Journal of New Theory
dergipark.org.tr/tr/pub/jnt
Open Access

ISSN: 2149-1402 (51) (2025) 1-9 doi.org/10.53570/jnt.1647686

Semiconformal Curvature Tensor on (κ, μ)-Paracontact Metric Manifolds

Ümit Yıldırım[1], Mustafa Arslan[2]

Article Info
Received: 26 Feb 2025
Accepted: 18 May 2025
Published: 30 Jun 2025
Research Article

Abstract— This paper investigates several properties of the semiconformal curvature tensor on a (κ, μ)-paracontact metric manifold. It first examines the results arising when such a manifold is both semiconformal and semisymmetric. Based on these findings, this study provides characterizations of the manifold. It then explores the derivative interactions between various curvature tensors and the semiconformal curvature tensor. According to the results, the present paper establishes the conditions under which a (κ, μ)-paracontact metric manifold reduces to a (κ, μ)-paracontact metric manifold.

Keywords — *Semiconformal curvature tensor, (κ, μ)-paracontact metric manifolds, semisymmetric*

Mathematics Subject Classification (2020) 53C15, 53D10

1. Introduction

In 1985, the first studies on paracontact structures and paracontact geometry were presented by Kaneyuki and Williams [1]. Later, paracontact geometry studies were continued by Zamkovoy [2] with his systematic studies on paracontact manifolds and submanifolds. The more general case of paracontact metric manifolds, (κ, μ)-paracontact metric manifolds and (κ, μ, ν)- paracontact metric manifolds continue to be studied by many authors [3–7].

Let M be a $(2n+1)$-dimensional smooth manifold, and let ϕ be a $(1,1)$-tensor field, ξ a characteristic vector field, and η a 1-form on M. The triple (ϕ, ξ, η) is called an almost paracontact structure on M if it satisfies the following conditions:

i. $\phi(\xi) = 0$, $\eta o \phi = 0$, and $\eta(\xi) = 1$

ii. $\phi^2 \varrho_1 = \varrho_1 - \eta(\varrho_1)\xi$ and $\varrho_1 \in \chi(M)$

iii. The tensor field ϕ induces an almost paracomplex structure on each fibre of $D = \ker(\eta)$, meaning that the eigendistributions $D^+\phi$ and $D^-\phi$ of ϕ, corresponding to the eigenvalues 1 and -1, respectively, have equal dimension n.

The (M, ϕ, ξ, η) quadruple together with the (ϕ, ξ, η) structure on M is called an almost paracontact manifold [2]. There exists a semi-Riemannian metric g on the almost paracontact metric manifold M such that

[1]umit.yildirim@amasya.edu.tr (Corresponding Author); [2]mustafa05musti@gmail.com
[1,2]Department of Mathematics, Faculty of Arts and Sciences, Amasya University, Amasya, Türkiye

$$g(\phi\varrho_1, \phi\varrho_2) = -g(\varrho_1, \varrho_2) + \eta(\varrho_1)\eta(\varrho_2), g(\varrho_1, \xi) = \eta(\varrho_1) \tag{1.1}$$

and

$$g(\phi\varrho_1, \varrho_2) + g(\varrho_1, \phi\varrho_2) = 0$$

Let $(M^{2n+1}, \phi, \xi, \eta)$ be an almost paracontact manifold. Then, the semi-Riemannian metric g is called an almost paracontact metric on $(M^{2n+1}, \phi, \xi, \eta)$, the structure (ϕ, ξ, η, g) is called an almost paracontact metric structure, and the quintet $(M^{2n+1}, \phi, \xi, \eta, g)$ is called an almost paracontact metric manifold [2]. If

$$d\eta(\varrho_1, \varrho_2) = \frac{1}{2}(\varrho_1\eta(\varrho_2) - \varrho_2\eta(\varrho_1) - \eta([\varrho_1, \varrho_2]))$$

is defined, then

$$\Phi(\varrho_1, \varrho_2) = g(\varrho_1, \phi\varrho_2) = d\eta(\varrho_1, \varrho_2)$$

