetilde{\theta}_1) I(\gamma)(u,\theta_0) du$$

where $\widetilde{\theta}_1 = (\theta_0,\theta_1)$. By the mathematical induction,

$$S_{I^n(\gamma)}(t,\widetilde{\theta}_n) = \left\| I^n(\gamma)'(t,\widetilde{\theta}_{n-1}) \right\| \cos\left(\int_0^t \frac{\det(I^n(\gamma)(u,\widetilde{\theta}_{n-1}), I^n(\gamma)'(u,\widetilde{\theta}_{n-1}), I^n(\gamma)''(u,\widetilde{\theta}_{n-1}))}{\left\| I^n(\gamma)'(u,\widetilde{\theta}_{n-1}) \right\|^2} du + \theta_n \right)$$

and

$$I(I^n(\gamma)(t,\widetilde{\theta}_{n-1}),\theta_n) = I^{n+1}(\gamma)(s,\widetilde{\theta}_n) = \int_0^t S_{I^n(\gamma)}(t,\widetilde{\theta}_n) I^n(\gamma)(u,\widetilde{\theta}_{n-1}) du$$

where $\widetilde{\theta}_n = (\theta_0,\theta_1,\ldots,\theta_n)$. Define

$$I^{-1}(\gamma)(t,\theta_0) := I(-\gamma)(t,\theta_0) = -I(\gamma)(t,-\theta_0)$$

Then,

$$I^{-n}(\gamma)(t,\widetilde{\theta}_{n-1}) = I^n(-\gamma)(t,\widetilde{\theta}_{n-1}) = -I^n(\gamma)(t,-\widetilde{\theta}_{n-1})$$

where $-\widetilde{\theta}_{n-1} = (-\theta_0,-\theta_1,\ldots,-\theta_{n-1})$. Consider the set

$$\mathbb{Z}(\gamma) = \left\{ \ldots, I^{-2}(\gamma), I^{-1}(\gamma), I^0(\gamma) = \gamma, I(\gamma), I^2(\gamma), \ldots \right\}$$

Then, it can be observed that $(\mathbb{Z}(\gamma),+,.)$ with the following addition and multiplication defined on $\mathbb{Z}(\gamma)$ is a ring:

$$I^n(\gamma) + I^m(\gamma) = I^{n+m}(\gamma)$$

and

$$I^n(\gamma).I^m(\gamma) = I^{nm}(\gamma)$$

where $n,m \in \mathbb{Z}$.

Throughout this paper, let $S_\gamma(t)$ and $I(\gamma)(t)$ denote $S_\gamma(t,0)$ and $I(\gamma)(t,0)$, respectively.

Theorem 3.3. In three dimensional Euclidean spaces, the curve γ is a spherical k-slant curve if and only if $I(\gamma)$ is a spherical $(k+1)$-slant curve.

Proof. Let γ be a regular curve with a coordinate neighborhood (I,γ). Hence,

$$\psi_0(t) = \gamma(t) = \frac{\alpha'(t)}{S_\gamma(t)} = \overline{\psi}_1(t)$$

Thus, γ is the tangent indicatrix of the curve $I(\gamma)$ and $\psi_k = \overline{\psi}_{k+1}$ where $\psi_k(t) = \frac{\psi'_{k-1}(t)}{\|\psi'_{k-1}(t)\|}$ and $\overline{\psi}_{k+1}(t) = \frac{\overline{\psi}'_k(t)}{\|\overline{\psi}'_k(t)\|}$. □

Example 3.4. Let $\gamma(t) = \left(-\frac{1}{\sqrt{2}}\sin 2t, \frac{1}{\sqrt{2}}\sin 2t, \cos 2t\right)$. Then, $\|\gamma'(t)\| = 2$, $\det(\gamma(t), \gamma'(t), \gamma''(t)) = 0$, $S_\gamma(t, \theta_0) = 2\cos(\theta_0)$, and

$$I(\gamma)(t,\theta_0) = \int 2\cos(\theta_0)\left(-\frac{1}{\sqrt{2}}\sin 2t, \frac{1}{\sqrt{2}}\sin 2t, \cos 2t\right) dt \tag{3.9}$$

It can be observed that the geodesic circle $\gamma(t)$ is the intersection of the plane $x + y = 0$ and the unit sphere $S^2(O, 1)$ where the center of the sphere is the origin. By integrating (3.9),

$$I(\gamma)(t,\theta_0) = \cos(\theta_0)\left(\frac{1}{\sqrt{2}}\cos(2t) + c_1, -\frac{1}{\sqrt{2}}\cos(2t) + c_2, \sin(2t) + c_3\right)$$

Here, $I(\gamma)(t, \theta_0)$ lies on sphere $S^2(M, r)$ where $M = (-c_1, -c_2, -c_3)$ and $r = |\cos(\theta_0)|$. In this case, $I(\gamma)$ lies on $S^2(O, 1)$ if and only if $c_1 = c_2 = c_3 = 0$ and $\cos(\theta_0) = \varepsilon = \pm 1$. Thus,

$$I(\gamma)(t) = \left(\frac{\varepsilon}{\sqrt{2}}\cos(2t), -\frac{\varepsilon}{\sqrt{2}}\cos(2t), \varepsilon\sin(2t)\right)$$

It can be observed that the geodesic circle $I(\gamma)$ is the intersection of the plane $x + y = 0$ and the sphere $S^2(O, 1)$ and $\langle\gamma(t), I(\gamma)(t)\rangle = 0$. Therefore, there are four types of arc, belong to a geodesic circle: $I(\gamma) = \gamma$, $I^2(\gamma) = \gamma$, $I^3(\gamma) = \gamma$, and $I^4(\gamma) = \gamma$.

Example 3.5. Consider the following steps, constructing all k-slant curves:

Step 1. Let $\gamma = S^1(\overrightarrow{a}, r)$ be a circle in the unit sphere centered at the origin where $\gamma(s) = (r\cos ws, r\sin ws, a)$, $w = \frac{1}{r}$, $\overrightarrow{a} = (0, 0, a)$, and $a^2 + r^2 = 1$. Hence,

$$S_\gamma(s, \theta_0) = \cos(aws + \theta_0)$$

and

$$I(\gamma)(s,\theta_0) = \int \cos(aws + \theta_0)(r\cos ws, r\sin ws, a)ds$$

where $\|\gamma'(s)\| = r$ and $\det(\gamma(s), \gamma'(s), \gamma''(s)) = aw$. Then, we have a curve $I(\gamma)$ as follows:

$$I(\gamma)(s,\theta_0) = \begin{pmatrix} \frac{r}{2}\left[\frac{1}{w(a+1)}\sin\left(w(a+1)s + \theta_0\right) + \frac{1}{w(a-1)}\sin\left(w(a-1)s + \theta_0\right)\right], \\ \frac{r}{2}\left[-\frac{1}{w(a+1)}\cos\left(w(a+1)s + \theta_0\right) + \frac{1}{w(a-1)}\cos\left(w(a-1)s + \theta_0\right)\right], \\ r\sin\left(aws + \theta_0\right) \end{pmatrix}$$

where $\|I(\gamma)(s, \theta_0)\| = 1$. In Figure 1, the graph of $I(\gamma)(s, \theta_0)$ is provided where $r = \frac{1}{2}$, $a = \frac{\sqrt{3}}{2}$, and $\theta_0 = 0$.

