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Toward the Determination of Vietoris-like Polynomials

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Abstract— This paper studies the relationship between polynomials and classical number sequences, focusing on their structural properties and mathematical significance. It explores a specific class of polynomials inspired by Vietoris' number sequences, referred to as Vietoris-like polynomials. The primary objective is to analyze their fundamental algebraic properties, recurrence relations, and special identities. The study employs algebraic methods to derive the recurrence relations and explicit formulas for these polynomials. Moreover, it establishes Catalan-like, Cassini-like, and d'Ocagne-like identities.

Keywords — *Polynomials, Vietoris' sequence, Vietoris-like polynomials*

Mathematics Subject Classification (2020) 11B83, 11K31

1. Introduction

Polynomial forms of number sequences, beginning with Fibonacci polynomials, hold an important place in various subfields of mathematics such as geometry and algebra [1–3]. Fibonacci and Lucas polynomials constitute significant recursive sequences with remarkable algebraic and combinatorial properties. These polynomials have been extensively studied for their theoretical importance and applicability in interdisciplinary fields such as coding theory, quantum computing, and symbolic computation. Their structural characteristics enable efficient formulations in both pure and applied mathematics. In particular, Fibonacci-type polynomials have considerable applications in number theory [4–7]. For any variable quantity x, the Fibonacci polynomial $F_n(x)$ is defined as

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad \text{for all } n \geq 2$$

with $F_0(x) = 0$ and $F_1(x) = 1$. With a similar idea, the Lucas polynomial $L_n(x)$ is defined as

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \quad \text{for all } n \geq 2$$

with $L_0(x) = 2$ and $L_1(x) = x$. For more details, see [8, 9].

In 1958, Vietoris used Appell polynomials in connection with positivity problems of trigonometric sums [10]. Positivity as an interdisciplinary subject was an active research field, and several works were conducted using Vietoris' results [11]. Later on, in [12], the authors studied Vietoris' number

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sequence $\{v_s\}_{s\geq 0}$ with s-th term formula

$$v_s = \frac{1}{2^s}\binom{s}{\lfloor \frac{s}{2} \rfloor} \tag{1.1}$$

where $\binom{s}{\lfloor \frac{s}{2} \rfloor}$ is the central binomial coefficient [13] and $\lfloor . \rfloor$ represents the floor function. This sequence is associated with the sequence A283208 in the Online Encyclopedia of Integer Sequences (OEIS) [14]. As can be observed from [13, 15–19], Vietoris' sequence is one of the members of rational sequences, and some terms are as follows:

$$1, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{5}{16}, \frac{5}{16}, \frac{35}{128}, \frac{35}{128}, \frac{63}{256}, \frac{63}{256}, \ldots$$

In addition, the sequence of Appell-Vietoris polynomials [20], namely $\{\mathbb{V}_n(x)\}_{n\geq 0}$, is defined. For this sequence,

$$\mathbb{V}_n(x) = \sum_{k=0}^{n} \mathbb{T}_k^n x^k = \sum_{k=0}^{n} \binom{n}{k} c_{n-k} x^k$$

where $\mathbb{T}_k^n$ and c_k are triangle, i.e., these numbers form a triangular array with $n+1$ rows, indexed from $k=0$ to $k=n$, and $k-$th term of the Vietoris sequence, respectively. In [20], it can be seen that Vietoris' sequence via the sequence of Appell-Vietoris polynomials for $x=0$.

In this paper, we investigate the following questions: Is it possible to determine a special type of Vietoris-like polynomials by considering the properties of Vietoris' numbers? If so, what relations, identities, and properties do they satisfy? What conditions must be imposed on Vietoris-like polynomials to obtain meaningful results? This paper aims to explore and provide answers to the questions posed.

The rest of this study is structured as follows: Section 2 introduces the fundamental concepts to be utilized throughout the paper. Section 3 defines special Vietoris-like polynomials, investigates some of their basic properties, and analyzes their recurrence relations, special equalities, and identities such as those of Catalan, Cassini, and d'Ocagne. Finally, Section 4 provides the conclusions.

2. Preliminaries

This section discusses the basic properties of Vietoris' number sequence $\{v_s\}_{s\geq 0}$ with the s-th element in (1.1), For more details, see [10–18]. Even members of $\{v_s\}_{s\geq 0}$ are as follows:

$$v_{2n} = \frac{1}{2^{2n}}\binom{2n}{n}, \quad n \geq 0$$

where $v_{2n} = v_{2n-1}$. The two-term recurrence relation for $\{v_{2n}\}_{n\geq 0}$ is as follows:

$$v_{2n+2} = \mathcal{L}(2n)v_{2n}, \quad n \geq 0 \tag{2.1}$$

where

$$\mathcal{L}(k) = \frac{k+1}{k+2}, \quad k \geq 0 \tag{2.2}$$

Thus, the expression for v_{2n} in terms of any v_{2k} is as follows:

$$v_{2n} = \prod_{l=1}^{n-k} \mathcal{L}(2n-2l)v_{2k}, \quad n > k$$

Similarly, v_{2n} in terms of v_0 is as follows:

$$v_{2n+2} = \prod_{i=0}^{n} \mathcal{L}(2i)v_0 = \frac{(2n+1)!!}{(2n+2)!!}$$

Here, the double factorial of a number is defined as the product of all positive integers up to this number that shares the same parity (odd or even) as itself. The three consecutive-term recurrence relation for $\{v_{2n}\}_{n\geq 0}$ is as follows [17]:

$$v_{2n+2} = \frac{1}{2}v_{2n+1} + \frac{1}{2}\mathcal{L}(2n)v_{2n}, \quad n \geq 0 \tag{2.3}$$

The characteristic equation for the recurrence relation in (2.3) is given by [17]:

$$t^2 - \frac{1}{2}t - \frac{1}{2}\mathcal{L}(2n) = 0$$

with roots

$$r_{2n}^{\dagger_1} = \frac{1}{4}\left(1 - \sqrt{1+8\mathcal{L}(2n)}\right) \quad \text{and} \quad r_{2n}^{\dagger_2} = \frac{1}{4}\left(1 + \sqrt{1+8\mathcal{L}(2n)}\right) \tag{2.4}$$

According to the roots in (2.4), Vietoris' number sequence provides the following Binet-like formula [17]:

$$v_{2n} = c_{2n}^{\dagger_1}\left(r_{2n}^{\dagger_1}\right)^{2n} + c_{2n}^{\dagger_2}\left(r_{2n}^{\dagger_2}\right)^{2n}$$

where

$$c_{2n}^{\dagger_1} = \frac{\left(r_{2n}^{\dagger_2}\right)^{2n} - v_2}{\left(r_{2n}^{\dagger_2}\right)^{2n} - \left(r_{2n}^{\dagger_1}\right)^{2n}} \prod_{k=1}^{n-1}\left(2r_{2k}^{\dagger_1} - 1\right)r_{2k}^{\dagger_1} \quad \text{and} \quad c_{2n}^{\dagger_2} = \frac{v_2 - \left(r_{2n}^{\dagger_1}\right)^{2n}}{\left(r_{2n}^{\dagger_2}\right)^{2n} - \left(r_{2n}^{\dagger_1}\right)^{2n}} \prod_{k=1}^{n-1}\left(2r_{2k}^{\dagger_2} - 1\right)r_{2k}^{\dagger_2}$$

