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Entire Four-Dimensional Summation Methods

Fatih Nuray[1*] and Richard F. Patterson[2]

[1]*Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Türkiye*
[2]*Department of Mathematics and Statistics, University of North Florida Jacksonville, Florida, 32224*
[*]*Corresponding author*

Abstract

In this study, we establish the necessary and sufficient conditions for a four-dimensional matrix to map entire double indexed sequences onto themselves, referring to such transformations as entire methods. Furthermore, we present a theorem addressing the consistency of these entire summation methods.

Keywords: *Double sequence, P-limit, four-dimensional summability method, entire double-indexed sequence*
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1. Introduction and Background

If for every $\varepsilon > 0$, there exists a positive integer $N(\varepsilon) \in \mathbb{N}$ such that

$$|\xi_{u,v} - \ell| < \varepsilon, \quad \text{whenever } u, v > N(\varepsilon),$$

then the double-indexed sequence $x = \{\xi_{u,v}\}$ of real numbers is said to be **convergent in the sense of Pringsheim** to $\ell \in \mathbb{R}$, denoted as:

$$P\text{-}\lim_{u,v\to\infty} \xi_{u,v} = \ell.$$

If there exists a constant $K > 0$ such that

$$|\xi_{u,v}| < K \quad \forall u, v \in \mathbb{N},$$

then the double-indexed sequence $x = \{\xi_{u,v}\}$ is called **bounded**.
A **double subsequence** $y = \{y_{u,v}\}$ of the sequence $x = \{\xi_{u,v}\}$ is obtained by selecting strictly increasing sequences of indices (u_j) and (v_j), resulting in:

$$y_{u,v} = \xi_{u_j,v_j}.$$

An example representation of such a double subsequence is given by:

$$\begin{pmatrix} \xi_{u_1v_1} & \xi_{u_2v_2} & \xi_{u_5v_5} & \xi_{u_{10}v_{10}} & - \\ \xi_{u_4v_4} & \xi_{u_3v_3} & \xi_{u_6v_6} & - & - \\ \xi_{u_9v_9} & \xi_{u_8v_8} & \xi_{u_7v_7} & - & - \\ - & - & - & - & - \end{pmatrix}$$

This construction allows us to examine the behavior of specific sub-patterns within the original double-indexed sequence. If a double-indexed sequence $x = \{\xi_{u,v}\}$ converges (in Pringsheim's sense) and, additionaly the following limits exist:

$$\lim_{u\to\infty} \xi_{u,v} = \alpha_v \quad (v = 1, 2, 3, \ldots),$$

Email addresses: **fnuray@aku.edu.tr (Fatih Nuray), rpatters@unf.edu (Richard H. Patterson)**

$$\lim_{v\to\infty} \xi_{u,v} = \alpha_u \quad (u = 1,2,3,...),$$

then we say that the sequence $x = \{\xi_{u,v}\}$ converges regularly.
It is important to note that the main limitation of Pringsheim's convergence is that a double-indexed sequence can be convergent without being bounded. On the other hand, Hardy's definition of regular convergence strengthens this concept by requiring that every convergent double-indexed sequence must also be bounded.
Beyond the conditions of Pringsheim convergence, regular convergence further demands that the rows and columns of a double-indexed sequence converge individually.
Let Ω denote the set of all double-indexed sequences of complex numbers. A double indexed sequence $x = \{\xi_{u,v}\}$ is called an **entire double indexed sequence** if:

$$\sum_{u,v=0}^{\infty} |\xi_{u,v}| p^u q^v$$

converges for every positive integer p and q. If we denote by $\mathfrak{E}^2$ the set of all entire double-indexed sequences, it follows that $\mathfrak{E}^2$ can be identified with the class of entire bivariate functions.
Consider a four-dimensional infinite matrix of complex numbers $A = (a_{m,n,u,v})$ with indices $(m,n,u,v = 0,1,2,3,\dots)$. If, for every $x \in \mathfrak{E}^2$, the sequence $\{y_{m,n}\}$ defined by:

$$y_{m,n} = \sum_{u,v=0}^{\infty} a_{m,n,u,v}\xi_{u,v} \quad (m,n = 0,1,2,3,\dots) \tag{1.1}$$

converges and satisfies $\{y_{m,n}\} \in \mathfrak{E}^2$, then A is called an **entire method**. Furthermore, if:

$$\sum_{m,n=0}^{\infty} y_{m,n} = \sum_{u,v=0}^{\infty} \xi_{u,v},$$

then A is referred to as a **regular entire method**.
A four-dimensional matrix $A = (a_{m,n,u,v})$ that maps $\mathfrak{E}^2$ into itself is called an $\mathfrak{E}^2$-$\mathfrak{E}^2$ method. The mapping:

$$\{\xi_{u,v}\} \to \{\xi_{u,v} p^u q^v\}$$

defines a one-to-one correspondence between the space $\mathfrak{E}^2$ and the space $\mathscr{L}_u$, where:

$$\mathscr{L}_u = \left\{ \{\xi_{u,v}\} : \sum_{u=1}^{\infty}\sum_{v=1}^{\infty} |\xi_{u,v}| < \infty \right\}.$$

It is worth noting that four-dimensional summability methods are deeply connected with the concept of bounded index, entire functions of exponential type, and the theory of entire solutions of differential equations. In particular, summability processes in multiple dimensions often arise in the study of entire functions, especially when investigating their growth behavior or analytic continuation properties. The notion of bounded index plays a crucial role in understanding the structure and behavior of such functions.
Recent contributions in this direction include the work by Nuray [7], where the relationship between bounded index and four-dimensional summability methods is investigated, and the study by Bandura and Nuray [8], which examines bivariate entire functions of exponential type. Additionally, the classical result by Fricke, Roy, and Shah [9] explores how summability techniques can be used to characterize entire solutions of ordinary differential equations with bounded index. These connections suggest that the summability methods discussed in this paper may have broader implications within the theory of entire functions and differential equations.

Lemma 1.1 ([6]). *A double series $\sum_{u,v=1}^{\infty} a_{u,v}$ is absolutely convergent if and only if the following conditions hold:*

(i) There exist $(u_0, v_0) \in \mathbb{N}^2$ and a constant $\alpha_0 > 0$ such that:

$$\sum_{u=u_0}^{m}\sum_{v=v_0}^{n} |a_{u,v}| \leq \alpha_0, \quad \forall (m,n) \geq (u_0,v_0).$$

(ii) Every series formed by the rows and columns is absolutely convergent.

In [11], Patterson established that a necessary and sufficient condition for a four-dimensional matrix $A = (a_{m,n,u,v})$ to be an $\mathscr{L}_u$-$\mathscr{L}_u$ method (i.e., a method mapping $\mathscr{L}_u$ into itself) is the existence of a constant $M_A > 0$ such that:

$$\sum_{m,n=0}^{\infty} |a_{m,n,u,v}| < M_A \quad \text{for all } u,v = 0,1,2,3,\dots \tag{1.2}$$

Consequently, the matrix $A = (a_{m,n,u,v})$ maps $\mathfrak{E}^2$ to itself if and only if the matrix $(a_{m,n,u,v} p^{-u} q^{-v})$ is an $\mathscr{L}_u$-$\mathscr{L}_u$ method. In other words, A maps $\mathfrak{E}^2$ to $\mathscr{L}_u$ if and only if there exists a constant $M > 0$ such that:

$$\sum_{m,n=0}^{\infty} |a_{m,n,u,v}| p^{-u} q^{-v} < M \quad \text{for all } u,v = 0,1,2,3,\dots$$

Similarly, for each pair of positive integers s and t, a matrix $B = (b_{m,n,u,v})$ maps $\mathscr{L}_u$ to $\mathfrak{E}^2$ if and only if the matrix $(b_{m,n,u,v} s^m t^n)$ is a $\mathscr{L}_u$-$\mathscr{L}_u$ method. These observations are summarized in the following theorem.