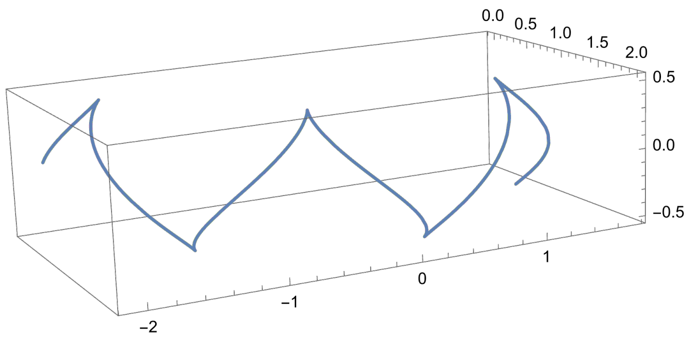

Figure 1. Graph of $I(\gamma)(s, 0)$ where $r = \frac{1}{2}$ and $a = \frac{\sqrt{3}}{2}$

Hence, it can be observed that this curve is a spherical helix. Furthermore, since tangent indicatrices of $I(\gamma)(s,\theta_0)$ are equal to $\gamma = S^1(a,r)$, all spherical helices are of the form $I(\gamma)(s,\theta_0)$, where axis of the helix is equal to $\overrightarrow{u} = (0,0,1)$. Blaschke [16,20] has established that all spherical helices are of the form $I(\gamma)(s,\theta_0)$, where the axis of the helix is equal to $\overrightarrow{u} = (0,0,1)$. Moreover, Blaschke [16,20] has shown that the projections of the spherical helices onto the plane xy are arcs of epicycloid.

Step 2. From (3.1),

$$S_{I(\gamma)}(s,\widetilde{\theta}_1) = \cos(aws+\theta_1)\cos\left(\frac{1}{aw}\cos(aws+\theta_1)+\theta_2\right)$$

Thus, we have a curve $I(I(\gamma)(s,\theta_0),\theta_1) = I^2(\gamma)(s,\widetilde{\theta}_1)$ as follows:

$$I^2(\gamma)(s,\widetilde{\theta}_1) = \int S_{I(\gamma)}(s,\widetilde{\theta}_1) I(\gamma)(s,\widetilde{\theta}_0)ds$$

In Figure 2, the graph of 1-slant curve is given where $r = \frac{1}{2}$, $a = \frac{\sqrt{3}}{2}$, and $\theta_0 = \theta_1 = 0$.

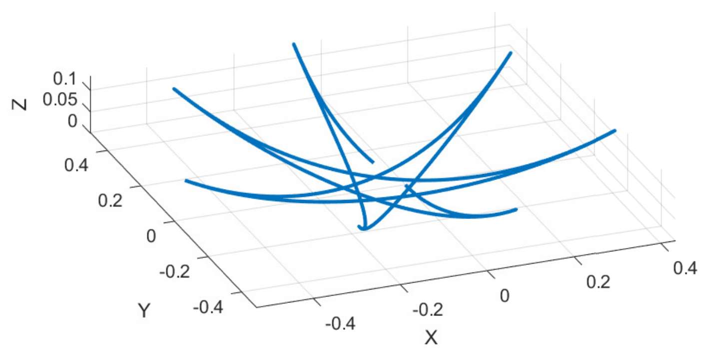

Figure 2. Graph of $I^2(\gamma)(s,\widetilde{\theta}_1)$ where $r = \frac{1}{2}$, $a = \frac{\sqrt{3}}{2}$, and $\widetilde{\theta}_1 = (0,0)$

Since tangent indicatrices of $I^2(\gamma)(s,\widetilde{\theta}_1)$ are equal to $I(\gamma)(s,\widetilde{\theta}_0)$, all spherical slant curves are of the form $I^2(\gamma)(s,\widetilde{\theta}_1)$ in which the axis of the slant curve is equal to $\overrightarrow{u} = (0,0,1)$.

Step 3. Let $I^2(\gamma)$ be a spherical curve where

$$I^2(\gamma)(s,\widetilde{\theta}_1) = \int S_{I(\gamma)}(s,\widetilde{\theta}_1) I(\gamma)(s,\widetilde{\theta}_0)ds$$

Then,

$$S_{I^2(\gamma)}(s,\widetilde{\theta}_2) = \cos(aws+\theta_0)\cos\left(\frac{1}{aw}\cos(aws+\theta_0)+\theta_1\right)\times\cos\left(\int \det(\gamma(s),\alpha(s),\alpha'(s))ds+\theta_2\right)$$

Thus, we have a curve $C^2(a,r)$ as follows:

$$I^3(\gamma)(s,\widetilde{\theta}_2) = \int S_{I^2(\gamma)}(s,\widetilde{\theta}_2) I^2(\gamma)(s,\widetilde{\theta}_1)ds$$

This curve is a spherical 2-slant curve. Since tangent indicatrices of $I^3(\gamma)$ are equal to $I^2(\gamma)$, all spherical 2-slant curves are of the form $I^3(\gamma)$ in which the axis of the 2-slant curve is equal to $\overrightarrow{u} = (0,0,1)$.

We have all 3-slant curves $I^4(\gamma)$ by the mathematical induction method. With similar method, we have all k-slant curves $I^{k+1}(\gamma)$ where axis of the k-slant curve is equal to $\overrightarrow{u} = (0,0,1)$, for all $k \in \mathbb{N}$. Consequently, we have the following corollary.

Corollary 3.6. In three dimensional Euclidean spaces, there exists a spherical k-slant curve, for all $k \in \mathbb{N}$. Furthermore, all spherical k-slant curves are of the form $I^{k+2}(\gamma)$, where the axis of the k-slant curve is equal to $\overrightarrow{u} = (0,0,1)$, for all $k \in \mathbb{N}$. If the geodesic circle which lies on the sphere is changed, then we have all spherical k-slant curves.

