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Topp-Leone Nadarajah-Haghighi distribution

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Abstract

In this paper, a three parameter model which can be used in modeling survival data, reliability problems and fatigue life studies has been studied. We derived explicit expressions for some of its statistical and mathematical identifying properties such as ordinary moments, generating function, incomplete moments and order statistics. The maximum likelihood estimations of model parameters were also obtained -being based on complete sample. We assessed the performance of the maximum likelihood estimators in terms of standard deviations, bias and mean squared errors by means of a simulation study. The usefulness of the model was illustrated by using a real data set. The proposed distribution provides better fits than some well-known generalized distributions under the same criteria of comparison.

Keywords: Topp-Leone distribution; Nadarajah-Haghighi distribution; maximum likelihood estimation; extended distributions.

Öz

Topp-Leone Nadarajah Haghighi Dağılımı

Bu çalışmada yaşam verileri, güvenirlik problemleri ve yorulma ömrü çalışmalarında kullanılabilecek üç parametreli bir model üzerinde çalışılmıştır. Bu modele ait momentler, üreten fonksiyonlar, tamamlanmamış momentler ve sıra istatistikleri gibi istatistiksel ve matematiksel özellikler türetilmiştir. Tam örneklem durumuna dayalı olarak, model parametrelerinin sıradan en çok olabilirlik tahminleri elde edilmiştir. Bir simülasyon çalışması ile model parametrelerinin en çok olabilirlik tahmin edicilerinin performansları, standart sapma, yan ve hata kareler ortalamaları ile değerlendirilmiştir. Modelin kullanışlılığı, bir gerçek veri setine dayalı olarak gösterilmiştir. Önerilen dağılım bazı karşılaştırma kriterleri altında, literatürde iyi iyi bilinen birtakım dağılımlardan daha iyi uyum sağlamıştır.

Anahtar sözcükler: *Topp-Leone dağılımı; Nadarajah-Haghighi dağılımı; en çok olabilirlik tahmini; genişletilmiş dağılımlar.*

1. Introduction

Recently, a new generalization of the exponential distribution, named Nadarajah-Haghighi (NH) distribution, as an alternative distribution to the gamma, Weibull and exponentiated-exponential distributions was proposed by [11]. The cumulative distribution function (cdf) of the NH distribution is given by

$$G(x,\eta,\beta)=1-e^{1-(1+\beta x)^{\eta}},x>0, \quad (1)$$

and the corresponding probability density function (pdf) is

$$g(x,\eta,\beta)=\eta\beta(1+\beta x)^{\eta-1}e^{1-(1+\beta x)^{\eta}},x>0, \quad (2)$$

where the parameter $\eta>0$ controls the shape of the distribution and $\beta>0$ is the scale parameter. [11] pointed out that the density function (2) has the attractive feature of always having the zero mode. They also showed that larger values of η in (2) will lead to faster decay of the upper tail.

We shall refer to the new distribution using (1) and (2) as the Topp Leone Nadarajah-Haghighi (TLNH) model using the Topp-Leone generated (TLG) family of distributions which was introduced by [14]. The pdf and cdf of the TLG family of distributions are given by

$$f(x;\alpha,\xi)=f(x)=2\alpha g(x;\xi)G(x;\xi)^{\alpha-1}\left[1-G(x;\xi)\right]\left[2-G(x;\xi)\right]^{\alpha-1} \quad (3)$$

and

$$F(x;\alpha,\xi)=F(x)=\left\{G(x;\xi)\left[2-G(x;\xi)\right]\right\}^{\alpha}, \quad (4)$$

where $\alpha>0$ is the shape parameter and ξ is the parameter vector of the baseline distribution G. By inserting (1) and (2) into (3) and (4), we can write the pdf and cdf of the TLNH model as

$$f(x)=f(x;\alpha,\eta,\beta)=2\alpha\eta\beta(1+\beta x)^{\eta-1}e^{2\left[1-(1+\beta x)^{\eta}\right]}\left[1-e^{2\left[1-(1+\beta x)^{\eta}\right]}\right]^{\alpha-1} \quad (5)$$

and

$$F(x)=F(x;\alpha,\eta,\beta)=\left[1-e^{2\left[1-(1+\beta x)^{\eta}\right]}\right]^{\alpha}, \quad (6)$$

respectively, and where $x>0,\alpha,\beta>0$.