By the roots in (2.4), the following holds: $r_0^{\dagger_2} = \frac{1+\sqrt{5}}{4}$ (half of the golden ratio), $r_{2n}^{\dagger_1} + r_{2n}^{\dagger_2} = \dfrac{1}{2}$, and $r_{2n}^{\dagger_1}r_{2n}^{\dagger_2} = -\frac{\mathcal{L}(2n)}{2}$ [17]. Using (2.1), (2.3) of order two for the even index is rewritten as [17]:

$$v_{2n+2} = \frac{1}{2}\mathcal{L}(2n)v_{2n} + \frac{1}{2}\mathcal{L}(2n)\mathcal{L}(2n-2)v_{2n-2}, \quad n \geq 1$$

The characteristic equation of this recurrence is as follows [17]:

$$t^2 - \frac{1}{2}\mathcal{L}(2n)t - \frac{1}{2}\mathcal{L}(2n)\mathcal{L}(2n-2) = 0$$

with roots

$$\mathbf{r}_{2n}^{\dagger_1} = \frac{\mathcal{L}(2n)}{4}\left(1 - \sqrt{1+8\frac{\mathcal{L}(2n-2)}{\mathcal{L}(2n)}}\right) \quad \text{and} \quad \mathbf{r}_{2n}^{\dagger_2} = \frac{\mathcal{L}(2n)}{4}\left(1 + \sqrt{1+8\frac{\mathcal{L}(2n-2)}{\mathcal{L}(2n)}}\right) \tag{2.5}$$

According to these roots, Vietoris' number sequence provides the Binet-like formula [17]:

$$v_{2n} = \mathbf{c}_{2n}^{\dagger_1}\left(\mathbf{r}_{2n}^{\dagger_1}\right)^{2n} + \mathbf{c}_{2n}^{\dagger_2}\left(\mathbf{r}_{2n}^{\dagger_2}\right)^{2n}$$

where

$$\mathbf{c}_{2n}^{\dagger_1} = \frac{(2n-1)!!\left(-\mathcal{L}(2n)\left(\mathbf{r}_{2n}^{\dagger_2}\right)^{2n} + \left(\mathbf{r}_{2n+2}^{\dagger_2}\right)^{2n+2}\right)}{2^n n!\left(\left(\mathbf{r}_{2n}^{\dagger_1}\right)^{2n}\left(\mathbf{r}_{2n+2}^{\dagger_2}\right)^{2n+2} - \left(\mathbf{r}_{2n}^{\dagger_2}\right)^{2n}\left(\mathbf{r}_{2n+2}^{\dagger_1}\right)^{2n+2}\right)}$$

and

$$\mathbf{c}_{2n}^{\dagger_2} = \frac{(2n-1)!!\left(\mathcal{L}(2n)\left(\mathbf{r}_{2n}^{\dagger_1}\right)^{2n} - \left(\mathbf{r}_{2n+2}^{\dagger_1}\right)^{2n+2}\right)}{2^n n!\left(\left(\mathbf{r}_{2n}^{\dagger_1}\right)^{2n}\left(\mathbf{r}_{2n+2}^{\dagger_2}\right)^{2n+2} - \left(\mathbf{r}_{2n}^{\dagger_2}\right)^{2n}\left(\mathbf{r}_{2n+2}^{\dagger_1}\right)^{2n+2}\right)}$$

By the roots in (2.5), the following hold: $\mathbf{r}_{2n}^{\dagger_1} + \mathbf{r}_{2n}^{\dagger_2} = \dfrac{\mathcal{L}(2n)}{2}$ and $\mathbf{r}_{2n}^{\dagger_1}\mathbf{r}_{2n}^{\dagger_2} = -\dfrac{\mathcal{L}(2n)\mathcal{L}(2n-2)}{2}$ [17]. Moreover, the generating function is given by [15]:

$$g(z) = \frac{\sqrt{1+z} - \sqrt{1-z}}{z\sqrt{1-z}} = \sum_{p=0}^{\infty} v_p z^p, \quad 0 < |z| < 1$$

3. Special Vietoris-like Polynomials

This section introduces special Vietoris-like polynomials and presents several of their properties.

Definition 3.1. For real variable x, the s-th element of Vietoris-like polynomial sequence $\{\mathcal{V}_s(x)\}_{s\geq 0}$ is defined by

$$\mathcal{V}_s(x) = \begin{cases} \mathcal{L}(s-1)\mathcal{V}_{s-1}(x), & \text{if } s \text{ is odd} \\ \frac{x+1}{2}\mathcal{L}(s-2)\mathcal{V}_{s-2}(x), & \text{if } s \text{ is even} \end{cases} \tag{3.1}$$

where $\mathcal{V}_0(x) = 1$.

The first few Vietoris-like polynomials are

$$1,\ \tfrac{1}{2},\ \tfrac{x+1}{4},\ \tfrac{3(x+1)}{16},\ \tfrac{3(x+1)^2}{32},\ \tfrac{5(x+1)^2}{64},\ \tfrac{5(x+1)^3}{128},\ \tfrac{35(x+1)^3}{1024},\ \tfrac{35(x+1)^4}{2048},\ \tfrac{63(x+1)^4}{4096},\ \tfrac{63(x+1)^5}{8192},\ \cdots \tag{3.2}$$

In particular, for $x = 1$, Vietoris-like polynomials are equal to Vietoris' sequence. For $x = -1, \mathcal{V}_s(x) = 0$ where $s \geq 2$. It can be observed the graphs of the first eleven elements of Vietoris-like polynomial sequence in Figure 1, for $-5 \leq x \leq 5$.

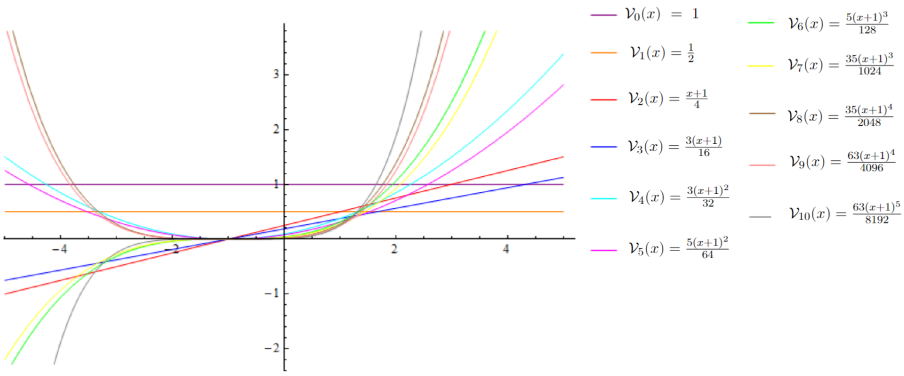

Figure 1. First eleven elements of Vietoris-like polynomial sequence

Corollary 3.2. Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence. Then, two-term recurrence relation, for $\{\mathcal{V}_{2n}(x)\}_{n\geq 0}$ is obtained from (3.1), for $s = 2n+2$ as follows:

$$\mathcal{V}_{2n+2}(x) = \frac{x+1}{2}\mathcal{L}(2n)\mathcal{V}_{2n}(x) \tag{3.3}$$

Moreover, even members can be also written using (2.2) such that:

$$\mathcal{V}_{2n}(x) = \left(\frac{x+1}{2}\right)^n \frac{1}{2^{2n}}\binom{2n}{n}, \quad n \geq 0 \tag{3.4}$$

Corollary 3.3. Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence. Considering (3.3) in terms of $\{\mathcal{V}_{2n}(x)\}_{n\geq 0}$,

$$\mathcal{V}_{2n+2}(x) = \left(\frac{x+1}{2}\right)^2 \mathcal{L}(2n)\mathcal{L}(2n-2)\mathcal{V}_{2n-2}(x), \quad n \geq 1 \tag{3.5}$$