Definition 3.7. In three dimensional Euclidean spaces, consider the set

$$S\left(S^1(\overrightarrow{a},r)\right) = \left\{\ldots, I^{-2}(\gamma), I^{-1}(\gamma), \gamma = S^1(\overrightarrow{a},r), I(\gamma), I^2(\gamma), I^3(\gamma), \ldots\right\}$$

We say that $S\left(S^1(\overrightarrow{a},r)\right)$ is a set of spherical slant curve. Hence, the set of all spherical k-slant curve, denoted by $\left(S^I\right)$, is given by

$$S^I = \bigcup_{\overrightarrow{a}\in D} S\left(S^1(\overrightarrow{a},r)\right)$$

where $D = \{\overrightarrow{a} = (a_1,a_2,a_3) \mid a_1^2 + a_2^2 + a_3^3 < 1\}$ and $\|\overrightarrow{a}\|^2 + r^2 = 1$.

In [21], the authors have demonstrated that a curve lies on the 2-sphere if and only if

$$\left(\left(\frac{1}{\kappa}\right)'\frac{1}{\tau}\right)' + \frac{\tau}{\kappa} = 0 \tag{3.10}$$

The solution of (3.10) is given by

$$\frac{1}{\kappa} = A\cos\left(\int_0^s \tau(u)du\right) + B\sin\left(\int_0^s \tau(u)du\right) \tag{3.11}$$

where $R = \sqrt{A^2+B^2}$ is the radius of a sphere [21–23]. From (3.11),

$$\frac{1}{\kappa} = R\left(\frac{A}{R}\cos\left(\int_0^s \tau(u)du\right) + \frac{B}{R}\sin\left(\int_0^s \tau(u)du\right)\right) \tag{3.12}$$

If $\cos\alpha_0 = \frac{A}{R}$, then $\sin\alpha_0 = \frac{-B}{R}$. From (3.12),

$$\frac{1}{\kappa} = R\cos\left(\int_0^s \tau(u)du + \alpha_0\right) \tag{3.13}$$

Let M be a unit-speed regular curve with a coordinate neighborhood (I,γ). In this paper, we suppose that $0 \in I$ without loss of generality. For all $s \in I$, the oscillating sphere of a curve is equal to the sphere in which the curve lies on the sphere. Furthermore, the oscillating circle lies on this sphere. From (3.13), $\frac{1}{\kappa_0} = \cos\alpha_0 = \frac{R_0}{R}$. If R is equal to 1, then

$$\cos\alpha_0 = R_0 = \frac{1}{\kappa_0}$$

where $R_0 = \frac{1}{\kappa_0} = \sup\left\{\frac{1}{\kappa(s)} \mid s \in I\right\}$. Hence, we can provide another proof of the characterization of spherical curves.

Proof. Let β be a regular spherical curve with a coordinate neighborhood (I,β), s be the arc-length parameter of the curve, and κ and τ be curvatures of the curves. Thus, we can define a spherical curve as $S_\gamma(s)\gamma(s) = \beta'(s)$ where

$$S_\gamma(s) = \|\gamma'(s)\|\cos\left(\int_0^s \frac{\det(\gamma(u),\gamma'(u),\gamma''(u))}{\|\gamma'(u)\|^2}du + \theta_0\right)$$

Since $S_\gamma(s)\gamma(s) = \beta'(s)$, $S_\gamma(s) = 1$ and $\gamma(s) = \beta'(s)$. Then,

$$1 = \kappa \cos\left(\int_0^s \tau(u)du + \theta_0\right) \tag{3.14}$$

Conversely, we suppose a curve K satisfies (3.14). Let K be a unit-speed regular curve with a coordinate neighborhood (I, γ). We can define a curve M with the coordinate neighborhood (I, γ) where $\gamma(s) = \beta'(s)$. From (3.14), $S_\gamma(s) = 1$ and $\int S_\gamma(s)\gamma(s)ds = \int \beta'(s)ds = \beta(s)$. From Lemma 3.1, K is a regular spherical curve. □

4. Construction of k-Slant Curves in $\mathbb{E}^3$

Lemma 3.1 can be applied to curve theory. Let γ be a unit-speed curve with a coordinate neighborhood (I, γ) and $\{T, N, B, \kappa, \tau\}$ be the Serret-Frenet apparatus of the curves. Hence, $\|\gamma'(s)\| = 1$ and tangent indicatrix of curve γ is $\sigma(s) = \gamma'(s) = T(s)$. Then, there exist a differentiable function $S_T : I \longrightarrow \mathbb{R}$ such that

$$\left\| \int S_T(s)\gamma'(s)ds \right\| = 1$$

where

$$S_T(s) = \kappa(s) \cos\left(\int_0^s \tau(u)du + \theta_0\right)$$

In this case, we can define a unit-speed curve β with a coordinate neighborhood (I, β) such that

$$I(D\gamma)(s, \theta_0) = \beta'(s) = \int S_T(s)\gamma'(s)ds$$

and

$$\beta''(s) = S_T(s)\gamma'(s) \tag{4.1}$$

Thus, the curve β is obtained as follows:

$$\beta(s) = D^{-1}I(D\gamma)(s, \theta_0) = J(\gamma)(s, \theta_0)$$

where

$$J(\gamma) = D^{-1}I(D\gamma)$$

and D is a derivative operator. Let $\overline{T}$, $\overline{N}$, and $\overline{B}$ be the Serret-Frenet vectors and $\overline{\kappa}$ and $\overline{\tau}$ be the curvature and torsion of a curve K, respectively, where

$$\overline{\kappa}(s) = S_T(s) = \kappa(s) \cos\left(\int_0^s \tau(u)du + \theta_0\right)$$

and

$$\overline{\tau}(s) = \kappa(s) \sin\left(\int_0^s \tau(u)du + \theta_0\right)$$

From (4.1),

$$\overline{N}(s) = \varepsilon T(s) \tag{4.2}$$

where $\varepsilon = \pm 1$. Without loss of generality, we suppose that

$$\overline{\kappa}(s) = \kappa(s) \cos\left(\int_0^s \tau(u)du + \theta_0\right)$$

From (4.1) and (4.2), it can be observed that the principal normal of β and the tangent of γ is colinear.

Theorem 4.1. In three dimensional Euclidean spaces, let γ be a unit speed curve with a coordinate neighborhood (I,γ). In this case, γ is a k-slant curve if and only if $J(\gamma)$ is a $(k+1)$-slant curve.