The model in (5) and (6) is a special case of the exponentiated generalized Nadarajah-Haghighi distribution in [15]. Below we provide some plots of the pdf and hazard rate function (hrf), defined by $\frac{f(x)}{1-F(x)}$ of the TLNH model, to show its flexibility. Figure 1(a) displays some density plots of the TLNH for some parameter values. Plots of the hrf of the TLNH model for selected parameter values are given in Figure 1(b). The cdf in (6) can be expressed as

$$F(x)=\sum_{k=0}^{\infty} t_k H_{\alpha+k,\eta,\beta}(x), \tag{7}$$

where $t_k=(-1)^k 2^{\alpha-k}\binom{\alpha}{k}$ and $H_{\alpha+k,\eta,\beta}(x)=\left[G(x;\eta,\beta)\right]^{\alpha+k}=\left[1-e^{1-(1+\beta x)^{\eta}}\right]^{\alpha+k}$ is the cdf of the exponentiated Nadarajah-Haghighi (ENH) distribution in [8] with power parameter $\alpha+k$.

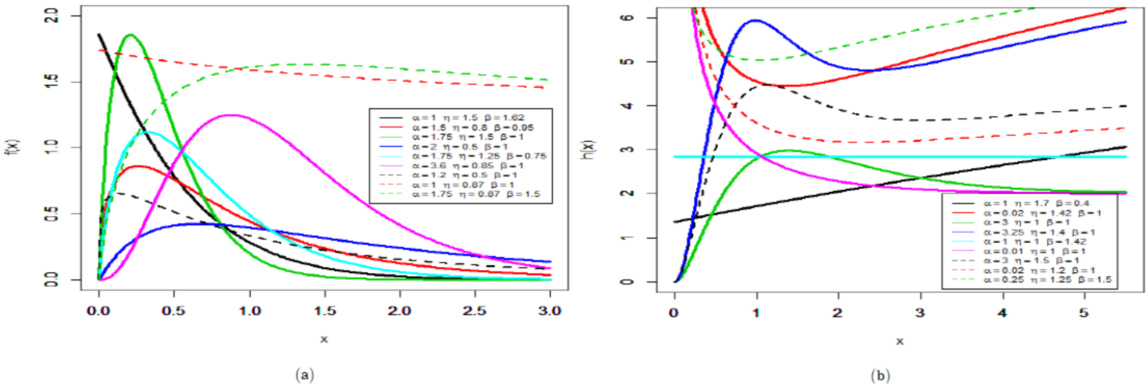

Figure 1. Plots of the TLNH pdf and hrf for some parameter values.

The corresponding TLNH density function is obtained by differentiating (7), to get

$$f(x)=\sum_{k=0}^{\infty} t_k h_{\alpha+k,\eta,\beta}(x), \tag{8}$$

where $h_{\alpha+k,\eta,\beta}(x)=(\alpha+k)\eta\beta(1+\beta x)^{\eta-1}\left[1-e^{1-(1+\beta x)^{\eta}}\right]^{\alpha+k-1} e^{1-(1+\beta x)^{\eta}}$ is the density of ENH density with power parameter $\alpha+k$. Thus, several of its structural properties can be obtained from Equation (8) and properties of the ENH distribution.

The new additional positive shape parameter α is sought to provide a more flexible distribution, and this is clearly shown in Figure 1. Several extensions of the NH model can be cited -such as the ENH or Lehmann Type 1-NH model by [1] and [8], transmuted Nadarajah-Haghighi model by [2], gamma Nadarajah-Haghighi model by [3] and [13], Kumaraswamy-Nadarajah-Haghighi model by [16], modified Nadarajah-Haghighi model by [6], exponentiated generalized Nadarajah-Haghighi model by [15], Marshall-Olkin Nadarajah-Haghighi model by [9] and beta Nadarajah-Haghighi model by [5] among others.