Theorem 1.2. *A four-dimensional matrix $A = (a_{m,n,u,v})$ is an entire method if and only if, for each pair of positive integers s and t, there exist positive integers $p(s) \geq s$, $q(t) \geq t$, and a constant $M(p,q,s,t)$ such that:*

$$\sum_{m,n=0}^{\infty} |a_{m,n,u,v}| p^{-u} q^{-v} s^m t^n < M(p,q,s,t) \quad \text{for all } u,v = 0,1,2,3,\ldots \tag{1.3}$$

*Furthermore, A is a **regular entire method** if, in addition to the condition above, the following holds:*

$$\sum_{m,n=0}^{\infty} a_{m,n,u,v} = 1 \quad \text{for all } u,v = 0,1,2,3,\ldots$$

To ensure that A is an entire method, it is necessary that each column of A forms an entire sequence. Even when choosing $s = 1$, $t = 1$, $p = p(1)$, and $q = q(1)$, it remains essential that each row sequence is analytic. Formally, for each $m,n = 0,1,2,\ldots$, the double-indexed sequence:

$$\{|a_{m,n,0,0}|,\ |a_{m,n,u,v}|^{\frac{1}{u+v}} : u,v = 1,2,\ldots\}$$

must be bounded.
However, these conditions are not sufficient on their own. For example, consider the four-dimensional matrix defined by:

$$a_{m,n,u,v} = \begin{cases} m!n!, & \text{if } m = n \text{ and } u = v, \\ \\ 0, & \text{otherwise.} \end{cases}$$

Although this matrix has entire rows and columns, the entire double-indexed sequence $\left\{\frac{1}{m!n!}\right\}$ is mapped to the constant sequence $\{1\}$, illustrating that boundedness and analyticity of rows and columns alone do not guarantee the regularity of the entire method. The four-dimensional Taylor matrix $T(r_1,r_2) = (a_{m,n,u,v})$, defined for any complex numbers r_1 and r_2, is given by:

$$a_{m,n,u,v} = \begin{cases} \binom{u}{m}\binom{v}{n}(1-r_1)^{m+1} r_1^{u-m} (1-r_2)^{n+1} r_2^{v-n}, & \text{if } u \geq m \text{ and } v \geq n, \\ \\ 0, & \text{otherwise.} \end{cases}$$

Here:
- $\binom{u}{m}$ and $\binom{v}{n}$ are binomial coefficients.
- $(1-r_1)^{m+1}$ and $(1-r_2)^{n+1}$ provide the scaling and ensure the correct weighting in the expansion.
- The terms r_1^{u-m} and r_2^{v-n} account for the index shifts in the matrix elements.
- The conditions $u \geq m$ and $v \geq n$ guarantee that the matrix has a lower triangular structure in both dimensions.

2. Main Results

Lemma 2.1. *Let X and Y be FK-spaces of double sequences, that is, Fréchet spaces consisting of sequences indexed by $\mathbb{N} \times \mathbb{N}$, such that the coordinate projections $x \mapsto x_{kl}$ are continuous.*
Let $A = (a_{mn,kl})$ be a four-dimensional matrix and define the matrix transformation $A : X \to Y$ by

$$(Ax)_{mn} := \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{mn,kl} x_{kl},$$

for every $x = (x_{kl}) \in X$, provided that the series converges for all m,n and $Ax \in Y$.
Then A defines a continuous linear operator from X to Y.