The (ϕ, ξ, η, g) quadruplet is called a paracontact metric structure, and the quintet (M, ϕ, ξ, η, g) is called a paracontact metric manifold [2]. If M is a $(2n+1)$-dimensional almost paracontact metric manifold and the vector field ξ in the structure (ϕ, ξ, η, g) is a Killing vector field concerning g. The paracontact structure on M is called a K-paracontact structure, and M is called a K-paracontact metric manifold [2]. On a paracontact metric manifold, h is a symmetric operator, and the following properties hold:

$$h\xi = 0, \quad h\phi = -\phi h, \quad \text{and} \quad \operatorname{Tr} h = \operatorname{Tr} \phi h = 0 \tag{1.2}$$

$$2h\varrho_1 = (L_\xi\phi)\varrho_1 = L_\xi\phi\varrho_1 - \phi L_\xi\varrho_1 = [\xi, \phi\varrho_1] - \phi[\xi, \varrho_1]$$

$$\nabla_{\varrho_1}\xi = -\phi\varrho_1 + \phi h\varrho_1$$

Here, L is the Lie derivative [2].

Motivated by the above studies, this work investigates some symmetry conditions of a (κ, μ)-paracontact metric manifold. This paper consists of four sections. The first section provides basic information about almost paracontact metric manifolds. The second section introduces basic definitions and properties of (κ, μ)-paracontact metric manifolds and semiconformal curvature tensors. The third section obtains the results when a (κ, μ)-paracontact metric manifold is semiconformally semisymmetric. The fourth section investigates the condition for a (κ, μ)-paracontact metric manifold to be semiconformally Ricci semisymmetric and characterizes the manifold according to the obtained results. The last section concludes the paper.

2. Preliminaries

This section presents some notions to be needed in the following sections.

Definition 2.1. [4] A paracontact metric manifold is said to be a (κ, μ)-paracontact manifold if the curvature tensor R satisfies the following conditions

$$R(\varrho_1, \varrho_2)\xi = \kappa(\eta(\varrho_2)\varrho_1 - \eta(\varrho_1)\varrho_2) + \mu(\eta(\varrho_2)h\varrho_1 - \eta(\varrho_1)h\varrho_2) \tag{2.1}$$

for all $\varrho_1, \varrho_2 \in \chi(M)$ and κ and μ are real constants.

Here, if $\mu = 0$, then the (κ, μ)-paracontact metric manifold is called $N(\kappa)$-paracontact metric manifold.

In a (κ, μ)-paracontact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ such that $n > 1$, the following relations hold:

$$h^2 = (\kappa + 1)\phi^2$$

$$(\nabla_{\varrho_1}\phi)\varrho_2 = -g(\varrho_1 - h\varrho_1, \varrho_2)\xi + \eta(\varrho_2)(\varrho_1 - h\varrho_1)$$

for $\kappa \neq -1$.

$$S(\varrho_1, \varrho_2) = (2(1-n) + \nu\mu)g(\varrho_1, \varrho_2) + (2(n-1) + \mu)g(h\varrho_1, \varrho_2) + (2(n-1) + n(2\kappa - \mu))\eta(\varrho_1)\eta(\varrho_2)$$

$$S(\varrho_1, \xi) = 2n\kappa\eta(\varrho_1) \tag{2.2}$$

$$Q\varrho_1 = (2(1-n) + n\mu)\varrho_1 + (2(n-1) + \mu)h\varrho_1 + (2(n-1) + n(2\kappa - \mu))\eta(\varrho_1)\xi$$

$$\phi\xi = 2n\kappa\xi$$

and

$$Q\phi - \phi Q = 2(2(n-1) + \mu)h\phi$$

In 2017, Kim [8] defined a curvature tensor of $(1,3)-$type that remains invariant under conharmonic transformation, called semiconformal curvature tensor, and obtained some results. The semiconformal curvature tensor is a generalization of the conformal curvature tensor and the conharmonic curvature tensor. Recently, many conditions of the semiconformal curvature tensor have been studied by many researchers [8–20]. The semiconformal curvature tensor P of $(1,3)-$type on a Riemann manifold (M^{2n+1}, g), $n > 1$, is as follows:

$$P(\varrho_1, \varrho_2)\varrho_3 = -(n-2)bC(\varrho_1, \varrho_2)\varrho_3 + (a + (n-2)b)H(\varrho_1, \varrho_2)\varrho_3 \tag{2.3}$$