The rest of the paper can be outlined as follows. In Section 2, we derive some mathematical properties of the new distribution. In Section 3, the model parameter is estimated by using maximum likelihood method. We assess the performance of the maximum likelihood estimators in terms of standard deviations, biases and mean squared errors by means of a simulation study as well as a real data application, which is given at the end of Section 3 to illustrate the flexibility of the new model. Some concluding remarks are presented in Section 4.

2. Statistical properties

Some statistical properties of the mentioned distribution are obtained in this Section.

2.1. Moments

The r^{th} ordinary moment of *X* is given by

$$\mu'_r = E(X^r) = \sum_{k=0}^{\infty} t_k \int_0^{\infty} x^r h_{\alpha+k,\eta,\beta}(x)\,dx,$$

Then we obtain

$$\mu'_r = E(X^r) = \sum_{k,j=0}^{\infty} \sum_{i=0}^{r} \zeta_{k,j,i}^{(\alpha+k,r)} \Gamma\left(\tfrac{i}{\eta}+1, j+1\right), \tag{9}$$

where $\zeta_{k,j,i}^{(\alpha+k,r)} = t_k(-1)^{r+j-i} \dfrac{\alpha+k}{(1+j)^{\frac{i}{1+\eta}} \beta^r} \binom{\alpha+k-1}{j} e^{1+j}$, and $\Gamma(a,z) = \int_z^{\infty} w^{a-1} e^{-w} dw$ is the incomplete gamma function. For $\alpha+k>0$ integer, the moments in (8) will be reduced to

$$\mu'_r = \sum_{k=0}^{\infty} \sum_{j=0}^{\alpha+k} \sum_{i=0}^{r} \zeta_{k,j,i}^{(\alpha+k,r)} \Gamma\left(\tfrac{i}{\eta}+1, j+1\right). \tag{10}$$

When *r*=1 in (9) and (10), we obtain the mean of *X*. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The r^{th} central moment of *X*, say μ_r , is

$$\mu_r = E[X-\mu'_1]^r = \sum_{h=0}^{r} (-1)^h \binom{r}{h} \mu'_r (\mu'_1)^{r-h}.$$

2.2. Generating function and incomplete moments

The moment generating function (mgf), $M_X(t) = E(e^{tX})$, of *X* can be derived from (9) or (10) as $M_X(t) = \sum_{k=0}^{\infty} t_k \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r$. The main applications of the first incomplete moment refer to the mean deviations and the functions of Bonferroni and Lorenz curves. These curves are very useful in economics, reliability studies, demography, insurance and medicine. The s^{th} incomplete moment, say $c_s(t)$, of *X* can be expressed from(9) as $c_s(t) = \sum_{k=0}^{\infty} t_k \int_{-\infty}^{t} x^s h_{\alpha+k,\eta,\beta}(x)\,dx = \sum_{k,j=0}^{\infty} \sum_{i=0}^{s} \zeta_{k,j,i}^{(\alpha+k,s)} \Gamma\left(\tfrac{i}{\eta}+1, (j+1)(1+t\beta)^{\eta}\right)$.

The mean deviations about the arithmetic mean, $\eta_1 = E(|X-\mu'_1|)$, and about the median, $\eta_2 = E(|X-M|)$, of *X* are given by $\eta_1 = 2\mu'_1 F(\mu'_1) - 2c_1(\mu'_1)$ and $\eta_2 = \mu'_1 - 2c_1(M)$, respectively, where *M* is the median, $F(\mu'_1)$ is calculated from (6), and $c_1(t)$ is the first incomplete moment given by the last equation with *s*=1.