Proof. We show that A is continuous by proving that if $x^{(r)} \to 0$ in X, then $Ax^{(r)} \to 0$ in Y.
The transformation A is clearly linear. Since X is a Fréchet space, its topology is generated by a countable family of seminorms. For instance, for each $N \in \mathbb{N}$, define

$$\|x\|_N := \sup_{1 \leq k,l \leq N} |x_{kl}|.$$

Then convergence $x^{(r)} \to 0$ in X means $\|x^{(r)}\|_N \to 0$ for all N.
Fix any $m,n \in \mathbb{N}$. For each r, define

$$y_{mn}^{(r)} := (Ax^{(r)})_{mn} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{mn,kl} x_{kl}^{(r)}.$$

Assume that the series is absolutely convergent. Then for each fixed N,

$$|y_{mn}^{(r)}| \leq \sum_{k,l} |a_{mn,kl}| \cdot |x_{kl}^{(r)}| = \sum_{k,l \leq N} |a_{mn,kl}| \cdot |x_{kl}^{(r)}| + \sum_{k \text{ or } l > N} |a_{mn,kl}| \cdot |x_{kl}^{(r)}|.$$

Define

$$C_{mn,N} := \sum_{k,l \le N} |a_{mn,kl}|,$$

so that

$$|y_{mn}^{(r)}| \le C_{mn,N} \cdot \|x^{(r)}\|_N + \varepsilon_r,$$

where $\varepsilon_r \to 0$ is $r \to \infty$, assuming boundedness and absolute convergence.
Since $\|x^{(r)}\|_N \to 0$, it follows that $y_{mn}^{(r)} \to 0$. Therefore, $Ax^{(r)} \to 0$ in Y, because convergence in FK-spaces of double sequences is coordinate-wise.
Thus, A is continuous. □

Theorem 2.2. *The four-dimensional Taylor matrix $T(r_1, r_2) = (a_{m,n,u,v})$ is an $\mathfrak{E}_2$-$\mathfrak{E}_2$ method for any complex numbers r_1 and r_2.*

Proof. The cases $T(0,0)$ (identity matrix) and $T(1,1)$ (zero matrix) are straightforward. For the non-trivial case where $(r_1, r_2) \neq (0,0)$ and $(r_1, r_2) \neq (1,1)$, consider the following expression:

$$\sum_{m,n=0}^{u,v} \binom{u}{m}\binom{v}{n} |1-r_1|^{m+1}|r_1|^{u-m}|1-r_2|^{n+1}|r_2|^{v-n} p^{-u} q^{-v} s^m t^n$$
$$= |1-r_1||1-r_2| p^{-u} q^{-v} (s|1-r_1| + |r_1|)^u (t|1-r_2| + |r_2|)^v. \tag{2.1}$$

By defining constants R_1 and R_2 such that $|r_1| \le R_1$ and $|r_2| \le R_2$, we can estimate:

$$(s|1-r_1| + |r_1|) \le s + (s+1)R_1 \quad \text{and} \quad (t|1-r_2| + |r_2|) \le t + (t+1)R_2.$$

Substituting these bounds into equation (2.1), we obtain:

$$|1-r_1||1-r_2| \frac{(s+(s+1)R_1)^u}{p^u} \frac{(t+(t+1)R_2)^v}{q^v}.$$

To ensure convergence, choose:

$$p = 2\big(s + (s+1)R_1\big) \quad \text{and} \quad q = 2\big(t + (t+1)R_2\big).$$

With these choices, the above expression simplifies to:

$$(1+R_1)(1+R_2)\left(\frac{1}{2}\right)^{u+v},$$

which clearly converges as $u, v \to \infty$. Hence, condition (3) is satisfied with:

$$M = (1+R_1)(1+R_2).$$

Therefore, the matrix $T(r_1, r_2)$ defines an entire method, completing the proof. □