Additionally, the term $\mathcal{V}_{2n}(x)$ in terms of any $\mathcal{V}_{2k}(x)$ is as follows:

$$\mathcal{V}_{2n}(x) = \prod_{l=1}^{n-k} \left(\frac{x+1}{2}\right)^{n-k} \mathcal{L}(2n-2l)\mathcal{V}_{2k}(x), \quad n > k \tag{3.6}$$

The following equality in terms of $\mathcal{V}_0(x)$ is obtained:

$$\mathcal{V}_{2n+2}(x) = \left(\frac{x+1}{2}\right)^{n+1} \prod_{i=0}^{n} \mathcal{L}(2i)\mathcal{V}_0(x) = \left(\frac{x+1}{2}\right)^{n+1} \frac{(2n+1)!!}{(2n+2)!!} \tag{3.7}$$

Proof. By putting $s = 2n+2$ and $s = 2n$ in (3.1), (3.3) and

$$\mathcal{V}_{2n}(x) = \frac{x+1}{2}\mathcal{L}(2n-2)\mathcal{V}_{2n-2}(x) \tag{3.8}$$

calculates, respectively. When (3.8) is substituted into (3.3), (3.5) is obtained. If this process continues, (3.6) is obtained. Moreover, for a particular value $k = 0$, (3.6) is transformed into (3.7). Here, it is clear that $\prod\limits_{i=0}^{n} \mathcal{L}(2i) = \frac{(2n+1)!!}{(2n+2)!!}$ via (2.2) □

Corollary 3.4. The three consecutive-term recurrence relation for $\{\mathcal{V}_{2n}(x)\}_{n\geq 0}$ is as follows:

$$\mathcal{V}_{2n+2}(x) = \frac{x}{2}\mathcal{V}_{2n+1}(x) + \frac{\mathcal{L}(2n)}{2}\mathcal{V}_{2n}(x) \tag{3.9}$$

Proof. From (3.3), $\mathcal{V}_{2n+2}(x) = \frac{x+1}{2}\mathcal{L}(2n)\mathcal{V}_{2n}(x)$. Then, it follows $\mathcal{V}_{2n+2}(x) = \frac{x}{2}\mathcal{L}(2n)\mathcal{V}_{2n}(x) + \frac{1}{2}\mathcal{L}(2n)\mathcal{V}_{2n}(x)$. From (3.1), $\mathcal{L}(2n)\mathcal{V}_{2n}(x)$ for $\mathcal{V}_{2n+1}$. This ultimately leads to the three-consecutive-term recurrence relation (3.9). □

Corollary 3.5. Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence. Then, two-term recurrence relation, for $\{\mathcal{V}_{2n+1}(x)\}_{n\geq 0}$ is obtained from (3.1), for $s = 2n+1$ as follows:

$$\mathcal{V}_{2n+1}(x) = \frac{x+1}{2}\mathcal{L}(2n)\mathcal{V}_{2n-1}(x) \tag{3.10}$$

Moreover, odd members can be also written using (2.2) such that:

$$\mathcal{V}_{2n-1}(x) = \left(\frac{x+1}{2}\right)^{n-1} \frac{1}{2^{2n}} \binom{2n}{n}, \quad n \geq 0$$

Corollary 3.6. Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence. Considering (3.10) in terms of $\{\mathcal{V}_{2n+1}(x)\}_{n\geq 0}$,

$$\mathcal{V}_{2n+1}(x) = \left(\frac{x+1}{2}\right)^2 \mathcal{L}(2n)\mathcal{L}(2n-2)\mathcal{V}_{2n-3}(x), \quad n \geq 1 \tag{3.11}$$

Additionally, by using (3.10), the term $\mathcal{V}_{2n+1}(x)$ in terms of any $\mathcal{V}_{2k+1}(x)$ is as follows:

$$\mathcal{V}_{2n-1}(x) = \prod_{l=1}^{n-k} \left(\frac{x+1}{2}\right)^{n-k} \mathcal{L}(2n-2l)\mathcal{V}_{2k-1}(x), \quad n > k \tag{3.12}$$

the term $\mathcal{V}_{2n+1}(x)$ in terms of $\mathcal{V}_0(x)$ is obtained as follows:

$$\mathcal{V}_{2n+1}(x) = \left(\frac{x+1}{2}\right)^{n} \prod_{i=0}^{n} \mathcal{L}(2i)\mathcal{V}_0(x) = \left(\frac{x+1}{2}\right)^{n} \frac{(2n+1)!!}{(2n+2)!!} \tag{3.13}$$

Proof. By putting $s = 2n+1$ and $s = 2n-1$ in (3.1), (3.10) and

$$\mathcal{V}_{2n-1}(x) = \frac{x+1}{2}\mathcal{L}(2n-2)\mathcal{V}_{2n-3}(x), \tag{3.14}$$

is calculated, respectively. When (3.14) is substituted into (3.10), (3.11) is obtained. If this process continues, (3.12) is obtained. Moreover, for a particular value $k = 0$, (3.12) is transformed into (3.13). Here, it is clear that $\prod_{i=0}^{n} \mathcal{L}(2i) = \frac{(2n+1)!!}{(2n+2)!!}$ via (2.2) □

Theorem 3.7 (Binet-like Formula-Form 1)**.** Let $\{\mathcal{V}_{2n}(x)\}_{n\geq 0}$ be Vietoris-like polynomial sequence. Then, for $n > 1$, it provides Binet-like formula:

$$\mathcal{V}_{2n}(x) = C^{\dagger_1}_{2n}(x)(R^{\dagger_1}_{2n}(x))^{2n} + C^{\dagger_2}_{2n}(x)(R^{\dagger_2}_{2n}(x))^{2n} \tag{3.15}$$

where

$$R^{\dagger_1}_{2n}(x) = \frac{1}{4}\left(x - \sqrt{x^2 + 8\mathcal{L}(2n)}\right),\ R^{\dagger_2}_{2n}(x) = \frac{1}{4}\left(x + \sqrt{x^2 + 8\mathcal{L}(2n)}\right) \tag{3.16}$$

and

$$\begin{cases} C^{\dagger_1}_{2n}(x) &= \left(\frac{x+1}{2}\right)^{n-1} \dfrac{(R^{\dagger_2}_{2n}(x))^{2n} - \mathcal{V}_2(x)}{(R^{\dagger_2}_{2n}(x))^{2n} - (R^{\dagger_1}_{2n}(x))^{2n}} \prod\limits_{k=1}^{n-1} (2R^{\dagger_1}_{2k}(x) - x)R^{\dagger_1}_{2k}(x) \\ C^{\dagger_2}_{2n}(x) &= \left(\frac{x+1}{2}\right)^{n-1} \dfrac{\mathcal{V}_2(x) - (R^{\dagger_1}_{2n}(x))^{2n}}{(R^{\dagger_2}_{2n}(x))^{2n} - (R^{\dagger_1}_{2n}(x))^{2n}} \prod\limits_{k=1}^{n-1} (2R^{\dagger_2}_{2k}(x) - x)R^{\dagger_2}_{2k}(x) \end{cases} \tag{3.17}$$