2.3. Moments of the residual life

The n^{th} moment of residual life, $m_n(t)=E\left[(X-t)^n \mid X>t\right], n=1,2,\ldots$, uniquely determines $F(x)$. The n^{th} moment of residual life of X is given by $m_n(t)=\frac{1}{1-F(t)}\int_t^{\infty}(x-t)^n\, dF(x)$. Therefore, $m_n(t)=\frac{1}{1-F(t)}\sum_{k,j=0}^{\infty}\sum_{i=0}^{n}\zeta_{k,j,i}^{(\alpha+k,s)^*}\Gamma\left(\frac{i}{\eta}+1,(j+1)(1+t\beta)^{\eta}\right)$, where $\zeta_{k,j,i}^{(\alpha+k,r)^*}=t_k^*\zeta_{k,j,i}^{(\alpha+k,r)}$ and $t_k^*=t_k\sum_{r=0}^{n}\binom{n}{r}(-t)^{n-r}$. The mean residual life function or the life expectation at age t can be defined by $m_1(t)$, which represents the expected additional life length for a unit which is alive at age t.

2.4. Moments of the reversed residual life

The n^{th} moment of residual life, $M_n(t)=E\left[(t-X)^n \mid X\le t\right], t>0, n=1,2,\ldots$, uniquely determines $F(x)$. The n^{th} moment of reversed residual life of X is given by $M_n(t)=\frac{1}{F(t)}\int_t^{\infty}(t-x)^n\, dF(x)$. Therefore, $M_n(t)=\frac{1}{F(t)}\sum_{k,j=0}^{\infty}\sum_{i=0}^{n}\zeta_{k,j,i}^{(\alpha+k,s)^{**}}\Gamma\left(\frac{i}{\eta}+1,(j+1)(1+t\beta)^{\eta}\right)$, where $\zeta_{k,j,i}^{(\alpha+k,r)^{**}}=t_k^{**}\zeta_{k,j,i}^{(\alpha+k,r)}$ and $t_k^{**}=t_k\sum_{r=0}^{n}(-1)^r\binom{n}{r}t^{n-r}$. The mean waiting time or mean inactivity time -being also called the mean reversed residual life function- is given by $M_1(t)$, and it represents the waiting time elapsed ever since the failure of an item on the condition that this failure had occurred in $(0,t)$.

2.5. Stress-strength modelling

The measure of reliability of industrial components has many applications especially in the area of lifetime testing and engineering -to name just a few. In stress-strength modeling, **R** = Pr(X_2 < X_1) is a measure of dependability of the system, when it is subjected to random stress X_2 and has strength X_1 (e.g. see Kotz et al., 2003). The system fails if and only if the applied stress is greater than its strength, and the component will function satisfactorily whenever $X_1 > X_2$. Herewith let X_1 and X_2 be two independent random variables with $TLNH(\alpha_1,\eta,\beta)$ and $TLNH(\alpha_2,\eta,\beta)$ distributions. Thus **R** can be expressed as

$$\mathbf{R}=\Pr(X_1>X_2)=\sum_{k,j=0}^{\infty}\frac{(-1)^{k+j}\,2^{\alpha_1+\alpha_2-k-j}}{(\alpha_2+j)(\alpha_2+\alpha_1+k+j)}\binom{\alpha_1}{j}\binom{\alpha_2}{j}.$$

2.6. Order Statistics

Let $X_1,X_2,\ldots,X_n$ be a random sample from the TLNH distribution and let $X_{(1)},X_{(2)},\ldots,X_{(n)}$ be the corresponding order statistics. The pdf of i'th order statistic, say $X_{i:n}$, can be written as

$$f_{i:n}(x)=\frac{f(x)}{\mathrm{B}(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}F(x)^{j+i-1}, \tag{11}$$

where $\mathrm{B}(\cdot,\cdot)$ is the beta function. Substituting (5) and (6) into equation (11) the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x)=\sum\nolimits_{j=0}^{n-i}\sum\nolimits_{w,d=0}^{\infty} t_{(j,w,d)}h_{w+d,\eta,\beta}(x),$$