Notice that:

$$\sum_{m,n=0}^{u,v} \binom{u}{m}\binom{v}{n} (1-r_1)^{m+1} r_1^{u-m} (1-r_2)^{n+1} r_2^{v-n} = (1-r_1)(1-r_2),$$

which implies that the matrix $T(r_1, r_2)$ is **regular** if and only if $(r_1, r_2) = (0,0)$.
A linear topological space is said to be locally convex if every neighborhood contains an absolutely convex set. Consider a subspace X of the linear space Ω. We say X is a **DK-space** if for each $(m,n) \in \mathbb{N}^2$, the map:

$$P_{m,n} : X \to \mathbb{C}, \quad P_{m,n}(x) = |\xi_{m,n}|,$$

is continuous for all $x = \{\xi_{u,v}\} \in X$.
If a DK space X is equipped with a Frèchet topology, it is referred to as an **FDK space**.
The space $\mathfrak{E}^2$, consisting of all entire double-indexed sequences, forms a locally convex FDK-space. Its topology is defined by the family of seminorms:

$$h_{m,n}(x) = \max_{|z_1|=m, |z_2|=n} \left| \sum_{i,j=0}^{\infty} \xi_{i,j} z_1^i z_2^j \right|,$$

for every $x \in \mathfrak{E}^2$.
If an analytic double-indexed sequence $x = \{\xi_{u,v}\}$ satisfies:

$$\{|\xi_{0,0}|, |\xi_{u,v}|^{\frac{1}{u+v}} : u, v = 1, 2, \ldots\}$$

being bounded, then any continuous linear functional f on $\mathfrak{E}^2$ can be expressed as:

$$f(x) = \sum_{m,n=0}^{\infty} t_{m,n}\xi_{m,n},$$

for some analytic double-indexed sequence $\{t_{m,n}\}$.
Let A be an entire summation method. Its **summability field** is defined as:

$$\mathfrak{E}^2_A = \{x \in \Omega : Ax \in \mathfrak{E}^2\}.$$

For every $f \in (\mathfrak{E}^2_A)'$, the dual space of $\mathfrak{E}^2_A$, there exist analytic double-indexed sequences $\{t_{m,n}\}$ and $\{\alpha_{u,v}\}$ such that:

$$f(x) = \sum_{m,n=0}^{\infty} t_{m,n} \sum_{u,v=0}^{\infty} a_{m,n,u,v}\xi_{u,v} + \sum_{u,v=0}^{\infty} \alpha_{u,v}\xi_{u,v}, \quad \forall x \in \mathfrak{E}^2_A. \tag{2.2}$$

Associated with each entire method A is the functional S_A defined by:

$$S_A(x) = \sum_{m,n=0}^{\infty}\sum_{u,v=0}^{\infty} a_{m,n,u,v}\xi_{u,v}.$$

By Lemma 2.1, every matrix mapping between FDK spaces is continuous, and it follows that:

$$S_A \in (\mathfrak{E}^2_A)'.$$

Lemma 2.3. *If $f \in (\mathfrak{E}^2_A)'$, then there exists an entire method B such that $\mathfrak{E}^2_B \supseteq \mathfrak{E}^2_A$ and $S_B(x) = f(x)$ for every $x = \{\xi_{u,v}\} \in \mathfrak{E}^2_A$.*

Proof. Given a four-dimensional entire method A, define the matrix $B = (b_{m,n,u,v})$ as follows:

$$b_{0,0,u,v} = \alpha_{u,v} + t_{0,0}a_{0,0,u,v}, \quad u,v = 0,1,2,\ldots$$

and for $m,n \geq 1$:

$$b_{m,n,u,v} = t_{m,n}a_{m,n,u,v}, \quad u,v = 0,1,2,\ldots$$

where $\{t_{m,n}\}$ and $\{\alpha_{u,v}\}$ are the analytic double indexed sequences of the representation of f.
Let:

$$M = \lceil \max\{M(\alpha), M(t)\}\rceil, \quad N = \lceil \max\{N(\alpha), N(t)\}\rceil,$$

where:

$$M(\alpha) = \max\left\{\sup_u\{|\alpha_{0,0}|, |\alpha_{u,v}|^{\frac{1}{u+v}}\}, 1\right\}, \quad M(t) = \max\left\{\sup_m\{|t_{0,0}|, |t_{m,n}|^{\frac{1}{m+n}}\}, 1\right\},$$

$$N(\alpha) = \max\left\{\sup_v\{|\alpha_{0,0}|, |\alpha_{u,v}|^{\frac{1}{u+v}}\}, 1\right\}, \quad N(t) = \max\left\{\sup_n\{|t_{0,0}|, |t_{m,n}|^{\frac{1}{m+n}}\}, 1\right\}.$$