where a and b are constants and not simultaneously zero, $C(\varrho_1, \varrho_2)\varrho_3$ denotes the conformal curvature tensor of $(1,3)-$type, and $H(\varrho_1, \varrho_2)\varrho_3$ denotes the conharmonic curvature tensor of $(1,3)-$type. The conformal curvature tensor of $(1,3)-$type and the conharmonic curvature tensor of $(1,3)-$type are given as:

$$\begin{aligned} C(\varrho_1, \varrho_2)\varrho_3 = {} & R(\varrho_1, \varrho_2)\varrho_3 - \tfrac{1}{n-2}(S(\varrho_2, \varrho_3)\varrho_1 - S(\varrho_1, \varrho_3)\varrho_2 + g(\varrho_2, \varrho_3)Q\varrho_1 - g(\varrho_1, \varrho_3)Q\varrho_2) \\ & + \tfrac{r}{(n-1)(n-2)}(g(\varrho_2, \varrho_3)\varrho_1 - g(\varrho_1, \varrho_3)\varrho_2) \end{aligned} \tag{2.4}$$

and

$$H(\varrho_1, \varrho_2)\varrho_3 = R(\varrho_1, \varrho_2)\varrho_3 - \frac{1}{n-2}(S(\varrho_2, \varrho_3)\varrho_1 - S(\varrho_1, \varrho_3)\varrho_2 + g(\varrho_2, \varrho_3)Q\varrho_1 - g(\varrho_1, \varrho_3)Q\varrho_2) \tag{2.5}$$

where r is scalar curvature, R is Riemann curvature tensor of $(1,3)-$type of the manifold M^{2n+1}, and S is the Ricci tensor of the manifold, given by $g(Q\varrho_1, \varrho_2) = S(\varrho_1, \varrho_2)$, where Q is the Ricci operator. Here, if (2.5) and (2.4) are used in (2.3), then (2.3) is reduced to the following form:

$$P(\varrho_1, \varrho_2)\varrho_3 = aH(\varrho_1, \varrho_2)\varrho_3 - \frac{br}{n-1}(g(\varrho_2, \varrho_3)\varrho_1 - g(\varrho_1, \varrho_3)\varrho_2) \tag{2.6}$$

Putting $Z = \xi$ and using (1.1), (2.1), and (2.2),

$$H(\varrho_1, \varrho_2)\xi = \left(\tfrac{-(\kappa+2)-n(\mu-2)}{2n-1}\right)(\eta(\varrho_2)\varrho_1 - \eta(\varrho_1)\varrho_2) + \left(\tfrac{(2\mu-2)(n-1)}{2n-1}\right)(\eta(\varrho_2)h\varrho_1 - \eta(\varrho_1)h\varrho_2) \tag{2.7}$$

Put $\varrho_2 = \xi$ in (2.7),

$$H(\varrho_1, \varrho_2)\xi = \left(\frac{-(\kappa+2) - n(\mu-2)}{2n-1}\right)(\varrho_1 - \eta(\varrho_1)\xi) + \left(\frac{(2\mu-2)(n-1)}{2n-1}\right)h\varrho_1 \tag{2.8}$$

Moreover, putting $\varrho_1 = \xi$ in (2.5),

$$H(\xi,\varrho_2)\varrho_3 = \left(\frac{2n-n\mu-4}{2n-1}\right)(g(\varrho_2,\varrho_3)\xi-\eta(\varrho_3)\varrho_2) - \left(\frac{2(n-1)(\mu-1)}{2n-1}\right)(g(h\varrho_2,\varrho_3)\xi-\eta(\varrho_3)h\varrho_2) \quad (2.9)$$

Putting $\varrho_1 = \xi$ in (2.6) and using (2.9),

$$P(\xi,\varrho_2)\varrho_3 = \left(\frac{a[2n-n\mu-4]}{2n-1} - \frac{br}{2n}\right)(g(\varrho_2,\varrho_3)\xi-\eta(\varrho_3)\varrho_2) - \left(\frac{2a(n-1)(\mu-1)}{2n-1}\right)(g(h\varrho_2,\varrho_3)\xi-\eta(\varrho_3)h\varrho_2) \quad (2.10)$$

Similarly, choosing $\varrho_3 = \xi$ in (2.6) and (2.8),

$$P(\varrho_1,\varrho_2)\xi = \left(\frac{-a(\kappa+2)-an(\mu-2)}{2n-1} - \frac{br}{2n}\right)(\eta(\varrho_2)\varrho_1-\eta(\varrho_1)\varrho_2) + \left(\frac{2a(n-1)(\mu-1)}{2n-1}\right)(\eta(\varrho_2)h\varrho_1-\eta(\varrho_1)h\varrho_2) \quad (2.11)$$

3. Semiconformally Semisymmetric (κ,μ)-paracontact Metric Manifold

This section introduces a semiconformally semisymmetric (κ,μ)-paracontact metric manifold.