where $t_{(j,w,d)}=(-1)^j\, wt_w\,\delta_{j+i-1,d}\left[(w+d)\mathrm{B}(i,n-i+1)\right]^{-1}$ and $\delta_{j+i-1,d}$ can be obtained recursively from $\delta_{j+i-1,d}=\frac{1}{d\,t_0}\sum\nolimits_{m=0}^{d} t_m\left[m(j+1)-d\right]\delta_{j+i-1,d-m}$ for $d\geq 1$ and $\delta_{j+i-1,0}=t^{j+i-1}$. Then, the pdf of the TLNH order statistics is a mixture of ENH density. Hence, by following Lemonte (2013), the moments of $X_{i:n}$ can be written as

$$E\left(X_{i:n}^{q}\right)=\sum\nolimits_{j=0}^{n-i}\sum\nolimits_{w,d,p=0}^{\infty}\sum\nolimits_{s=0}^{q}\zeta_{j,w,d,p,s}^{(\alpha+k,q)}\Gamma\left(\tfrac{s}{\eta}+1,p+1\right), \tag{12}$$

where $\zeta_{j,w,d,p,s}^{(\alpha+k,q)}=t_{j,w,d}\zeta_{p,s}^{(\alpha+k,q)}$. For $w+d>0$ integer, (12) can be reduced to

$$E\left(X_{i:n}^{q}\right)=\sum\nolimits_{j=0}^{n-i}\sum\nolimits_{w,d,p=0}^{\infty}\sum\nolimits_{p=0}^{w+k}\sum\nolimits_{s=0}^{q}\zeta_{j,w,d,p,s}^{(\alpha+k,q)}\ \Gamma\left(\tfrac{s}{\eta}+1,p+1\right).$$

3. Maximum likelihood estimations of the model parameters

In this section, we estimate the parameters of the TLNH distribution by the method of maximum likelihood estimation (MLE). Let $X_1,X_2,\ldots,X_n$ be a random sample from the TLNH distribution with observed values $x_1,x_2,\ldots,x_n$, and $\mathbf{\Psi}=(\alpha,\eta,\beta)^T$ be the vector of the model parameters. The log-likelihood function of $\mathbf{\Psi}$ may be expressed as

$$\begin{aligned}\ell &= n\log 2+n\log\alpha+n\log\eta+n\log\beta+(\eta-1)\sum_{i=1}^{n}\log(1+\beta x_i)+\sum_{i=1}^{n}(1-(1+\beta x_i)^{\eta})+(\alpha-1)\sum_{i=1}^{n}\log q_i\\ &\quad+\sum_{i=1}^{n}\log(1-q_i)+(\alpha-1)\sum_{i=1}^{n}\log(2-q_i),\end{aligned} \tag{13}$$

where $q_i=1-\exp\left\{1-(1+\beta x_i)^{\eta}\right\}$. By following the normal routine of parameter estimations for the MLE of α,η,β, we differentiate equation (13) with respect α,η,β, in order to obtain the score vector $\left(U_\alpha=\frac{\partial\ell}{\partial\alpha},U_\eta=\frac{\partial\ell}{\partial\eta},U_\beta=\frac{\partial\ell}{\partial\beta}\right)^T$. The elements of the score vector are given by

$$U_\alpha=\frac{n}{\alpha}+\sum_{i=1}^{n}\log q_i+\sum_{i=1}^{n}\log(2-q_i),$$

$$U_\eta=\frac{n}{\eta}+\sum_{i=1}^{n}\log(1+\beta x_i)-\sum_{i=1}^{n}(1+\beta x_i)^{\eta}\log\left((1+\beta x_i)^{\eta}\right)+(\alpha-1)\sum_{i=1}^{n}\frac{m_i}{q_i}-\sum_{i=1}^{n}\frac{m_i}{1-q_i}-(\alpha-1)\sum_{i=1}^{n}\frac{m_i}{2-q_i},$$