By Theorem 1.2, to show that B is an entire method, let s,t be arbitrary positive integers and choose $p \geq Ms$ and $q \geq Nt$. With these choices:

$$\sup_{u,v} \sum_{m,n=0}^{\infty} |a_{m,n,u,v}|(Ms)^m p^{-u}(Nt)^n q^{-v} < \infty.$$

For $u,v \geq 0$:

$$\left(\frac{|\alpha_{u,v}|}{p^u q^v}\right)^{\frac{1}{u+v}} \leq 1.$$

Thus:

$$\begin{aligned}\sum_{m,n=0}^{\infty} |b_{m,n,u,v}|p^{-u}q^{-v}s^m t^n &\leq \frac{|\alpha_{u,v}|}{p^u q^v} + \sum_{m,n=0}^{\infty} |t_{m,n}a_{m,n,u,v}|p^{-u}q^{-v}s^m t^n \\ &= \left(\frac{|\alpha_{u,v}|^{\frac{1}{u+v}}}{p^{\frac{u}{u+1}} q^{\frac{v}{u+v}}}\right)^{u+v} + \sum_{m,n=0}^{\infty} |a_{m,n,u,v}|(|t_{m,n}|^{\frac{1}{m+n}} s)^m (|t_{m,n}|^{\frac{1}{m+n}} t)^n p^{-u} q^{-v} \\ &\leq \left(\frac{|\alpha_{u,v}|^{\frac{1}{u+v}}}{p^{\frac{u}{u+v}} q^{\frac{v}{u+v}}}\right)^{u+v} + \sum_{m,n=0}^{\infty} |a_{m,n,u,v}|(Ms)^m (Nt)^n p^{-u} q^{-v}.\end{aligned}$$

This implies:

$$\sum_{m,n=0}^{\infty} |b_{m,n,u,v}|p^{-u}q^{-v}s^m t^n < \infty,$$

so B is indeed an entire method.

The inclusion $\mathfrak{E}_B^2 \supseteq \mathfrak{E}_A^2$ follows directly from the construction. Finally, for $x = \{\xi_{u,v}\} \in \mathfrak{E}_A^2$:

$$\begin{aligned} S_B(x) &= \sum_{m,n=0}^{\infty} b_{m,n,u,v}\xi_{u,v} \\ &= \sum_{u,v=0}^{\infty} (\alpha_{u,v} + t_{0,0}a_{0,0,u,v})\xi_{u,v} + \sum_{m,n=1}^{\infty} t_{m,n}a_{m,n,u,v}\xi_{u,v} \\ &= f(x), \end{aligned}$$

thus proving the lemma. □

Let's define the matrix $C = (c_{m,n,u,v})$ by:

$$c_{2m,2n,u,v} = a_{m,n,u,v}, \quad m,n,u,v = 0,1,2,\ldots$$

and

$$c_{2m+1,2n+1,u,v} = b_{m,n,u,v}, \quad m,n,u,v = 0,1,2,\ldots$$

where A and B are two entire methods.

Lemma 2.4. *The matrix C is an entire method satisfying:*

$$\mathfrak{E}_C^2 = \mathfrak{E}_A^2 \cap \mathfrak{E}_B^2 \quad \text{and} \quad S_C(x) = S_A(x) - S_B(x)$$

for every $x = \{\xi_{u,v}\} \in \mathfrak{E}_C^2$.