Definition 3.1. A (κ,μ)-paracontact metric manifold M^{2n+1} is said to be semiconformally semisymmetric if the semiconformal curvature tensor satisfies the condition

$$R(\varrho_1,\varrho_2)P = 0$$

for all vector fields ϱ_1 and ϱ_2 on M.

Theorem 3.2. Let M be a $(2n+1)-$dimensional (κ,μ)-paracontact metric manifold. Then, M is semiconformally semisymmetric if and only if at least one of the following statements is true:

i. M is a $(\kappa,1)$-paracontact metric manifold.

ii. The semiconformal curvature tensor P of M reduces to the form

$$P(\varrho_1,\varrho_2)\varrho_3 = \frac{-br}{2n-1}(g(\varrho_2,\varrho_3)\varrho_1 - g(\varrho_1,\varrho_3)\varrho_2)$$

for all $\varrho_1,\varrho_2,\varrho_3 \in \chi(M)$.

iii. $(\kappa+1)(\kappa-\mu(\kappa+1))h^2 + (\mu-\kappa)h = 0.$

Proof. Assume that the $(2n+1)-$dimensional (κ,μ)-paracontact metric manifold M is semiconformally semisymmetric. Then, for all $\varrho_1,\varrho_2,\varrho_3,\varrho_4,\varrho_5 \in \chi(M)$,

$$\begin{aligned}(R(\varrho_1,\varrho_2)P)(\varrho_4,\varrho_5,\varrho_3) &= R(\varrho_1,\varrho_2)P(\varrho_4,\varrho_5)\varrho_3 - P(R(\varrho_1,\varrho_2)\varrho_4,\varrho_5)\varrho_3 - P(\varrho_4,R(\varrho_1,\varrho_2)\varrho_5)\varrho_3 \\ &\quad -P(\varrho_4,\varrho_5)R(\varrho_1,\varrho_2)\varrho_3 \\ &= 0\end{aligned} \quad (3.1)$$

Choosing $\varrho_3 = \xi$ in (3.1),

$$R(\varrho_1,\varrho_2)P(\varrho_4,\varrho_5)\xi - P(R(\varrho_1,\varrho_2)\varrho_4,\varrho_5)\xi - P(\varrho_4,R(\varrho_1,\varrho_2)\varrho_5)\xi - P(\varrho_4,\varrho_5)R(\varrho_1,\varrho_2)\xi = 0 \quad (3.2)$$

Using (2.1) and (2.11) in (3.2),

$$\begin{aligned}0 &= A\eta(\varrho_5)R(\varrho_1,\varrho_2)\varrho_4 - A\eta(\varrho_4)R(\varrho_1,\varrho_2)\varrho_5 + B\eta(\varrho_5)R(\varrho_1,\varrho_2)h\varrho_4 \\ &\quad -B\eta(\varrho_4)R(\varrho_1,\varrho_2)h\varrho_5 - A\eta(\varrho_5)R(\varrho_1,\varrho_2)\varrho_4 + A\eta(R(\varrho_1,\varrho_2)\varrho_4)\varrho_5 \\ &\quad -B\eta(\varrho_5)hR(\varrho_1,\varrho_2)\varrho_4 + B\eta(R(\varrho_1,\varrho_2)\varrho_4)h\varrho_5 - A\eta(R(\varrho_1,\varrho_2)\varrho_5)\varrho_4 \\ &\quad +A\eta(\varrho_4)R(\varrho_1,\varrho_2)\varrho_5 - B\eta(R(\varrho_1,\varrho_2)\varrho_5)h\varrho_4 + B\eta(\varrho_4)hR(\varrho_1,\varrho_2)\varrho_5 \\ &\quad -\kappa\eta(\varrho_2)P(\varrho_4,\varrho_5)\varrho_1 + \kappa\eta(\varrho_1)P(\varrho_4,\varrho_5)\varrho_2 - \mu\eta(\varrho_2)P(\varrho_4,\varrho_5)h\varrho_1 \\ &\quad +\mu\eta(\varrho_1)P(\varrho_4,\varrho_5)h\varrho_2\end{aligned}$$

where

$$A=\frac{-a(\kappa+2)-an(\mu-2)}{2n-1}-\frac{br}{2n}\quad\text{and}\quad B=\frac{2a(n-1)(\mu-1)}{2n-1}$$