$$U_\beta=\frac{n}{\beta}+\sum_{i=1}^{n}\frac{x_i}{1+\beta x_i}-\eta\sum_{i=1}^{n}x_i(1+\beta x_i)^{\eta-1}+(\alpha-1)\sum_{i=1}^{n}\frac{v_i}{q_i}-\sum_{i=1}^{n}\frac{v_i}{1-q_i}-(\alpha-1)\sum_{i=1}^{n}\frac{v_i}{2-q_i},$$

where $m_i = (1+\beta x_i)^{\eta} \log(1+\beta x_i) e^{1-(1+\beta x_i)^{\eta}}$ and $v_i = x_i (1+\beta x_i)^{\eta} e^{1-(1+\beta x_i)^{\eta}}$. By setting the non-linear system of equations $U_\alpha = U_\eta = U_\beta = 0$ and solving them simultaneously, the MLE of parameters are obtained. These equations cannot be solved analytically, However, statistical software can be used to solve them numerically by using iterative methods such as the Newton-Raphson type algorithms. For interval estimation of the model parameters, we require the observed information matrix

$$J(\boldsymbol{\psi}) = -\begin{pmatrix} U_{\alpha\alpha} & U_{\alpha\eta} & U_{\alpha\beta} \\ U_{\eta\alpha} & U_{\eta\eta} & U_{\eta\beta} \\ U_{\beta\alpha} & U_{\beta\eta} & U_{\beta\beta} \end{pmatrix}.$$

Under standard regularity conditions when $n \to \infty$, the distribution of $\hat{\boldsymbol{\psi}}$ can be approximated by a multivariate normal $N_3\left(0, J(\hat{\boldsymbol{\psi}})^{-1}\right)$ distribution to construct approximate confidence intervals for the parameters. Here, $J(\hat{\boldsymbol{\psi}})$ is the total observed information matrix evaluated at $\hat{\boldsymbol{\psi}}$.

3.1. Simulation Study

In this Section, we perform the simulation study to see the performance of MLEs of TLNH distribution. The random number generation is obtained with its quantile function (qf). We note that the u'th qf of the TLNH is given by $x_u = \frac{1}{\beta}\left\{\left[1-0.5\log\left(1-u^{1/\alpha}\right)\right]^{1/\eta} - 1\right\}, 0 < u < 1$. Hence, if U has uniform random variable on (0, 1), then X_U has the TLNH random variable.

We generated N=1000 samples of sizes 20, 50 and 100 from TLNH distribution with its qf. Then we computed the empirical means, standard deviations (SD), biases and, mean squared errors (MSE) of the MLEs with $Bias_{\hat{h}} = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{h}_i - h\right)$ and $MSE_{\hat{h}} = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{h}_i - h\right)^2$, where $h = \alpha, \eta, \beta$. All results were obtained by using optim's CG routine in the R programme. The results of this simulation are reported in Table 1. Table 1 lists the empirical mean, standard deviations, biases by increasing sample size. MSEs decrease, as expected, when the sample size increases.

Table 1. The empirical means, Biases, MSEs and, SDs for the selected TLNH distribution's parameters.