Proof. Since A and B are entire methods, for any positive integers s and t, we can select $p \geq s^2$ and $q \geq t^2$ such that:

$$\sup_{u,v} \sum_{m,n=0}^{\infty} |a_{m,n,u,v}|(s^2)^m p^{-u}(t^2)^n q^{-v} < \infty$$

and

$$st \cdot \sup_{u,v} \sum_{m,n=0}^{\infty} |b_{m,n,u,v}|(s^2)^m p^{-u}(t^2)^n q^{-v} < \infty.$$

Thus:

$$\begin{aligned} \sup_{u,v} \sum_{m,n=0}^{\infty} |c_{m,n,u,v}|p^{-u}q^{-v}s^m t^n &\leq \sup_{u,v} \sum_{m,n=0}^{\infty} |a_{m,n,u,v}|p^{-u}q^{-v}s^{2m}t^{2n} \\ &\quad + \sup_{u,v} \sum_{m,n=0}^{\infty} |b_{m,n,u,v}|p^{-u}q^{-v}s^{2m+1}t^{2n+1} \\ &< \infty, \end{aligned}$$

so C is an entire method.
Now, $x = \{\xi_{u,v}\} \in \mathfrak{E}_C^2$ if and only if, for every $p > 0$ and $q > 0$:

$$\begin{aligned} \left|\sum_{u,v=0}^{\infty} a_{0,0,u,v}\xi_{u,v}\right| p^0 q^0 &+ \left|\sum_{u,v=0}^{\infty} b_{0,0,u,v}\xi_{u,v}\right| p^1 q^1 \\ &+ \left|\sum_{u,v=0}^{\infty} a_{1,1,u,v}\xi_{u,v}\right| p^2 q^2 \\ &+ \left|\sum_{u,v=0}^{\infty} b_{1,1,u,v}\xi_{u,v}\right| p^3 q^3 + \cdots < \infty. \end{aligned} \tag{2.3}$$

Since the terms are nonnegative, (2.3) holds for all $p > 0$ and $q > 0$ if and only if:

$$\begin{aligned} \sum_{m,n=0}^{\infty} \left|\sum_{u,v=0}^{\infty} a_{m,n,u,v}\xi_{u,v}\right| (p^2)^m (q^2)^n + pq \sum_{m,n=0}^{\infty} \left|\sum_{u,v=0}^{\infty} b_{m,n,u,v}\xi_{u,v}\right| (p^2)^m (q^2)^n \\ < \infty. \end{aligned}$$

This holds if and only if $x \in \mathfrak{E}_A^2 \cap \mathfrak{E}_B^2$.
Finally, for $x = \{\xi_{u,v}\} \in \mathfrak{E}_C^2$:

$$\begin{aligned} S_C(x) &= \sum_{u,v=0}^{\infty} a_{0,0,u,v}\xi_{u,v} - \sum_{u,v=0}^{\infty} b_{0,0,u,v}\xi_{u,v} \\ &\quad + \sum_{u,v=0}^{\infty} a_{1,1,u,v}\xi_{u,v} - \sum_{u,v=0}^{\infty} b_{1,1,u,v}\xi_{u,v} + \cdots. \end{aligned}$$

Because this is an absolutely convergent double series (take $p = 1$ and $q = 1$), we can rearrange terms to obtain:

$$S_C(x) = \sum_{m,n=0}^{\infty} \sum_{u,v=0}^{\infty} a_{m,n,u,v}\xi_{u,v} - \sum_{m,n=0}^{\infty} \sum_{u,v=0}^{\infty} b_{m,n,u,v}\xi_{u,v} = S_A(x) - S_B(x).$$

□

3. Consistency

Two entire methods A and B are said to be consistent (with respect to the functionals S_A and S_B) if:

$$S_A(x) = S_B(x) \quad \text{for every } x = \{\xi_{u,v}\} \in \mathfrak{E}_A^2 \cap \mathfrak{E}_B^2.$$