Thus,

$$\begin{aligned}0 =\ & B\eta(\varrho_5)R(\varrho_1,\varrho_2)h\varrho_4-B\eta(\varrho_4)R(\varrho_1,\varrho_2)h\varrho_5+A\eta(R(\varrho_1,\varrho_2)\varrho_4)\varrho_5\\ &-B\eta(\varrho_5)hR(\varrho_1,\varrho_2)\varrho_4+B\eta(R(\varrho_1,\varrho_2)\varrho_4)h\varrho_5-A\eta(R(\varrho_1,\varrho_2)\varrho_5)\varrho_4\\ &-B\eta(R(\varrho_1,\varrho_2)\varrho_5)h\varrho_4+B\eta(\varrho_4)hR(\varrho_1,\varrho_2)\varrho_5-\kappa\eta(\varrho_2)P(\varrho_4,\varrho_5)\varrho_1\\ &+\kappa\eta(\varrho_1)P(\varrho_4,\varrho_5)\varrho_2-\mu\eta(\varrho_2)P(\varrho_4,\varrho_5)h\varrho_{11}+\mu\eta(\varrho_1)P(\varrho_4,\varrho_5)h\varrho_2\end{aligned}\tag{3.3}$$

Putting $Y=U=\xi$ in (3.3) and using (1.1), (1.2), (2.1), (2.10), and (2.11),

$$\begin{aligned}0 =\ & B\kappa g(\varrho_1,\varrho_5)\xi-B\kappa\eta(\varrho_5)\varrho_1+Bg(h\varrho_1,\varrho_5)\xi-B\mu\eta(\varrho_5)h\varrho_1-A\eta(\varrho_1)\varrho_5+B\kappa\eta(\varrho_1)h\varrho_5\\ &-B\eta(\varrho_1)h\varrho_5+A\kappa\eta(\varrho_1)\varrho_5-B\kappa g(h\varrho_5,\varrho_1)\xi+B\kappa\eta(\varrho_1)h\varrho_5+B\kappa\eta(\varrho_1)h\varrho_5-\mu Bg(h\varrho_5,h\varrho_1)\xi\end{aligned}\tag{3.4}$$

In (3.4), taking inner product with $\xi\in\chi(M)$,

$$B\left(\kappa\left(g(\varrho_1,\varrho_5)-\eta(\varrho_5)\eta(\varrho_1)-g(h\varrho_5,\varrho_1)\right)+\mu\left(g(h\varrho_1,\varrho_5)-g(h\varrho_5,h\varrho_1)\right)\right)=0\tag{3.5}$$

This equation is satisfied for the following three cases:

i. $B=\frac{2a(n-1)(\mu-1)}{2n-1}=0$ and $\kappa\left(g(\varrho_1,\varrho_5)-\eta(\varrho_5)\eta(\varrho_1)-g(h\varrho_5,\varrho_1)\right)+\mu\left(g(h\varrho_1,\varrho_5)-g(h\varrho_5,h\varrho_1)\right)\neq 0$

ii. $B=\frac{2a(n-1)(\mu-1)}{2n-1}\neq 0$ and $\kappa\left(g(\varrho_1,\varrho_5)-\eta(\varrho_5)\eta(\varrho_1)-g(h\varrho_5,\varrho_1)\right)+\mu\left(g(h\varrho_1,\varrho_5)-g(h\varrho_5,h\varrho_1)\right)=0$

iii. $B=\frac{2a(n-1)(\mu-1)}{2n-1}=0$ and $\kappa\left(g(\varrho_1,\varrho_5)-\eta(\varrho_5)\eta(\varrho_1)-g(h\varrho_5,\varrho_1)\right)+\mu\left(g(h\varrho_1,\varrho_5)-g(h\varrho_5,h\varrho_1)\right)=0$