The proof of Theorem 4.1 is the same as that of Theorem 3.3. Therefore, it can be observed that

i. γ is a planar curve if and only if $J(\gamma)$ is a general helix

ii. γ is a general helix if and only if $J(\gamma)$ is a slant curve

Note 4.2. In three dimensional Euclidean spaces, let γ be a unit-speed regular curve with a coordinate neighborhood (I,γ). From (4.1), if the curve γ is a planar curve, then we have a helix in the first step, i.e., $J(\gamma)$. In the second step, we have a 1-slant curve (slant curve). In the third step, we have 2-slant curve, i.e., $J^2(\gamma)$. If this procedure is continued, in the $k+1$. step, we have k-slant curve, i.e., $J^{k+1}(\gamma)$. Hence, the following set can be obtained:

$$\mathbb{Z}(\gamma)=\left\{\ldots,J^{-2}(\gamma),J^{-1}(\gamma),\gamma,J(\gamma),J^2(\gamma),\ldots\right\}$$

We said that this set is a slant curve chain taken by a planar curve γ..

Definition 4.3. The set $\mathbb{Z}(\gamma)=\{\ldots,J^{-2}(\gamma),J^{-1}(\gamma),\gamma,J(\gamma),J^2(\gamma),\ldots\}$ in Note 4.2 is called a slant curve chain generated by a planar curve γ.

Example 4.4. In three dimensional Euclidean spaces, let $\gamma=S^1$ be a circle provided by $\gamma(s)=(r\cos ws,-r\sin ws,0)$ where $w=\frac{1}{r}$. Hence, $\kappa=w$ and $\tau=0$.

Step 1. Let $\overline{\kappa}(s)=\epsilon w\cos c_0=A$ and $\overline{\tau}(s)=\overline{\epsilon}w\sin c_0=B$ where A and B are constants. Moreover,

$$\beta''(s)=A(-\sin ws,-\cos ws,0) \tag{4.3}$$

By integrating (4.3), we have the curve β as

$$\beta(s)=\left(Ar^2\cos(\epsilon ws),Ar^2\sin(\epsilon ws),bws+c_1\right)$$

Because β is a unit speed curve, $b=r\sin c_0$. If $a=Ar^2=\epsilon r\cos c_0$ and $c_1=0$, then

$$J(\gamma)(s)=(a\cos(ws),a\sin(ws),bws)$$

where $r=\sqrt{a^2+b^2}$.

Step 2. If $J(\gamma)(s)=(a\cos ws,a\sin ws,bws)$, then $\kappa=aw^2$ and $\tau=bw^2$. From (4.3), $\overline{\kappa}(s)=\epsilon aw^2\cos\left(bw^2s\right)$ and $\overline{\tau}(s)=\overline{\epsilon}aw^2\sin\left(bw^2s\right)$. Hence,

$$J^2(\gamma)''(s)=\epsilon aw^2\cos\left(bw^2s\right)(-aw\sin ws,aw\cos ws,bw)$$

and

$$J^2(\gamma)''(s)=\left(-\epsilon a^2w^3\cos\left(bw^2s\right)\sin ws,\epsilon a^2w^3\cos\left(bw^2s\right)\cos ws,\epsilon abw^3\cos\left(bw^2s\right)\right) \tag{4.4}$$

By integrating (4.4),

$$J^2(\gamma)'(s)=\epsilon aw^2\begin{pmatrix}\frac{a}{2}\left[\frac{1}{1+bw}\cos(w(1+bw)s)+\frac{1}{1-bw}\cos(w(1-bw)s)\right],\\ \frac{a}{2}\left[\frac{1}{1+bw}\sin(w(1+bw)s)+\frac{1}{1-bw}\sin(w(1-bw)s)\right],\\ \frac{1}{w}\sin\left(bw^2s\right)\end{pmatrix} \tag{4.5}$$

Therefore, $\left\|J^2(\gamma)'(s)\right\|=1$. If we integrate (4.5), then we have the curve $J^2(\gamma)(s)=(x(s),y(s),z(s))$ where

$$x(s)=\frac{\varepsilon a^2w}{2}\left[\frac{1}{(1+bw)^2}\sin(w(1+bw)s)+\frac{1}{(1-bw)^2}\sin(w(1-bw)s)\right]$$

$$y(s) = -\frac{\varepsilon a^2 w}{2}\left[\frac{1}{(1+bw)^2}\cos\left(w(1+bw)s\right) + \frac{1}{(1-bw)^2}\cos\left(w(1-bw)s\right)\right]$$
$$z(s) = -\frac{\varepsilon a}{bw}\cos\left(bw^2 s\right)$$

and $\varepsilon = \pm 1$.

Thus,

$$x^2 + y^2 - \frac{b^2}{a^2}z^2 = \frac{b^2}{a^4 w^4} \tag{4.6}$$

If γ is a unit circle, i.e., if $r = 1$, then $w = \frac{1}{r} = 1$. From (4.6),

$$x^2 + y^2 - \frac{b^2}{a^2}z^2 = \frac{b^2}{a^4}$$

Thus, we obtain similar solutions as in [1, 8].

Step 3. If

$$J^2(\gamma)(s) = \begin{pmatrix} \frac{a^2 w}{2}\left[\frac{1}{(1+bw)^2}\sin\left(w(1+bw)s\right) + \frac{1}{(1-bw)^2}\sin\left(w(1-bw)s\right)\right], \\ -\frac{a^2 w}{2}\left[\frac{1}{(1+bw)^2}\cos\left(w(1+bw)s\right) + \frac{1}{(1-bw)^2}\cos\left(w(1-bw)s\right)\right], \\ -\frac{a}{bw}\cos\left(bw^2 s\right) \end{pmatrix}$$

then $\kappa(s) = aw^2\cos\left(bw^2 s\right)$ and $\tau(s) = aw^2\sin\left(bw^2 s\right)$. From (4.3),

$$\overline{\kappa}(s) = \epsilon a w^2 \cos\left(bw^2 s\right)\cos\left(\frac{a}{b}\cos\left(bw^2 s\right)\right)$$

and

$$\overline{\tau}(s) = \overline{\epsilon} a w^2 \cos\left(bw^2 s\right)\sin\left(\frac{a}{b}\cos\left(bw^2 s\right)\right)$$