Proof. Considering (3.9), characteristic equation for $\{\mathcal{V}_{2n}(x)\}_{n\geq 0}$ is written by

$$t^2 - \frac{1}{2}xt - \frac{1}{2}\mathcal{L}(2n) = 0 \tag{3.18}$$

Thus, its roots $R^{\dagger_1}_{2n}(x)$ and $R^{\dagger_2}_{2n}(x)$ are

$$R^{\dagger_1}_{2n}(x) = \frac{1}{4}\left(x - \sqrt{x^2 + 8\mathcal{L}(2n)}\right) \quad \text{and} \quad R^{\dagger_2}_{2n}(x) = \frac{1}{4}\left(x + \sqrt{x^2 + 8\mathcal{L}(2n)}\right) \tag{3.19}$$

By (3.17),

$$\begin{aligned} C^{\dagger_1}_{2n}(x)(R^{\dagger_1}_{2n}(x))^{2n} + C^{\dagger_2}_{2n}(x)(R^{\dagger_2}_{2n}(x))^{2n} &= \left(\frac{x+1}{2}\right)^{n-1} \frac{\left(\left(R^{\dagger_2}_{2n}(x)\right)^{2n} - \mathcal{V}_2(x)\right)\prod\limits_{k=1}^{n-1}(2R^{\dagger_1}_{2k}(x) - x)R^{\dagger_1}_{2k}(x)\left(R^{\dagger_1}_{2n}(x)\right)^{2n}}{\left(R^{\dagger_2}_{2n}(x)\right)^{2n} - \left(R^{\dagger_1}_{2n}(x)\right)^{2n}} \\ &+ \left(\frac{x+1}{2}\right)^{n-1} \frac{\left(\mathcal{V}_2(x) - \left(R^{\dagger_1}_{2n}(x)\right)^{2n}\right)\prod\limits_{k=1}^{n-1}(2R^{\dagger_2}_{2k}(x) - x)R^{\dagger_2}_{2k}(x)\left(R^{\dagger_2}_{2n}(x)\right)^{2n}}{\left(R^{\dagger_2}_{2n}(x)\right)^{2n} - \left(R^{\dagger_1}_{2n}(x)\right)^{2n}} \\ &= \left(\frac{x+1}{2}\right)^{n-1} \frac{\left(R^{\dagger_2}_{2n}(x)\right)^{2n}\left(R^{\dagger_1}_{2n}(x)\right)^{2n}\prod\limits_{k=1}^{n-1}(2R^{\dagger_1}_{2k}(x) - x)R^{\dagger_1}_{2k}(x) - \left(R^{\dagger_1}_{2n}(x)\right)^{2n}\left(R^{\dagger_2}_{2n}(x)\right)^{2n}\prod\limits_{k=1}^{n-1}(2R^{\dagger_2}_{2k}(x) - x)R^{\dagger_2}_{2k}(x)}{\left(R^{\dagger_2}_{2n}(x)\right)^{2n} - \left(R^{\dagger_1}_{2n}(x)\right)^{2n}} \\ &+ \left(\frac{x+1}{2}\right)^{n-1} \frac{-\mathcal{V}_2(x)\left(R^{\dagger_1}_{2n}(x)\right)^{2n}\prod\limits_{k=1}^{n-1}(2R^{\dagger_1}_{2k}(x) - x)R^{\dagger_1}_{2n}(2k) + \mathcal{V}_2(x)\left(R^{\dagger_2}_{2n}(x)\right)^{2n}\prod\limits_{k=1}^{n-1}(2R^{\dagger_2}_{2k}(x) - x)R^{\dagger_2}_{2k}(x)}{\left(R^{\dagger_2}_{2n}(x)\right)^{2n} - \left(R^{\dagger_1}_{2n}(x)\right)^{2n}} \\ &= \left(\frac{x+1}{2}\right)^{n-1} \frac{\left(R^{\dagger_2}_{2n}(x)\right)^{2n}\left(R^{\dagger_1}_{2n}(x)\right)^{2n}\prod\limits_{k=1}^{n-1}\left(2\left(R^{\dagger_1}_{2k}(x)\right)^2 - xR^{\dagger_1}_{2k}(x)\right) - \left(2\left(R^{\dagger_2}_{2k}(x)\right)^2 - xR^{\dagger_2}_{2k}(x)\right)}{\left(R^{\dagger_2}_{2n}(x)\right)^{2n} - \left(R^{\dagger_1}_{2n}(x)\right)^{2n}} \\ &+ \mathcal{V}_2(x)\left(\frac{x+1}{2}\right)^{n-1} \frac{-\left(R^{\dagger_1}_{2n}(x)\right)^{2n}\prod\limits_{k=1}^{n-1}\left(2\left(R^{\dagger_1}_{2k}(x)\right)^2 - xR^{\dagger_1}_{2k}(x)\right) + \left(R^{\dagger_2}_{2n}(x)\right)^{2n}\prod\limits_{k=1}^{n-1}\left(2\left(R^{\dagger_2}_{2k}(x)\right)^2 - xR^{\dagger_2}_{2k}(x)\right)}{\left(R^{\dagger_2}_{2n}(x)\right)^{2n} - \left(R^{\dagger_1}_{2n}(x)\right)^{2n}} \end{aligned}$$

Since $R^{\dagger_1}_{2n}(x)$ and $R^{\dagger_2}_{2n}(x)$ in (3.19) satisfies (3.18), then

$$2\left(R^{\dagger_1}_{2k}(x)\right)^2 - xR^{\dagger_1}_{2k}(x) = \mathcal{L}(2k) \quad \text{and} \quad 2\left(R^{\dagger_2}_{2k}(x)\right)^2 - xR^{\dagger_2}_{2k}(x) = \mathcal{L}(2k)$$

Then,

$$C_{2n}^{\dagger_1}(x)(R_{2n}^{\dagger_1}(x))^{2n} + C_{2n}^{\dagger_2}(x)(R_{2n}^{\dagger_2}(x))^{2n} = \mathcal{V}_2(x)\left(\frac{x+1}{2}\right)^{n-1} \frac{-\left(R_{2n}^{\dagger_1}(x)\right)^{2n}\prod\limits_{k=1}^{n-1}\mathcal{L}(2k) + \left(R_{2n}^{\dagger_2}(x)\right)^{2n}\prod\limits_{k=1}^{n-1}\mathcal{L}(2k)}{\left(R_{2n}^{\dagger_2}(x)\right)^{2n} - \left(R_{2n}^{\dagger_1}(x)\right)^{2n}}$$
$$= \mathcal{V}_2(x)\left(\frac{x+1}{2}\right)^{n-1}\prod_{k=1}^{n-1}\mathcal{L}(2k)$$

Furthermore, using (3.6), for $k=1$, the equality $\mathcal{V}_{2n}(x) = \prod\limits_{l=1}^{n-1}\left(\frac{x+1}{2}\right)^{n-1}\mathcal{L}(2n-2l)\mathcal{V}_2(x)$ is obtained, and thus (3.15) is valid. $\square$