Here, if $B=\frac{2a(n-1)(\mu-1)}{2n-1}=0$, then $\mu=1$ and M is reduced to a $(\kappa,1)$ manifold. If $a=0$, then from (2.6)

$$P(X\varrho_1,\varrho_2)\varrho_3=-\frac{br}{2n-1}(g(\varrho_2,\varrho_3)\varrho_1-g(\varrho_1,\varrho_3)\varrho_2)$$

Finally, from (3.5),

$$\kappa(g(\varrho_1,\varrho_5)-\eta(\varrho_5)\eta(\varrho_1)-g(h\varrho_5,\varrho_1))+\mu(g(h\varrho_1,\varrho_5)-g(h\varrho_5,h\varrho_1))=0$$

Since h is symmetric, then

$$\kappa(g(\varrho_1,\varrho_5)-\eta(\varrho_5)\eta(\varrho_1)-g(h\varrho_5,\varrho_1))+\mu(g(h\varrho_1,\varrho_5)-g(h^2\varrho_5,\varrho_1))=0\tag{3.6}$$

Using (1.1) in (3.6),

$$\kappa(g(\phi\varrho_1,\phi\varrho_5)-g(h\varrho_1,\varrho_5))+\mu(g(h\varrho_1,\varrho_5)-(\kappa+1)g(\phi^2\varrho_1,\varrho_5))=0$$

and thus

$$\kappa\left(-g(\phi^2\varrho_1,\varrho_5)-g(h\varrho_1,\varrho_5)\right)+\mu\left(g(h\varrho_1,\varrho_5)-(\kappa+1)g(\phi^2\varrho_1,\varrho_5)\right)=0\tag{3.7}$$

From (3.7),

$$g(\phi^2\varrho_1,\varrho_5)[-\kappa-\mu(\kappa+1)]+g(h\varrho_1,\varrho_5)(\mu-\kappa)=0$$

and thus

$$(\kappa+1)(-\kappa-\mu(\kappa+1))g(h\varrho_1,h\varrho_5)+(\mu-\kappa)g(h\varrho_1,\varrho_5)=0$$

Therefore,

$$(\kappa+1)(-\kappa-\mu(\kappa+1))h^2+(\mu-\kappa)h=0$$

□

Moreover, if the trace of (3.6) is considered,

$$2n\kappa - \mu trh^2 = 2n\kappa - (\kappa+1)\mu\phi^2 = 2n\kappa - 2n(\kappa+1)\mu = 0$$

It means that

$$\kappa - (\kappa+1)\mu = 0$$

If $\kappa+1=0$ then $\kappa=0$. However, this contradicts $\kappa+1=0$. So $\kappa+1$ can never vanish (zero). Hence,

$$\mu = \frac{\kappa}{\kappa+1}$$

Here, $\mu=0$ if and only if $\kappa=0$. In this case, $R(X,Y)\xi=0$. Using Zamkovoy classification, M is $H^{2n}\times\mathbb{R}$.

4. Semiconformally Ricci Semisymmetric (κ,μ)-Paracontact Metric Manifold

This section defines a semiconformally Ricci semisymmetric (κ,μ)-paracontact metric manifold.

Definition 4.1. A (κ,μ)-paracontact metric manifold M^{2n+1} is said to be semiconformally Ricci semisymmetric if the semiconformal curvature tensor satisfies the condition

$$P(\varrho_1,\varrho_2)S=0$$

for all vector fields ϱ_1 and ϱ_2 on M.

Definition 4.2. A (κ,μ)-paracontact metric manifold M^{2n+1} is said to be $\eta-$Einstein manifold if its Ricci tensor S satisfies the condition

$$S(\varrho_1,\varrho_2)=\alpha g(\varrho_1,\varrho_2)+\beta\eta(\varrho_1)\eta(\varrho_2)$$

for all vector fields ϱ_1 and ϱ_2 and some real constants α and β. For $\beta=0$, it reduces to an Einstein manifold.

Theorem 4.3. Let M be a $(2n+1)-$dimensional (κ,μ)-paracontact metric manifold. Then, M is semiconformally semisymmetric if and only if M is an Einstein manifold.