Thus,

$$J^3(\gamma)(s)''(s) = \overline{\kappa}(s)\begin{pmatrix} \frac{a^2 w^2}{2}\left[\frac{1}{1+bw}\cos\left(w(1+bw)s\right) + \frac{1}{1-bw}\cos\left(w(1-bw)s\right)\right], \\ \frac{a^2 w^2}{2}\left[\frac{1}{1+bw}\sin\left(w(1+bw)s\right) + \frac{1}{1-bw}\sin\left(w(1-bw)s\right)\right], \\ \frac{1}{w}\sin\left(bw^2 s\right) \end{pmatrix}$$

Furthermore, from [24], if $t = e^{i\left(\frac{\pi}{2}-\phi\right)}$, then

$$\cos(x\cos\phi) = J_0(x) + 2\sum_{k=1}^{\infty}(-1)^k J_{2k}(x)\cos(2k\phi)$$

and

$$\sin(x\cos\phi) = 2\sum_{k=1}^{\infty}(-1)^k J_{2k-1}(x)\cos\left((2k-1)\phi\right)$$

where J_n is the Bessel function defined by

$$J_n(x) = \sum_{k=1}^{\infty}(-1)^k \frac{\Gamma(n+1)}{2^{2k}k!\Gamma(n+k+1)}x^{2k+n}$$

or

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos\left(n\phi - x\sin\phi\right)d\phi$$

where n is an integer. If $\phi(s) = bw^2 s$ and $x = \frac{a}{b}$, then

$$\cos\left(\frac{a}{b}\cos\left(bw^2 s\right)\right) = J_0\left(\frac{a}{b}\right) + \sum_{k=1}^{\infty} 2(-1)^k J_{2k}\left(\frac{a}{b}\right)\cos\left(2kbw^2 s\right) \tag{4.7}$$

and

$$\sin\left(\frac{a}{b}\cos\left(bw^2 s\right)\right) = 2\sum_{k=1}^{\infty}(-1)^k J_{2k-1}\left(\frac{a}{b}\right)\cos\left((2k-1)bw^2 s\right) \tag{4.8}$$

From (4.7),

$$\overline{\kappa}(s) = \epsilon a w^2 \cos\left(bw^2 s\right)\left(J_0\left(\frac{a}{b}\right) + \sum_{k=1}^{\infty} 2(-1)^k J_{2k}\left(\frac{a}{b}\right)\cos\left(2kbw^2 s\right)\right)$$

Since $J^3(\gamma)(s)''(s) = (x''(s), y''(s), z''(s))$, then

$$x''(s) = \varepsilon \frac{a^3 w^4}{4}\left(\frac{1}{1+bw}\cos\left(w(1+2bw)s\right) + \frac{1}{1-bw}\cos\left(w(1-2bw)s\right) + \frac{2}{a^2w^2}\cos\left(ws\right)\right)\left(J_0(\tfrac{a}{b}) + \sum_{k=1}^{\infty} 2(-1)^k J_{2k}(\tfrac{a}{b})\cos(2kbw^2 s)\right)$$

$$y''(s) = \varepsilon \frac{a^3 w^4}{4}\left(J_0(\tfrac{a}{b}) + \sum_{k=1}^{\infty} 2(-1)^k J_{2k}(\tfrac{a}{b})\cos(2kbw^2 s)\right)\left(\frac{1}{1+bw}\sin\left(w(1+2bw)s\right) + \frac{1}{1-bw}\sin\left(w(1-2bw)s\right) + \frac{2}{a^2w^2}\sin\left(ws\right)\right)$$

and

$$z''(s) = \varepsilon a w \cos\left(bw^2 s\right)\sin\left(bw^2 s\right)\cos\left(\frac{a}{b}\cos\left(bw^2 s\right)\right)$$

Thus,

$$\begin{aligned} x''(s) =\ & \varepsilon \frac{a^3 w^4}{4} J_0(\tfrac{a}{b})\left(\tfrac{1}{1+bw}\cos\left(w(1+2bw)s\right) + \tfrac{2}{a^2w^2}\cos\left(ws\right) + \tfrac{1}{1-bw}\cos\left(w(1-2bw)s\right)\right) \\ & + \varepsilon \frac{a^3 w^4}{4}\Bigg[\tfrac{1}{1+bw}\sum_{k=1}^{\infty}(-1)^k J_{2k}(\tfrac{a}{b})\left(\cos\left(\left(2bw\left(k+1\right)+1\right)ws\right) + \cos\left(\left(2bw\left(k-1\right)-1\right)ws\right)\right) \\ & + \tfrac{1}{1-bw}\sum_{k=1}^{\infty}(-1)^k J_{2k}(\tfrac{a}{b})\left(\cos\left(\left(2bw\left(k-1\right)+1\right)ws\right) + \cos\left(\left(2bw\left(k+1\right)-1\right)ws\right)\right) \\ & + \tfrac{1}{a^2w^2}\sum_{k=1}^{\infty} 2(-1)^k J_{2k}(\tfrac{a}{b})\left(\cos\left(\left(2kbw+1\right)ws\right) + \cos\left(\left(2kbw-1\right)ws\right)\right)\Bigg] \\ y''(s) =\ & \varepsilon \frac{a^3 w^4}{4} J_0(\tfrac{a}{b})\left(\tfrac{1}{1+bw}\sin\left(w(1+2bw)s\right) + \tfrac{2}{a^2w^2}\sin\left(ws\right) + \tfrac{1}{1-bw}\sin\left(w(1-2bw)s\right)\right) \\ & + \varepsilon \frac{a^3 w^4}{4}\Bigg[\tfrac{1}{1+bw}\sum_{k=1}^{\infty}(-1)^k J_{2k}(\tfrac{a}{b})\left(\sin\left(\left(2bw\left(k+1\right)+1\right)ws\right) + \sin\left(\left(2bw\left(k-1\right)-1\right)ws\right)\right) \\ & + \tfrac{1}{1-bw}\sum_{k=1}^{\infty}(-1)^k J_{2k}(\tfrac{a}{b})\left(\sin\left(\left(2bw\left(k-1\right)+1\right)ws\right) + \sin\left(\left(2bw\left(k+1\right)-1\right)ws\right)\right) \\ & + \tfrac{1}{a^2w^2}\sum_{k=1}^{\infty}(-1)^k J_{2k}(\tfrac{a}{b})\left(\sin\left(\left(2kbw+1\right)ws\right) + \sin\left(\left(2kbw-1\right)ws\right)\right)\Bigg] \end{aligned}$$

and

$$z''(s) = \varepsilon a w \cos\left(bw^2 s\right)\sin\left(bw^2 s\right)\cos\left(\frac{a}{b}\cos\left(bw^2 s\right)\right)$$