Parameters	n=20			n=50			n=100		
(α,η,β)	$\hat{\alpha}$	$\hat{\eta}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\eta}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\eta}$	$\hat{\beta}$
(0.5, 0.5, 0.5)	0.5671	0.8111	0.6742	0.5216	0.6599	0.5710	0.5155	0.5931	0.5438
SD	0.1985	0.4802	0.7290	0.1090	0.3223	0.4383	0.0798	0.2294	0.3270
Bias	0.0671	0.3111	0.1742	0.0216	0.1599	0.0710	0.0155	0.0931	0.0438
MSE	0.0439	0.3271	0.5612	0.0123	0.1293	0.1970	0.0066	0.0612	0.1087
(1, 1, 1)	1.1601	1.4038	1.1003	1.0647	1.2311	1.1255	1.0221	1.1325	1.0060
SD	0.4496	0.6770	0.8947	0.2322	0.5222	0.7319	0.1429	0.3390	0.4349
Bias	0.1601	0.4038	0.1003	0.0647	0.2311	0.1255	0.0221	0.1325	0.0060
MSE	0.2275	0.6210	0.8097	0.0580	0.3258	0.5509	0.0209	0.1323	0.1890
(0.5, 2, 1)	0.5587	2.2562	1.2077	0.5275	2.1585	1.1231	0.5116	2.0408	1.0489
SD	0.1624	0.7386	0.7295	0.0931	0.5968	0.6408	0.0602	0.3166	0.2976
Bias	0.0588	0.2562	0.2077	0.0210	0.1114	0.1511	0.0165	0.0408	0.0489
MSE	0.0298	0.6107	0.5748	0.0098	0.3681	0.3977	0.0037	0.1018	0.0909
(2, 2, 2)	2.3427	2.4780	2.1038	2.1113	2.0616	2.0957	2.0674	2.1471	2.0645
SD	0.8543	1.0058	1.2183	0.4397	0.4549	0.6493	0.3098	0.5663	0.7216
Bias	0.3427	0.4780	0.1038	0.1123	0.0616	0.0957	0.0674	0.1471	0.0645
MSE	0.8467	1.2391	1.4937	0.2055	0.2105	0.4304	0.1004	0.3421	0.5244

(0.5, 5, 50)	0.5527	5.4948	50.0517	0.5236	5.2312	50.0325	0.5087	5.0813	50.0121
SD	0.1566	1.3000	0.2251	0.0869	0.8361	0.4462	0.0577	0.5218	0.0941
Bias	0.0527	0.4948	0.0517	0.0236	0.2312	0.0325	0.0087	0.0813	0.0121
MSE	0.0273	1.9331	0.0533	0.0081	0.7518	0.1999	0.0034	0.2786	0.0090

3.2. A real data application

We furthermore present an application based on the real data set to show the flexibility of the TLNH distribution. We compare TLNH with generalized exponential (GE) by [7], beta exponential (BE) by [12], Kumaraswamy exponential (KwE) by [4] and, NH distributions under the estimated log-likelihood $\hat{\ell}$ value, Kolmogorov-Smirnov (K-S) statistics, Akaike information criterion (AIC), Consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC). The pdfs of KwE, BE, GE distributions are given as follows (for $x > 0$ and $\alpha, \eta, \beta > 0$):

$$f_{KwE}(x;\alpha,\eta,\beta) = \eta\alpha\beta\left(1-e^{-\beta x}\right)^{\alpha-1}\left[1-\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\eta-1} e^{-\beta x}, \quad f_{BE}(x;\alpha,\eta,\beta) = \beta\left(1-e^{-\beta x}\right)^{\alpha-1} e^{-\beta\eta x}\mathrm{B}^{-1}(\alpha,\eta)$$

and $f_{GE}(x;\alpha,\beta) = \alpha\beta\left(1-e^{-\beta x}\right)^{\alpha-1} e^{-\beta x}$, where $\mathrm{B}(\cdot,\cdot)$ is the beta function.

The real data set represents the remission times (in months) of a random sample of 128 bladder cancer patients [10] as: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69. This data is known as unimodal hrf shaped in the literature. We also analyzied this data set for the ENH model. By using the data set, we obtained K-S statistics and its p-value as 0.0442 and 0.9636 respectively. These results are compatible with [8].

The results of this application are listed in Table 2. These results show that the TLNH distribution has the lowest AIC, CAIC, BIC, HQIC and K-S values and has the biggest estimated log-likelihood and p-value of the K-S statistics among all the fitted models. Hence, it could be chosen as the best model under these criteria. We also plot the estimated pdf and cdf of the TLNH for the data set in Figure 2. Clearly, the TLNH distribution provides a closer fit to the empirical pdf and cdf.

Table 2. The MLEs (standard errors within the parentheses), $\hat{\ell}$, AIC, CAIC, BIC, HQIC and K-S statistics [p-value] for remission data.