Example 3.8. Calculate $\mathcal{V}_6(x)$ and $\mathcal{V}_8(x)$ using Binet-like Formula-Form 1 for $n=3$ and $n=4$, respectively. Through (3.16) and (3.17),

$$R_6^{\dagger_1}(x) = \frac{\left(x-\sqrt{x^2+7}\right)^6}{4096}$$
$$R_6^{\dagger_2}(x) = \frac{\left(x+\sqrt{x^2+7}\right)^6}{4096}$$
$$C_6^{\dagger_1}(x) = \frac{(x+1)^2\left(x+\sqrt{x^2+6}\right)\left(x+\sqrt{x^2+\frac{20}{3}}\right)\left(-x+\frac{1}{2}\left(x+\sqrt{x^2+6}\right)\right)\left(-x+\frac{1}{2}\left(x+\sqrt{x^2+\frac{20}{3}}\right)\right)\left(\frac{1}{4}(-1-x)+\frac{\left(x+\sqrt{x^2+7}\right)^6}{4096}\right)}{64\left(-\frac{\left(x-\sqrt{x^2+7}\right)^6}{4096}+\frac{\left(x+\sqrt{x^2+7}\right)^6}{4096}\right)}$$

and

$$C_6^{\dagger_2}(x) = \frac{(x+1)^2\left(x-\sqrt{x^2+6}\right)\left(x-\sqrt{x^2+\frac{20}{3}}\right)\left(-x+\frac{1}{2}\left(x-\sqrt{x^2+6}\right)\right)\left(-x+\frac{1}{2}\left(x-\sqrt{x^2+\frac{20}{3}}\right)\right)\left(\frac{1+x}{4}-\frac{\left(x-\sqrt{x^2+7}\right)^6}{4096}\right)}{64\left(-\frac{\left(x-\sqrt{x^2+7}\right)^6}{4096}+\frac{\left(x+\sqrt{x^2+7}\right)^6}{4096}\right)}$$

Then, $\mathcal{V}_6(x) = C_6^{\dagger_1}(x)(R_6^{\dagger_1}(x))^6 + C_6^{\dagger_2}(x)(R_6^{\dagger_2}(x))^6 = \frac{5(x+1)^3}{128}$. It can also be observed that $\mathcal{V}_6(x)$ via (3.2). Similarly, through (3.16) and (3.17),

$$R_8^{\dagger_1}(x) = \frac{\left(x-\sqrt{x^2+\frac{36}{5}}\right)^8}{65536}$$
$$R_8^{\dagger_2}(x) = \frac{\left(x+\sqrt{x^2+\frac{36}{5}}\right)^8}{65536}$$
$$C_8^{\dagger_1}(x) = \frac{(x+1)^3\left(-x+\sqrt{x^2+6}\right)\left(x+\sqrt{x^2+6}\right)\left(-\frac{x}{2}+\frac{1}{2}\sqrt{x^2+\frac{20}{3}}\right)\left(x+\sqrt{x^2+\frac{20}{3}}\right)\left(-x+\sqrt{x^2+7}\right)\left(x+\sqrt{x^2+7}\right)\left(\frac{1}{4}(-x-1)+\frac{\left(x+\sqrt{x^2+\frac{36}{5}}\right)^8}{65536}\right)}{2048\left(-\frac{\left(x-\sqrt{x^2+\frac{36}{5}}\right)^8}{65536}+\frac{\left(x+\sqrt{x^2+\frac{36}{5}}\right)^8}{65536}\right)}$$

and

$$C_8^{\dagger_2}(x) = \frac{(x+1)^3\left(-x-\sqrt{x^2+6}\right)\left(x-\sqrt{x^2+6}\right)\left(x-\sqrt{x^2+\frac{20}{3}}\right)\left(-\frac{x}{2}-\frac{1}{2}\sqrt{x^2+\frac{20}{3}}\right)\left(-x-\sqrt{x^2+7}\right)\left(x-\sqrt{x^2+7}\right)\left(\frac{1+x}{4}-\frac{\left(x-\sqrt{x^2+\frac{36}{5}}\right)^8}{65536}\right)}{2048\left(-\frac{\left(x-\sqrt{x^2+\frac{36}{5}}\right)^8}{65536}+\frac{\left(x+\sqrt{x^2+\frac{36}{5}}\right)^8}{65536}\right)}$$

Then, it follows that $\mathcal{V}_8(x) = C_8^{\dagger_1}(x)(R_8^{\dagger_1}(x))^8 + C_8^{\dagger_2}(x)(R_8^{\dagger_2}(x))^8 = \frac{35(x+1)^4}{2048}$. It can also be checked via (3.2).

Remark 3.9. The following hold for $R_{2n}^{\dagger_1}(x)$ and $R_{2n}^{\dagger_2}(x)$:

i. $R_0^{\dagger_2}(1) = \frac{1+\sqrt{5}}{4}$, which is half of the golden ratio *ii.* $R^{\dagger_1}_{2n}(x) + R^{\dagger_2}_{2n}(x) = \frac{x}{2}$

iii. $R^{\dagger_1}_{2n}(x) R^{\dagger_2}_{2n}(x) = -\frac{\mathcal{L}(2n)}{2}$

Theorem 3.10 (Binet-like Formula-Form 2)**.** Let $\{\mathcal{V}_{2n}(x)\}_{n\geq 0}$ be Vietoris-like polynomial sequence. Then, it provides Binet-like formula

$$\mathcal{V}_{2n}(x) = \mathcal{C}^{\dagger_1}_{2n}(x)(\mathcal{R}^{\dagger_1}_{2n}(x))^{2n} + \mathcal{C}^{\dagger_2}_{2n}(x)(\mathcal{R}^{\dagger_2}_{2n}(x))^{2n} \tag{3.20}$$

where

$$\begin{cases} \mathcal{R}^{\dagger_1}_{2n}(x) = \dfrac{\mathcal{L}(2n)}{4}\left(x - \sqrt{x^2 + \dfrac{4(x+1)\mathcal{L}(2n-2)}{\mathcal{L}(2n)}}\right) \\ \mathcal{R}^{\dagger_2}_{2n}(x) = \dfrac{\mathcal{L}(2n)}{4}\left(x + \sqrt{x^2 + \dfrac{4(x+1)\mathcal{L}(2n-2)}{\mathcal{L}(2n)}}\right) \end{cases} \tag{3.21}$$

and

$$\begin{cases} \mathcal{C}^{\dagger_1}_{2n}(x) = \left(\dfrac{x+1}{2}\right)^n \dfrac{(2n-1)!!\left(-\mathcal{L}(2n)(\mathcal{R}^{\dagger_2}_{2n}(x))^{2n} + (\mathcal{R}^{\dagger_2}_{2n+2}(x))^{2n+2}\right)}{2^n n!\left((\mathcal{R}^{\dagger_1}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_2}_{2n+2}(x))^{2n+2} - (\mathcal{R}^{\dagger_2}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_1}_{2n+2}(x))^{2n+2}\right)} \\ \mathcal{C}^{\dagger_2}_{2n}(x) = \left(\dfrac{x+1}{2}\right)^n \dfrac{(2n-1)!!\left(\mathcal{L}(2n)(\mathcal{R}^{\dagger_1}_{2n}(x))^{2n} - (\mathcal{R}^{\dagger_1}_{2n+2}(x))^{2n+2}\right)}{2^n n!\left((\mathcal{R}^{\dagger_1}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_2}_{2n+2}(x))^{2n+2} - (\mathcal{R}^{\dagger_2}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_1}_{2n+2}(x))^{2n+2}\right)} \end{cases} \tag{3.22}$$

Proof. Considering (3.5), the characteristic equation of Vietoris-like polynomials is as follows:

$$t^2 - \frac{x}{2}\mathcal{L}(2n)t - \frac{x+1}{2}\frac{\mathcal{L}(2n)\mathcal{L}(2n-2)}{2} = 0$$

Thus, its roots $R^{\dagger_1}_{2n}(x)$ and $\mathcal{R}^{\dagger_2}_{2n}(x)$ are as follows:

$$\mathcal{R}^{\dagger_1}_{2n}(x) = \frac{\mathcal{L}(2n)}{4}\left(x - \sqrt{x^2 + \frac{4(x+1)\mathcal{L}(2n-2)}{\mathcal{L}(2n)}}\right)$$

and

$$\mathcal{R}^{\dagger_2}_{2n}(x) = \frac{\mathcal{L}(2n)}{4}\left(x + \sqrt{x^2 + \frac{4(x+1)\mathcal{L}(2n-2)}{\mathcal{L}(2n)}}\right)$$