If we integrate the above equations, then

$$\begin{aligned} x'(s) =\ & \varepsilon \frac{a^3 w^3}{4} J_0(\tfrac{a}{b})\left(\tfrac{1}{(1+bw)(1+2bw)}\sin\left(w(1+2bw)s\right) + \tfrac{2}{a^2w^2}\sin\left(ws\right) + \tfrac{1}{(1-bw)(1-2bw)}\sin\left(w(1-2bw)s\right)\right) \\ & + \varepsilon \frac{a^3 w^3}{4}\Bigg[\tfrac{1}{1+bw}\sum_{k=1}^{\infty}(-1)^k J_{2k}(\tfrac{a}{b})\left(\tfrac{1}{(2bw(k+1)+1)}\sin\left(\left(2bw\left(k+1\right)+1\right)ws\right) + \tfrac{1}{(2bw(k-1)-1)}\sin\left(\left(2bw\left(k-1\right)-1\right)ws\right)\right) \\ & + \tfrac{1}{1-bw}\sum_{k=1}^{\infty}(-1)^k J_{2k}(\tfrac{a}{b})\left(\tfrac{1}{(2bw(k-1)+1)}\sin\left(\left(2bw\left(k-1\right)+1\right)ws\right) + \tfrac{1}{(2bw(k+1)-1)}\sin\left(\left(2bw\left(k+1\right)-1\right)ws\right)\right) \\ & + \tfrac{1}{a^2w^2}\sum_{k=1}^{\infty} 2(-1)^k J_{2k}(\tfrac{a}{b})\left(\tfrac{1}{(2kbw+1)}\sin\left(\left(2kbw+1\right)ws\right) + \tfrac{1}{(2kbw-1)}\sin\left(\left(2kbw-1\right)ws\right)\right)\Bigg] \end{aligned}$$

$$\begin{aligned}
y'(s) \;=\; & -\varepsilon \tfrac{a^3w^3}{4} J_0(\tfrac{a}{b}) \left(\tfrac{1}{(1+bw)(1+2bw)} \cos\left(w(1+2bw)s\right) + \tfrac{2}{a^2w^2} \cos\left(ws\right) + \tfrac{1}{(1-bw)(1-2bw)} \cos\left(w(1-2bw)s\right) \right) \\
& -\varepsilon \tfrac{a^3w^3}{4} \left[\tfrac{1}{1+bw} \sum_{k=1}^{\infty} (-1)^k J_{2k}(\tfrac{a}{b}) \left(\tfrac{1}{(2bw(k+1)+1)^2} \cos\left((2bw\,(k+1)+1)\,ws\right) + \tfrac{1}{(2bw(k-1)-1)^2} \cos\left((2bw\,(k-1)-1)\,ws\right) \right) \right. \\
& + \tfrac{1}{1-bw} \sum_{k=1}^{\infty} (-1)^k J_{2k}(\tfrac{a}{b}) \left(\tfrac{1}{(2bw(k-1)+1)^2} \cos\left((2bw\,(k-1)+1)\,ws\right) + \tfrac{1}{(2bw(k+1)-1)^2} \cos\left((2bw\,(k+1)-1)\,ws\right) \right) \\
& \left. + \tfrac{1}{a^2w^2} \sum_{k=1}^{\infty} 2(-1)^k J_{2k}(\tfrac{a}{b}) \left(\tfrac{1}{(2kbw+1)^2} \cos\left((2kbw+1)\,ws\right) + \tfrac{1}{(2kbw-1)^2} \cos\left((2kbw-1)\,ws\right) \right) \right]
\end{aligned}$$

and

$$z'(s) = \frac{\varepsilon b^3 w^4}{a} \left(\cos\left(\frac{a}{b} \cos\left(bw^2 s\right) \right) + \frac{a}{b} \cos\left(bw^2 s\right) \sin\left(\frac{a}{b} \cos\left(bw^2 s\right) \right) \right)$$

where $(x'(s))^2 + (y'(s))^2 + (z'(s))^2 = 1$. From (4.7) and (4.8),

$$z'(s) = \tfrac{\epsilon b^3 w^4}{a} \left(J_0(\tfrac{a}{b}) + \sum_{k=1}^{\infty} 2(-1)^k J_{2k}(\tfrac{a}{b}) \cos(2kbw^2 s) + \tfrac{2a}{b} \cos\left(bw^2 s\right) \sum_{k=1}^{\infty} (-1)^k J_{2k-1}(\tfrac{a}{b}) \cos\left((2k-1)bw^2 s\right) \right)$$

Hence,

$$z'(s) = -\tfrac{\epsilon b}{aw} \left(J_0(\tfrac{a}{b}) + \sum_{k=1}^{\infty} 2(-1)^k J_{2k}(\tfrac{a}{b}) \cos(2kbw^2 s) + \tfrac{a}{b} \sum_{k=1}^{\infty} (-1)^k J_{2k-1}(\tfrac{a}{b}) \left(\cos\left(kbw^2 s\right) + \cos\left(2(k-1)bw^2 s\right) \right) \right)$$

If we integrate the above equations, then we have a curve $J^3(\gamma)(s) = (x(s), y(s), z(s))$ where

$$\begin{aligned}
x(s) \;=\; & -\varepsilon \tfrac{a^3w^2}{4} J_0(\tfrac{a}{b}) \left(\tfrac{1}{(1+bw)(1+2bw)^2} \cos\left(w(1+2bw)s\right) + \tfrac{2}{a^2w^2} \cos\left(ws\right) + \tfrac{1}{(1-bw)(1-2bw)^2} \cos\left(w(1-2bw)s\right) \right) \\
& -\varepsilon \tfrac{a^3w^2}{4} \left[\tfrac{1}{1+bw} \sum_{k=1}^{\infty} (-1)^k J_{2k}(\tfrac{a}{b}) \left(\tfrac{1}{(2bw(k+1)+1)^2} \cos\left((2bw\,(k+1)+1)\,ws\right) + \tfrac{1}{(2bw(k-1)-1)^2} \cos\left((2bw\,(k-1)-1)\,ws\right) \right) \right. \\
& + \tfrac{1}{1-bw} \sum_{k=1}^{\infty} (-1)^k J_{2k}(\tfrac{a}{b}) \left(\tfrac{1}{(2bw(k-1)+1)^2} \cos\left((2bw\,(k-1)+1)\,ws\right) + \tfrac{1}{(2bw(k+1)-1)^2} \cos\left((2bw\,(k+1)-1)\,ws\right) \right) \\
& \left. - \tfrac{1}{a^2w^2} \sum_{k=1}^{\infty} 2(-1)^k J_{2k}(\tfrac{a}{b}) \left(\tfrac{1}{(2kbw+1)^2} \cos\left((2kbw+1)\,ws\right) + \tfrac{1}{(2kbw-1)^2} \cos\left((2kbw-1)\,ws\right) \right) \right]
\end{aligned}$$