Proof. Suppose that M is a $(2n+1)-$dimensional (κ,μ)-paracontact metric manifold that is semiconformally semisymmetric. Then, for all $\varrho_1,\varrho_2,\varrho_3,\varrho_4\in\chi(M)$,

$$P(\varrho_1,\varrho_2)S(\varrho_3,\varrho_4)=0 \tag{4.1}$$

From (4.1),

$$S(P(\varrho_1,\varrho_2)\varrho_3,\varrho_4)+S(\varrho_3,P(\varrho_1,\varrho_2)\varrho_4)=0 \tag{4.2}$$

Choosing $\varrho_1=\varrho_3=\xi$ in (4.2),

$$S(P(\xi,\varrho_2)\xi,\varrho_4)+2n\kappa\eta(P(\xi,\varrho_2)\varrho_4)=0 \tag{4.3}$$

Using (2.10) and (2.11) in (4.3),

$$\begin{aligned}0=&\left(\tfrac{a(2n-n\mu-4)}{2n-1}-\tfrac{br}{2n}\right)\eta(\varrho_2)S(\xi,\varrho_4)-\left(\tfrac{a(2n-n\mu-4)}{2n-1}-\tfrac{br}{2n}\right)S(\varrho_2,\varrho_4)+\left(\tfrac{2a(n-1)(\mu-1)}{2n-1}\right)S(h\varrho_2,\varrho_4)\\&+\left(\tfrac{a(2n-n\mu-4)}{2n-1}-\tfrac{br}{2n}\right)2n\kappa g(\varrho_2,\varrho_4)-\left(\tfrac{a(2n-n\mu-4)}{2n-1}\tfrac{br}{2n}\right)2n\kappa\eta(\varrho_2)\eta(\varrho_4)-\left(\tfrac{2a(n-1)(\mu-1)}{2n-1}\right)2n\kappa g(h\varrho_2,\varrho_4)\end{aligned}$$

which implies

$$0 = \left(\tfrac{a(2n-n\mu-4)}{2n-1} - \tfrac{br}{2n}\right) S(\varrho_2, \varrho_4) - \left(\tfrac{a(2n-n\mu-4)}{2n-1} - \tfrac{br}{2n}\right) 2n\kappa g(\varrho_2, \varrho_4) + \left(\tfrac{2a(n-1)(\mu-1)}{2n-1}\right) S(h\varrho_2, \varrho_4) - \left(\tfrac{2a(n-1)(\mu-1)}{2n-1}\right) 2n\kappa g(hY\varrho_2, \varrho_4) \tag{4.4}$$

Choosing $Y = hY$ in (4.4),

$$0 = \left(\tfrac{a(2n-n\mu-4)}{2n-1} - \tfrac{br}{2n}\right) S(h\varrho_2, \varrho_4) - \left(\tfrac{a(2n-n\mu-4)}{2n-1} - \tfrac{br}{2n}\right) 2n\kappa(h\varrho_2, \varrho_4) + \left(\tfrac{2a(n-1)(\mu-1)}{2n-1}\right) (\kappa+1)S(\varrho_2, \varrho_4) - \left(\tfrac{2a(n-1)(\mu-1)}{2n-1}\right) (\kappa+1)(\kappa+1)2n\kappa g(\varrho_2, \varrho_4) \tag{4.5}$$

From (4.4) and (4.5),

$$S(\varrho_2, \varrho_4) = 2n\kappa g(\varrho_2, \varrho_4)$$

This shows that a (κ, μ)-paracontact metric manifold with $(2n+1)$-dimensional semiconformally Ricci semisymmetric is an Einstein manifold. □

5. Conclusion

This paper obtained significant and manifold-characterizing results for the semiconformally semisymmetry case of a (κ, μ)-paracontact metric manifold. First, it can be observed that the (κ, μ)-paracontact metric manifold reduces to a more special case of the $(\kappa, 1)-$ paracontact metric manifold. Another result of the theorem is that the semiconformal curvature tensor on the manifold is reduced to the following form:

$$P(\varrho_1, \varrho_2)\varrho_3 = -\frac{br}{2n}[g(\varrho_2, \varrho_3)\varrho_1 - g(\varrho_1, \varrho_3)\varrho_2]$$

When this case is considered, if

$$P(\varrho_1, \varrho_2)\varrho_3 = -\frac{br}{2n}[g(\varrho_2, \varrho_3)\varrho_1 - g(\varrho_1, \varrho_3)\varrho_2]$$

then the manifold is said to be reduced to the real space form with constant section curvature. However, there is no classification expressed in the literature regarding the condition obtained. It is an open problem whether a classification, such as semiconformal real space form, can be made by examining this situation. In addition, as another result of the theorem, the following relation is obtained:

$$(\kappa+1)(-\kappa - \mu(\kappa+1))h^2 + (\mu - \kappa)h = 0$$

A second open problem is to determine to which special case of a (κ, μ)-paracontact metric manifold this relation reduces.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

References

[1] S. Kaneyuki, F. L. Williams, *Almost paracontact and parahodge structures on manifolds*, Nagoya Mathematical Journal 99 (1985) 173–187.