By using (3.22), calculate $\mathcal{C}^{\dagger_1}_{2n}(x)(\mathcal{R}^{\dagger_1}_{2n}(x))^{2n} + \mathcal{C}^{\dagger_2}_{2n}(x)(\mathcal{R}^{\dagger_2}_{2n}(x))^{2n}$ as:

$$= \left(\frac{x+1}{2}\right)^n \frac{(2n-1)!!\left(-\mathcal{L}(2n)(\mathcal{R}^{\dagger_2}_{2n}(x))^{2n} + (\mathcal{R}^{\dagger_2}_{2n+2}(x))^{2n+2}\right)(\mathcal{R}^{\dagger_1}_{2n}(x))^{2n}}{2^n n!\left((\mathcal{R}^{\dagger_1}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_2}_{2n+2}(x))^{2n+2} - (\mathcal{R}^{\dagger_2}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_1}_{2n+2}(x))^{2n+2}\right)}$$

$$+ \left(\frac{x+1}{2}\right)^n \frac{(2n-1)!!\left(\mathcal{L}(2n)(\mathcal{R}^{\dagger_1}_{2n}(x))^{2n} - (\mathcal{R}^{\dagger_1}_{2n+2}(x))^{2n+2}\right)(\mathcal{R}^{\dagger_2}_{2n}(x))^{2n}}{2^n n!\left((\mathcal{R}^{\dagger_1}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_2}_{2n+2}(x))^{2n+2} - (\mathcal{R}^{\dagger_2}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_1}_{2n+2}(x))^{2n+2}\right)}$$

$$= \left(\frac{x+1}{2}\right)^n \frac{(2n-1)!!\left((\mathcal{R}^{\dagger_1}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_2}_{2n+2}(x))^{2n+2} - (\mathcal{R}^{\dagger_2}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_1}_{2n+2}(x))^{2n+2}\right)}{2^n n!\left((\mathcal{R}^{\dagger_1}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_2}_{2n+2}(x))^{2n+2} - (\mathcal{R}^{\dagger_2}_{2n}(x))^{2n}(\mathcal{R}^{\dagger_1}_{2n+2}(x))^{2n+2}\right)}$$

$$= \left(\frac{x+1}{2}\right)^n \frac{(2n-1)!!}{2^n n!}$$

$$= \left(\frac{x+1}{2}\right)^n \frac{(2n-1)!!}{(2n)!!}$$

Using (3.7), (3.20) is obtained. □

Example 3.11. Calculate $\mathcal{V}_6(x)$ with Binet-like Formula-Form 2 for $n = 3$. Through (3.21) and (3.22),

$$\mathcal{R}_6^{\dagger_1}(x) = \frac{117649}{64}\left(x - \sqrt{x^2 + \frac{80\,(x+1)}{21}}\right)^6$$

$$\mathcal{R}_6^{\dagger_2}(x) = \frac{117649}{64}\left(x - \sqrt{x^2 + \frac{80\,(x+1)}{21}}\right)^6$$

$$\mathcal{C}_6^{\dagger_1} = \frac{1953125(x+1)^3\left(\frac{1679616\left(3x+\sqrt{9x^2+35x+35}\right)^8}{390625} - \frac{823543}{512}\left(x+\sqrt{x^2+\frac{80(x+1)}{21}}\right)^6\right)}{4608\left(\left(3x+\sqrt{9x^2+35x+35}\right)^8\left(-21x+\sqrt{21}\sqrt{21x^2+80x+80}\right)^6 - \left(-3x+\sqrt{9x^2+35x+35}\right)^8\left(21x+\sqrt{21}\sqrt{21x^2+80x+80}\right)^6\right)}$$

and

$$\mathcal{C}_6^{\dagger_2}(x) = \frac{1953125(x+1)^3\left(-\frac{1679616\left(-3x+\sqrt{9x^2+35x+35}\right)^8}{390625} + \frac{823543}{512}\left(x-\sqrt{x^2+\frac{80(x+1)}{21}}\right)^6\right)}{4608\left(\left(3x+\sqrt{9x^2+35x+35}\right)^8\left(-21x+\sqrt{21}\sqrt{21x^2+80x+80}\right)^6 - \left(-3x+\sqrt{9x^2+35x+35}\right)^8\left(21x+\sqrt{21}\sqrt{21x^2+80x+80}\right)^6\right)}$$

Then, $\mathcal{V}_6(x) = \mathcal{C}_6^{\dagger_1}(x)(\mathcal{R}_6^{\dagger_1}(x))^6 + \mathcal{C}_6^{\dagger_2}(x)(\mathcal{R}_6^{\dagger_2}(x))^6 = \frac{5(x+1)^3}{128}$. It can be checked via (3.2). It can also be observed that $\mathcal{R}(x)$ and $\mathcal{C}(x)$ values obtained in this example are different from $R(x)$ and $C(x)$ values found in Example 3.8.

Remark 3.12. The following hold for $\mathcal{R}_{2n}^{\dagger_1}(x)$ and $\mathcal{R}_{2n}^{\dagger_2}(x)$:

i. $\mathcal{R}_{2n}^{\dagger_1}(x) + \mathcal{R}_{2n}^{\dagger_2}(x) = \dfrac{\mathcal{L}(2n)x}{2}$

ii. $\mathcal{R}_{2n}^{\dagger_1}(x)\mathcal{R}_{2n}^{\dagger_2}(x) = -\dfrac{\mathcal{L}(2n)\mathcal{L}(2n-2)(x+1)}{4}$

Remark 3.13. By setting $x = 1$ in the previously obtained results, the concepts related to Vietoris' number sequence $\{v_s\}_{s\geq 0}$ can be observed.

It can be observed that Theorem 3.7 presents Binet-like formula based on the three consecutive-term recurrence relation (3.9). Theorem 3.10 adapts the recurrence relation (3.1) into (3.9) and also derives Binet-like formula again. This leads to two alternative expressions, referred to as Form 1 and Form 2, for the Binet-like formula.

3.1. Some Identities for Vietoris-like Polynomials

This subsection investigates several identities for Vietoris-like polynomial sequence $\{\mathcal{V}_s(x)\}_{s\geq 0}$.

Proposition 3.14. Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence. Then, the following properties hold:

i. $\mathcal{V}_{2n}(x) + \mathcal{V}_{2n-1}(x) = \dfrac{x+3}{2}\mathcal{L}(2n-2)\mathcal{V}_{2n-2}(x)$

ii. $\mathcal{V}_{2n}(x) - \mathcal{V}_{2n-1}(x) = \dfrac{x-1}{2}\mathcal{L}(2n-2)\mathcal{V}_{2n-2}(x)$

iii. $\mathcal{V}_{2n+1}(x) + \mathcal{V}_{2n-1}(x) = \left(\frac{x+1}{2}\mathcal{L}(2n) + 1\right)\mathcal{L}(2n-2)\mathcal{V}_{2n-2}(x)$

iv. $\mathcal{V}_{2n+1}(x) - \mathcal{V}_{2n-1}(x) = \left(\frac{x+1}{2}\mathcal{L}(2n) - 1\right)\mathcal{L}(2n-2)\mathcal{V}_{2n-2}(x)$

v. $\mathcal{V}_{2n}(x) + \mathcal{V}_{2n+2}(x) = \left(\frac{x+1}{2}\mathcal{L}(2n) + 1\right)\mathcal{V}_{2n}(x) = \left(\frac{\mathcal{V}_{2n+2}(x)}{\mathcal{V}_{2n}(x)} + 1\right)\mathcal{V}_{2n}(x)$

vi. $\mathcal{V}_{2n}(x)-\mathcal{V}_{2n+2}(x)=\left(\frac{x+1}{2}\mathcal{L}(2n)-1\right)\mathcal{V}_{2n}(x)=\left(\frac{\mathcal{V}_{2n+2}(x)}{\mathcal{V}_{2n}(x)}-1\right)\mathcal{V}_{2n}(x)$