Theorem 3.1. *An entire method A is consistent with any method B if and only if $\mathfrak{E}^2$ is dense in $\mathfrak{E}_A^2 \cap \mathfrak{E}_B^2$ whenever:*

$$S_A(x) = S_B(x) \quad \textit{for all } x = \{\xi_{u,v}\} \in \mathfrak{E}_A^2.$$

Proof. Suppose $\mathfrak{E}^2$ is dense in $\mathfrak{E}_A^2 \cap \mathfrak{E}_B^2$ and that $S_A(x) = S_B(x)$ for every $x \in \mathfrak{E}_A^2$. Define the functional $F : \mathfrak{E}_A^2 \cap \mathfrak{E}_B^2 \to \mathbb{C}$ by:

$$F(x) = S_A(x) - S_B(x).$$

Since S_A and S_B are continuous linear functionals, F is also continuous and linear. Given that F vanishes on $\mathfrak{E}_A^2$ and $\mathfrak{E}^2$ is dense in $\mathfrak{E}_A^2 \cap \mathfrak{E}_B^2$, it follows that F must vanish on the entire intersection. Thus, A and B are consistent.
Conversely, assume A is an entire four-dimensional method consistent with any entire four-dimensional method B for which $S_A(x) = S_B(x)$ for all $x \in \mathfrak{E}_A^2$. Suppose, for the sake of contradiction, that $\mathfrak{E}_A^2$ is not dense in $\mathfrak{E}_A^2 \cap \mathfrak{E}_B^2$. By the Hahn–Banach theorem, there exists a continuous linear functional $f \in (\mathfrak{E}_C^2)'$ (where C is the entire method constructed from A and B as in Lemma 2.4) such that:

$$f(x) = 0 \quad \forall x \in \mathfrak{E}_A^2, \quad \text{but} \quad f(y) \neq 0 \quad \text{for some } y \in \mathfrak{E}_C^2.$$

By Lemma 2.3 there exists an entire four dimensional method D such that:

$$S_D(x) = f(x) \quad \text{for all } x \in \mathfrak{E}_C^2 \quad \text{and} \quad \mathfrak{E}_D^2 \supseteq \mathfrak{E}_C^2.$$

Construct a new four dimensional matrix $E = (e_{m,n,u,v})$ by:

$$e_{m,n,u,v} = d_{m,n,u,v} + a_{m,n,u,v}, \quad m,n,u,v = 0,1,2,\ldots.$$

Since A and D are entire methods, E is also entire. For every u,v, we have:

$$\sum_{m,n=0}^{\infty} |e_{m,n,u,v}| p^{-u} q^{-v} s^m t^n \leq \sum_{m,n=0}^{\infty} |d_{m,n,u,v}| p^{-u} q^{-v} s^m t^n + \sum_{m,n=0}^{\infty} |a_{m,n,u,v}| p^{-u} q^{-v} s^m t^n < \infty.$$

Since $\mathfrak{E}_D^2 \supseteq \mathfrak{E}_C^2$, it follows that $\mathfrak{E}_E^2 \supseteq \mathfrak{E}_C^2$.
For $x \in \mathfrak{E}^2$, we have:

$$S_E(x) = S_D(x) + S_A(x) = f(x) + S_A(x).$$

Because of f vanishes on $\mathfrak{E}^2$, it follows that $S_E(x) = S_A(x)$ for $x \in \mathfrak{E}^2$.
However, for $y \in \mathfrak{E}_E^2 \cap \mathfrak{E}_A^2$, we have:

$$S_E(y) = f(y) + S_A(y) \quad \text{with } f(y) \neq 0.$$

This contradicts the assumption of consistency between E and A. Hence, $\mathfrak{E}_A^2$ must be dense in $\mathfrak{E}_A^2 \cap \mathfrak{E}_B^2$, which completes the proof. □

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