Model	$\hat{\alpha}$	$\hat{\eta}$	$\hat{\beta}$	$-\hat{\ell}$	AIC	CAIC	BIC	HQIC	K-S [p-value]
TLNH	1.6528 (0.3310)	0.5524 (0.1201)	0.1864 (0.0913)	410.3979	826.7957	826.9893	835.3518	830.2721	0.0429 [0.9722]
GE	1.2179 (0.1488)		0.1211 (0.0135)	413.0776	830.1552	832.3487	840.7113	835.6316	0.0725 [0.5113]
BE	1.1726 (0.1312)	26.8142 (0.6327)	0.0046 (0.0006)	413.3671	832.7342	835.0594	846.1423	839.3693	0.0733 [0.4973]
KwE	1.4512 (0.0240)	0.2816 (0.0274)	0.4105 (0.0154)	412.4602	830.9204	831.1139	839.4765	834.3968	0.0713 [0.5330]
NH		0.9227 (0.1515)	0.1216 (0.0344)	414.2255	832.4510	832.5470	838.1550	834.7686	0.0919 [0.2296]

4. Conclusions

In this paper, a three parameter model which can be used in modeling survival data, reliability problems and fatigue life studies has been studied. We derived explicit expressions for some of its statistical and mathematical quantities including the ordinary moments, generating function, incomplete moments, moment of residual life and reversed residual life. The model parameters were estimated by using maximum likelihood method based on complete sample. We assessed the performance of the maximum likelihood estimators in terms of standard deviations, bias and mean squared errors by means of a simulation study. We observed that the TLNH distribution provides better fits than generalized exponential, beta exponential, Kumaraswamy exponential and Nadarajah-Haghighi distributions on the real data set. We hope that the TLNH model will attract wider applications in areas such as survival tests and lifetime data, hydrology, meteorology, engineering and others. As a future work we plan to consider the bivariate and the multivariate extensions of the TLNH distribution in particular with the copula based construction method, trivariate reduction etc.

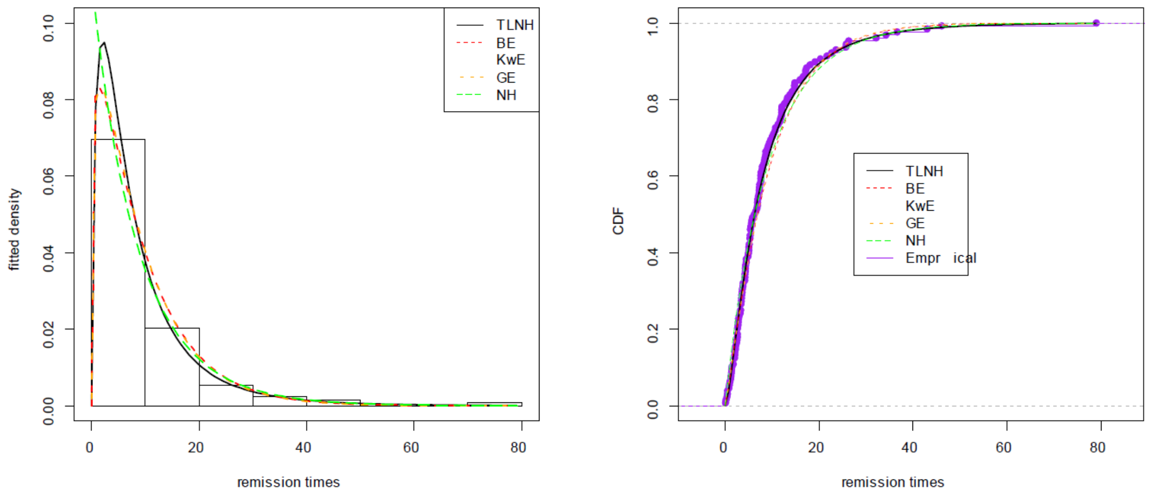

Figure 2. The estimated pdfs (left) and cdfs (right) for the remission data.

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