$$\begin{aligned}
y(s) \;=\; & -\varepsilon \tfrac{a^3w^2}{4} J_0(\tfrac{a}{b}) \left(\tfrac{1}{(1+bw)(1+2bw)^2} \sin\left(w(1+2bw)s\right) + \tfrac{2}{a^2w^2} \sin\left(ws\right) + \tfrac{1}{(1-bw)(1-2bw)^2} \sin\left(w(1-2bw)s\right) \right) \\
& -\varepsilon \tfrac{a^3w^2}{4} \left[\tfrac{1}{1+bw} \sum_{k=1}^{\infty} (-1)^k J_{2k}(\tfrac{a}{b}) \left(\tfrac{1}{(2bw(k+1)+1)} \sin\left((2bw\,(k+1)+1)\,ws\right) + \tfrac{1}{(2bw(k-1)-1)} \sin\left((2bw\,(k-1)-1)\,ws\right) \right) \right. \\
& + \tfrac{1}{1-bw} \sum_{k=1}^{\infty} (-1)^k J_{2k}(\tfrac{a}{b}) \left(\tfrac{1}{(2bw(k-1)+1)} \sin\left((2bw\,(k-1)+1)\,ws\right) + \tfrac{1}{(2bw(k+1)-1)} \sin\left((2bw\,(k+1)-1)\,ws\right) \right) \\
& \left. + \tfrac{1}{a^2w^2} \sum_{k=1}^{\infty} 2(-1)^k J_{2k}(\tfrac{a}{b}) \left(\tfrac{1}{(2kbw+1)} \sin\left((2kbw+1)\,ws\right) + \tfrac{1}{(2kbw-1)} \sin\left((2kbw-1)\,ws\right) \right) \right]
\end{aligned}$$

and

$$z(s) = -\tfrac{\varepsilon b}{aw} \left[J_0(\tfrac{a}{b}) s + \sum_{k=1}^{\infty} \tfrac{(-1)^k J_{2k}(\frac{a}{b})}{kbw^2} \sin(2kbw^2 s) + \tfrac{a}{b^2w^2} \sum_{k=1}^{\infty} (-1)^k J_{2k-1}(\tfrac{a}{b}) \left(\tfrac{1}{k} \sin\left(kbw^2 s\right) + \tfrac{1}{2(k-1)} \sin\left(2(k-1)bw^2 s\right) \right) \right]$$

Therefore, we have a 2-slant curve $J^3(\gamma)(s)$.

Definition 4.5. In three dimensional Euclidean spaces, if $\gamma = S^1(\overrightarrow{a}, r)$, then the set

$$\mathbb{Z}(\gamma) = \left\{ \ldots, J^{-2}(\gamma), J^{-1}(\gamma), \gamma, J(\gamma), J^2(\gamma), \ldots \right\}$$

is said to be constant precession curve chain. Hence, the set of all the N_k-constant procession curves, denoted by $\mathbb{Z}^N$, is as follows:

$$\mathbb{Z}^N = \bigcup_{\substack{\overrightarrow{a} \in E^3 \\ r>0}} \mathbb{Z}\left(S^1(\overrightarrow{a}, r)\right)$$

Furthermore, the set of all the k-slant curves in three dimensional Euclidean spaces is provided by $\mathbb{Z}^J \cup S^I$ where

$$\mathbb{Z}^J = \bigcup_{\gamma \text{ is a planar curve}} \mathbb{Z}(\gamma)$$

5. k-Slant Curves and Magnetic Curves

Let M be $2n+1$-smooth manifold. If there exists a structure (ϕ, ξ, η, g) such that for all $X, Y \in \chi(M)$, $\phi^2(X) = -X + \eta(X)\xi$, $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$, $\eta(X) = g(X, \xi)$, $\eta(\xi) = 1$, $\phi(\xi) = 0$, and $\eta \circ \phi = 0$, then it is said that (M, ϕ, ξ, η, g) is an almost contact metric manifold where ϕ, ξ, and η are type tensors $(1,0)$, $(1,0)$, and $(0,1)$, respectively, and g is a metric tensor [25]. In an almost contact metric manifold, it is said that Φ is a fundamental form of the almost contact metric manifold where $\Phi(X,Y) = g(\phi X, Y)$ [25]. Let (M, ϕ, ξ, η, g) be a 3-dimensional almost contact metric manifold. The extended of the cross-product is defined as follows [26]:

$$X \wedge Y = -g(\phi X, Y) - \eta(Y)\phi X + \eta(X)\phi Y, \quad \text{for all } X, Y \in \chi(M)$$

Thus, $\phi(X) = \xi \wedge X$. In three dimensional Euclidean spaces, the set $V = \{(v_1, v_2, 0) : v_1, v_2 \in R\}$ is a subspace of R^3. Hence, we can define a natural projection from $\mathbb{R}^3$ to V by $\pi(v_1, v_2, v_3) = (v_1, v_2, 0)$ and an almost complex map on V given by $J(v_1, v_2, 0) = (-v_2, v_1, 0)$. Consider $\phi = J \circ \pi$, $\eta = dz$, and $\xi = \frac{\partial}{\partial z}$. Then, $(R^3, \phi, \xi, \eta, g)$ is an almost contact metric manifold where g is the standard Euclid metric [26]. In this case, $X \wedge Y = X \times Y$ where $\times$ is the ordinary cross product [26]. Let γ be a unit-speed regular curve with a coordinate neighborhood (I, γ) and $\{T, N, B, \kappa, \tau\}$ be the Serret-Frenet apparatus of the curve. Hence, $T \wedge N = B$, $N \wedge B = T$, and $B \wedge T = N$. Then, the following equations hold:

$$\phi(T) = \eta(B)N - \eta(N)B$$

$$\phi(N) = \eta(T)B - \eta(B)T$$

and

$$\phi(B) = \eta(N)T - \eta(T)N$$

where $\xi = \eta(T)T + \eta(N)N + \eta(B)B$ and $\eta(T)^2 + \eta(N)^2 + \eta(B)^2 = 1$ [26]. Let ξ be a magnetic field and Φ be a close 2-form on M^3 where $\Phi(X,Y) = g(\phi X, Y)$ and $\phi(X) = \xi \wedge X$. Here, ϕ is the Lorentz force of Φ. If the following Landau-Hall equation is satisfied, then the curve γ is the magnetic curve of (M, g, Φ) where ∇ is a Levi-Civita connection of g [27]:

$$\nabla_T T = \phi(T) = \xi \times T$$

In this case, $(M^3, \phi, \xi, \eta, g)$ is an almost contact metric manifold, and Φ is the fundamental form of this manifold. From the Landau-Hall equation,

$$\begin{pmatrix} \phi(T) \\ \phi(N) \\ \phi(B) \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & w \\ 0 & -w & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