Proof. Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence.

i. From (3.8) and (3.1), $\mathcal{V}_{2n}(x)=\frac{x+1}{2}\mathcal{L}(2n-2)\mathcal{V}_{2n-2}(x)$ and $\mathcal{V}_{2n-1}(x)=\mathcal{L}(2n-2)\mathcal{V}_{2n-2}(x)$, respectively. The proof is completed when these equations are added side by side.

ii. From (3.3) and (3.8), $\mathcal{V}_{2n}(x)+\mathcal{V}_{2n+2}(x)=\left(\frac{x+1}{2}\mathcal{L}(2n)+1\right)\mathcal{V}_{2n}(x)$. Since $\frac{\mathcal{V}_{2n+2}(x)}{\mathcal{V}_{2n}(x)}=\frac{x+1}{2}\mathcal{L}(2n)$, the desired result is obtained.

The other proofs are similar. □

Proposition 3.15 (Catalan-like Identity)**.** Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence. For $s>t$ and $\mathcal{K}=\mathcal{V}_{s+t}(x)\mathcal{V}_{s-t}(x)-(\mathcal{V}_s(x))^2$, the following relation is valid: For all $n\geq 1$ and $m>1$,

$$\mathcal{K}=\begin{cases}\left(\prod_{l=1}^{t/2}\left(\frac{x+1}{2}\right)^{t/2}\mathcal{L}(s+t-2l)-\prod_{l=1}^{t/2}\left(\frac{x+1}{2}\right)^{t/2}\mathcal{L}(s-2l)\right)\mathcal{V}_s(x)\mathcal{V}_{s-t}(x), & s=2n \text{ and } t=2m\\ \left(\mathcal{L}(s-t-1)\prod_{l=1}^{(t+1)/2}\left(\frac{x+1}{2}\right)^{(t-1)/2}\mathcal{L}(s+t+1-2l)-\prod_{l=1}^{(t+1)/2}\left(\frac{x+1}{2}\right)^{(t+1)/2}\mathcal{L}(s-2l)\right)\mathcal{V}_{s-t-1}(x)\mathcal{V}_s(x), & s=2n \text{ and } t=2m-1\\ \left(\prod_{l=1}^{t/2}\left(\frac{x+1}{2}\right)^{(t-4)/2}\mathcal{L}(s+t+1-2l)-\prod_{l=1}^{t/2}\left(\frac{x+1}{2}\right)^{(t-4)/2}\mathcal{L}(s+1-2l)\right)\mathcal{V}_{s+1}(x)\mathcal{V}_{s-t+1}(x), & s=2n-1 \text{ and } t=2m\\ \left(\prod_{l=1}^{\frac{t-1}{2}}\left(\frac{x+1}{2}\right)^{\frac{t-1}{2}}\mathcal{L}(s+t-2l)-\prod_{l=1}^{\frac{t+1}{2}}\left(\frac{x+1}{2}\right)^{\frac{t-3}{2}}\mathcal{L}(s+1-2l)\right)\mathcal{V}_{s+1}(x)\mathcal{V}_{s-t}(x), & s=2n-1 \text{ and } t=2m-1\end{cases}$$

Proof. Consider (3.6). For $s=2n$ and $t=2m$,

$$\begin{aligned}\mathcal{V}_{s+t}(x)\mathcal{V}_{s-t}(x)-(\mathcal{V}_s(x))^2 =&\mathcal{V}_{2n+2m}(x)\mathcal{V}_{2n-2m}(x)-(\mathcal{V}_{2n}(x))^2\\ =&\prod_{l=1}^{m}\left(\frac{x+1}{2}\right)^m\mathcal{L}(2n+2m-2l)\mathcal{V}_{2n}(x)\mathcal{V}_{2n-2m}(x)\\ &-\prod_{l=1}^{m}\left(\frac{x+1}{2}\right)^m\mathcal{L}(2n-2l)\mathcal{V}_{2n}(x)\mathcal{V}_{2n-2m}(x)\\ =&\left(\prod_{l=1}^{m}\left(\frac{x+1}{2}\right)^m\mathcal{L}(2n+2m-2l)-\prod_{l=1}^{m}\left(\frac{x+1}{2}\right)^m\mathcal{L}(2n-2l)\right)\mathcal{V}_{2n}(x)\mathcal{V}_{2n-2m}(x)\\ =&\left(\prod_{l=1}^{t/2}\left(\frac{x+1}{2}\right)^{t/2}\mathcal{L}(s+t-2l)-\prod_{l=1}^{t/2}\left(\frac{x+1}{2}\right)^{t/2}\mathcal{L}(s-2l)\right)\mathcal{V}_s(x)\mathcal{V}_{s-t}(x)\end{aligned}$$

For $s=2n$ and $t=2m-1$, using (3.1),

$$\begin{aligned}\mathcal{V}_{s+t}(x)\mathcal{V}_{s-t}(x)-(\mathcal{V}_s(x))^2 =&\mathcal{V}_{2n+2m-1}(x)\mathcal{V}_{2n-2m+1}(x)-(\mathcal{V}_{2n}(x))^2\\ =&\frac{2\mathcal{V}_{2n+2m}(x)}{x+1}\frac{2\mathcal{V}_{2n-2m+2}(x)}{x+1}-(\mathcal{V}_{2n}(x))^2\\ =&\prod_{l=1}^{m}\left(\frac{x+1}{2}\right)^{m-2}\mathcal{L}(2n+2m-2l)\mathcal{V}_{2n}(x)\mathcal{V}_{2n-2m+2}(x)\\ &-\mathcal{V}_{2n}(x)\prod_{l=1}^{m-1}\left(\frac{x+1}{2}\right)^{m-1}\mathcal{L}(2n-2l)\mathcal{V}_{2n-2m+2}(x)\\ =&\left(\prod_{l=1}^{(t+1)/2}\left(\frac{x+1}{2}\right)^{(t-3)/2}\mathcal{L}(s+t+1-2l)-\prod_{l=1}^{(t-1)/2}\left(\frac{x+1}{2}\right)^{(t-1)/2}\mathcal{L}(s-2l)\right)\mathcal{V}_{s-t+1}(x)\mathcal{V}_s(x)\end{aligned}$$

For $s = 2n-1$ and $t = 2m$,

$$\begin{aligned}
\mathcal{V}_{s+t}(x)\mathcal{V}_{s-t}(x) - (\mathcal{V}_n(x))^2 =& \mathcal{V}_{2n+2m-1}(x)\mathcal{V}_{2n-2m-1}(x) - (\mathcal{V}_{2n-1}(x))^2 \\
=& \frac{2\mathcal{V}_{2n+2m}(x)}{x+1}\frac{2\mathcal{V}_{2n-2m}(x)}{x+1} - \left(\frac{2\mathcal{V}_{2n}(x)}{x+1}\right)^2 \\
=& \left(\prod_{l=1}^{m}\left(\frac{x+1}{2}\right)^{m-2}\mathcal{L}(2n+2m-2l) - \prod_{l=1}^{m}\left(\frac{x+1}{2}\right)^{m-2}\mathcal{L}(2n-2l)\right)\mathcal{V}_{2n}(x)\mathcal{V}_{2n-2m}(x) \\
=& \left(\prod_{l=1}^{t/2}\left(\frac{x+1}{2}\right)^{(t-4)/2}\mathcal{L}(s+t+1-2l) - \prod_{l=1}^{t/2}\left(\frac{x+1}{2}\right)^{(t-4)/2}\mathcal{L}(s+1-2l)\right)\mathcal{V}_{s+1}(x)\mathcal{V}_{s-t+1}(x)
\end{aligned}$$