[2] S. Zamkovoy, *Canonical connections on paracontact manifolds*, Annals of Global Analysis and Geometry 36 (1) (2009) 37–60.

[3] B. C. Montano, I. K. Erken, C. Murathan, *Nullity conditions in paracontact geometry*, Differential Geometry and Its Applications 30 (6) (2012) 665–693.

[4] B. C. Montano, L. Di Terlizzi, *Geometric structures associated to a contact metric (κ, μ)-space*, Pacific Journal of Mathematics 246 (2) (2010) 257–292.

[5] G. Calvaruso, *Homogeneous paracontact metric three-manifolds*, Illinois Journal of Mathematics 55 (2) (2011) 697–718.

[6] I. K. Erken, *Generalized $(\bar{\kappa} \neq -1, \bar{\mu})$-paracontact metric manifolds with $\xi(\bar{\mu}) = 0$*, International Electronic Journal of Geometry 8 (1) (2015) 77–93.

[7] I. K. Erken, C. Murathan, *A study of three-dimensional paracontact (κ, μ, ν)-spaces*, International Journal of Geometric Methods in Modern Physics 14 (07) (2017) 1750106.

[8] J. Kim, *On pseudo semiconformally symmetric manifolds*, Bulletin of the Korean Mathematical Society 54 (1) (2017) 177–186.

[9] J. Kim, *A type of conformal curvature tensor*, Far East Journal of Mathematical Sciences 99 (1) (2016) 61–74.

[10] J. P. Singh, M. Khatri, *On almost pseudo semiconformally symmetric manifolds*, Differential Geometry-Dynamical Systems 22 (2020) 233–253.

[11] M. Ali, N. A. Pundeer, Y. Y. J. Suh, *Proper semiconformal symmetries of spacetimes with divergence-free semiconformal curvature tensor*, Filomat 33 (16) (2019) 5191–5198.

[12] N. A. Pundeer, M. Ali, M. Bilal, *A spacetime admitting semi-conformal curvature tensor*, Balkan Journal of Geometry and Its Applications 27 (1) (2022) 130–137.

[13] S. K. Yadav, *A note on spacetimes in f(R)-Gravity*, Annals of Communications in Mathematics 6 (2) (2023) 99–108.

[14] S. K. Hui, A. Patra, A. Patra, *On generalized weakly semi-conformally symmetric manifolds*, Communications of the Korean Mathematical Society 36 (4) (2021) 771–782.

[15] S. Shenawy, A. Rabie, U. C. De, C. Mantica, N. Bin Turki, *Semi-conformally flat singly warped product manifolds and applications*, Axioms 12 (12) (2023) 1078.

[16] U. C. De, C. Dey, *Lorentzian manifolds: A characterization with semiconformal curvature tensor*, Communications of the Korean Mathematical Society 34 (3) (2019) 911–920.

[17] U. C. De, Y. J. Suh, *On weakly semiconformally symmetric manifolds*, Acta Mathematica Hungarica 157 (2) (2019) 503–521.

[18] A. Barman, *Some properties of a semi-conformal curvature tensor on a Riemannian manifold*, The Mathematics Student 91 (1-2) (2022) 201–208.

[19] B. Y. Chen, U. C. De, N. B. Turki, A. A. Syied, *Warped product manifolds: Characterizations through the symmetry of the semiconformal curvature tensor and applications*, International Journal of Geometric Methods in Modern Physics 22 (02) (2025) 2450281.

[20] J. P. Singh, M. Khatri, *On semi-conformal curvature tensor in (κ, μ)-contact metric manifold*, in: M. Tosun (Ed.), Conference Proceeding of 18th International Geometry Symposium, Conference Proceeding Science and Technology, 4 (2), 2021, pp. 215–225.