Therefore, the curve γ is magnetic curve if and only if $\xi = wT + \kappa B$ [28]. Bozkurt et al. [29] have defined a new type Landau-Hall equation as follows:

$$\nabla_T N = \phi(N) = \xi \times N$$

where N is a normal vector field along the curve. If

$$\nabla_T N = \phi(N) = \xi \times N$$

then γ is an $N-$magnetic curve of (M, g, Φ) [29]. From the Landau-Hall equation,

$$\begin{pmatrix} \phi(T) \\ \phi(N) \\ \phi(B) \end{pmatrix} = \begin{pmatrix} 0 & \kappa & \Omega \\ -\kappa & 0 & \tau \\ -\Omega & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

Then, the curve γ is an $N-$magnetic curve if and only if $\xi = \tau T + \kappa B - \Omega N$ [29]. Therefore, Bozkurt et al. [29] have defined a B-magnetic curve of (M, g, Φ) [29]. If

$$\nabla_T B = \phi(B) = \xi \times B$$

then the curve γ is a B-magnetic curve of (M, g, Φ) [29]. From the Landau-Hall equation,

$$\begin{pmatrix} \phi(T) \\ \phi(N) \\ \phi(B) \end{pmatrix} = \begin{pmatrix} 0 & w & 0 \\ -w & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

Therefore, the curve γ is a B-magnetic curve if and only if $\xi = \tau T + wB$ [29]. Similarly, we can define the generalized Landau-Hall equation as follows:

$$\nabla_T Z = \phi(Z) = \xi \times Z$$

Hence, it is said that the curve γ is an Z-magnetic curve of (M, g, Φ). From the generalized Landau-Hall equation, we have the following theorem.

Theorem 5.1. The curve γ is an Z-magnetic curve of (M, g, Φ) if and only if

$$\begin{pmatrix} Z_1' \\ Z_2' \\ Z_3' \end{pmatrix} = \begin{pmatrix} 0 & \kappa - \xi_3 & \xi_2 \\ -(\kappa - \xi_3) & 0 & \tau - \xi_1 \\ -\xi_2 & -(\tau - \xi_1) & 0 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}$$

where $\xi = \xi_1 T + \xi_2 N + \xi_3 B$ and $Z = Z_1 T + Z_2 N + Z_3 B$

If $\nabla_T N_k = \phi(N_k) = \xi \times N_k$, then it is said that the curve γ is an N_k-magnetic curve. The Lorentz force in the Serret-Frenet frame of γ_k is given by

$$\begin{pmatrix} \phi(T_k) \\ \phi(N_k) \\ \phi(B_k) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_k & \Omega_{k+1} \\ -\kappa_k & 0 & \tau \\ -\Omega_{k+1} & -\tau_k & 0 \end{pmatrix} \begin{pmatrix} T_k \\ N_k \\ B_k \end{pmatrix}$$

and the Serret-Frenet formulas of the curves are given by

$$\begin{pmatrix} T_k' \\ N_k' \\ B_k' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_k & 0 \\ -\kappa_k & 0 & \tau_k \\ 0 & -\tau_k & 0 \end{pmatrix} \begin{pmatrix} T_k \\ N_k \\ B_k \end{pmatrix}$$

Thus, the curve γ is an N_k-magnetic curve of the magnetic field ξ if and only if

$$\xi = \tau_k T_k - \Omega_{k+1} N_k + \kappa_k B_k \in \mathrm{Ker}\phi$$

If we derive ξ along the curve, then

$$\nabla_T \xi = (\tau_k' + \Omega_{k+1}\kappa_k)\, T_k - \Omega_{k+1}' N_k + (\kappa_k' - \Omega_{k+1}\tau_k)\, B_k$$

If ξ is a constant vector field along the curve, then

$$\tau_k' = -\Omega_{k+1}\kappa_k \tag{5.1}$$

and

$$\kappa_k' = \Omega_{k+1}\tau_k \tag{5.2}$$

where Ω_{k+1} is a constant. From (5.1) and (5.2),

$$\kappa_k = R\cos\left(\Omega_{k+1}s + c_0\right) \tag{5.3}$$

and

$$\tau_k = R\sin\left(\Omega_{k+1}s + c_0\right) \tag{5.4}$$

where R is a constant. From (5.3) and (5.4), $\gamma_k = J^2(S^1(\overrightarrow{a}, r))$ and $\gamma = J^{k+2}(S^1(\overrightarrow{a}, r))$. Furthermore, Ramiz et al. [19] have defined N_k-constant procession curves in 3-Euclidean spaces. In this spaces, the Darboux vectors of γ_k is defined as

$$W_k = \tau_k T_k + \kappa_k B_k$$

and

$$A_k = W_k \pm \Omega_{k+1} N_k$$

where Ω_{k+1} is a constant. Then, the curve γ is said to be an N_k-constant procession curve in three dimensional Euclidean spaces if there exists a constant angle between W_k and fixed direction A_k. From Theorem 4 in [19], the following statements are equivalent:

i. The curve γ is an N_k-constant procession curve

ii. $\kappa_k = R\cos\left(\Omega_{k+1}s + c_0\right)$ and $\tau_k = R\sin\left(\Omega_{k+1}s + c_0\right)$

where Ω_{k+1} and c_0 are constants. Consequently, the following theorem is obtained:

Theorem 5.2. In three dimensional Euclidean spaces, an N_k-magnetic curve is an N_k-constant procession curve if and only if $\xi = \tau_k T_k - \Omega_{k+1} N_k + \kappa_k B_k$ is a constant vector field along the curve.

6. Conclusion

In this study, we proposed a method for constructing k-slant curves from spherical curves in three dimensional Euclidean space. We showed that spherical k-slant and N_k-constant precession curves can be derived from circles, offering a novel proof for characterizing spherical curves. Furthermore, we introduced the concept of spherical prime curves and applied k-slant curves to magnetism theory, highlighting their significance in the analysis of magnetic curves. Generalizing k-slant curves to higher-dimensional spaces or non-Euclidean geometries is worth studying for future research. Moreover, their potential applications can be investigated, particularly in fields like quantum mechanics or fluid dynamics.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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