For $s = 2n-1$ and $t = 2m-1$,

$$\begin{aligned}
\mathcal{V}_{s+t}(x)\mathcal{V}_{s-t}(x) - (\mathcal{V}_s(x))^2 &= \mathcal{V}_{2n+2m-2}(x)\mathcal{V}_{2n-2m}(x) - (\mathcal{V}_{2n-1}(x))^2 \\
&= \mathcal{V}_{2n+2m-2}(x)\mathcal{V}_{2n-2m}(x) - \left(\frac{2\mathcal{V}_{2n}(x)}{x+1}\right)^2 \\
&= \left(\prod_{l=1}^{m-1}\left(\frac{x+1}{2}\right)^{m-1}\mathcal{L}(2n+2m-2-2l)\prod_{l=1}^{m}\left(\frac{x+1}{2}\right)^{m-2}\mathcal{L}(2n-2l)\right)\mathcal{V}_{2n}(x)\mathcal{V}_{2n-2m}(x) \\
&= \left(\prod_{l=1}^{\frac{t-1}{2}}\left(\frac{x+1}{2}\right)^{\frac{t-1}{2}}\mathcal{L}(s+t-2l) - \prod_{l=1}^{\frac{t+1}{2}}\left(\frac{x+1}{2}\right)^{\frac{t-3}{2}}\mathcal{L}(s+1-2l)\right)\mathcal{V}_{s+1}(x)\mathcal{V}_{s-t}(x)
\end{aligned}$$

□

The above proposition is also valid for $s > t > 2$.

Example 3.16. Considering (3.2), we compute $\mathcal{V}_{10}(x)\mathcal{V}_2(x) - \mathcal{V}_6(x)^2 = \frac{13(x+1)^6}{32768}$, where $s = 6$ and $t = 4$. Besides, using the above formula, we obtain the same results

$$\begin{aligned}
\mathcal{V}_{10}(x)\mathcal{V}_2(x) - \mathcal{V}_6(x)^2 =& \left(\prod_{l=1}^{2}\left(\tfrac{x+1}{2}\right)^2\mathcal{L}(10-2l) - \prod_{l=1}^{2}\left(\tfrac{x+1}{2}\right)^2\mathcal{L}(6-2l)\right)\mathcal{V}_6(x)\mathcal{V}_2(x) \\
=& \frac{13(x+1)^6}{32768}
\end{aligned}$$

Similarly, for $s = 6$ and $t = 3$,

$$\begin{aligned}
\mathcal{V}_9(x)\mathcal{V}_3(x) - \mathcal{V}_6(x)^2 =& \left(\mathcal{L}(2)\prod_{l=1}^{2}\left(\tfrac{x+1}{2}\right)\mathcal{L}(10-2l) - \prod_{l=1}^{2}\left(\tfrac{x+1}{2}\right)^2\mathcal{L}(6-2l)\right)\mathcal{V}_6(x)\mathcal{V}_2(x) \\
=& -\frac{(x+1)^5(-89+100x)}{65536}
\end{aligned}$$

For $s = 9$ and $t = 4$,

$$\begin{aligned}
\mathcal{V}_{13}(x)\mathcal{V}_5(x) - \mathcal{V}_9(x)^2 =& \left(\prod_{l=1}^{2}\mathcal{L}(14-2l) - \prod_{l=1}^{2}\mathcal{L}(10-2l)\right)\mathcal{V}_{10}(x)\mathcal{V}_6(x) \\
=& \frac{321(1+x)^8}{16777216}
\end{aligned}$$

For $s = 5$ and $t = 3$,

$$\begin{aligned}
\mathcal{V}_8(x)\mathcal{V}_2(x) - \mathcal{V}_5(x)^2 =& \left(\tfrac{x+1}{2}\mathcal{L}(6) - \prod_{l=1}^{2}\mathcal{L}(6-2l)\right)\mathcal{V}_6(x)\mathcal{V}_2(x) \\
=& \frac{5(x+1)^4(-3+7x)}{8192}
\end{aligned}$$

Proposition 3.17. For $n \geq 1$,

$$\mathcal{V}_{s+2}(x)\mathcal{V}_{s-2}(x) - (\mathcal{V}_s(x))^2 = \begin{cases} \frac{x+1}{2}\frac{2}{s(s+2)}\mathcal{V}_s(x)\mathcal{V}_{s-2}(x), & s = 2n \\ \frac{x+1}{2}\frac{2}{(s+1)(s+3)}\mathcal{V}_s(x)\mathcal{V}_{s-2}(x), & s = 2n-1 \end{cases}$$

Moreover, for $x = 1$, considering (3.1), we obtain the following result as in [18]:

$$v_{s+2}v_{s-2} - (v_s)^2 = \begin{cases} \frac{2}{s(s+2)}v_s v_{s-2}, & s = 2n \\ \frac{2}{(s+1)(s+3)}v_{s+1}v_{s-1}, & s = 2n-1 \end{cases}$$

Proposition 3.18 (Cassini-like Identity)**.** Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence. Then,

$$\mathcal{V}_{s+1}(x)\mathcal{V}_{s-1}(x) - (\mathcal{V}_s(x))^2 = \begin{cases} \frac{x+1}{2}\left(\mathcal{L}(s) - \frac{x+1}{2}\right)(\mathcal{V}_{s-1}(x))^2, & s = 2n \\ \mathcal{L}(s-1)\left(\frac{x+1}{2} - \mathcal{L}(s-1)\right)(\mathcal{V}_{s-1}(x))^2, & s = 2n-1 \end{cases}$$

where $n \geq 1$.

Proof. Consider (3.12). For $s = 2n$,

$$\begin{aligned} \mathcal{V}_{s+1}(x)\mathcal{V}_{s-1}(x) - (\mathcal{V}_s(x))^2 &= \mathcal{V}_{2n+1}(x)\mathcal{V}_{2n-1}(x) - (\mathcal{V}_{2n}(x))^2 \\ &= \frac{x+1}{2}\mathcal{L}(2n)\mathcal{V}_{2n-1}(x)\mathcal{V}_{2n-1}(x) - \left(\frac{x+1}{2}\mathcal{V}_{2n-1}\right)^2 \\ &= \frac{x+1}{2}\left(\mathcal{L}(2n) - \frac{x+1}{2}\right)(\mathcal{V}_{2n-1}(x))^2 \\ &= \frac{x+1}{2}\left(\mathcal{L}(s) - \frac{x+1}{2}\right)(\mathcal{V}_{s-1}(x))^2 \end{aligned}$$

For $s = 2n-1$,

$$\begin{aligned} \mathcal{V}_{s+1}(x)\mathcal{V}_{s-1}(x) - (\mathcal{V}_s(x))^2 &= \mathcal{V}_{2n}(x)\mathcal{V}_{2n-2}(x) - (\mathcal{V}_{2n-1}(x))^2 \\ &= \frac{x+1}{2}\mathcal{L}(2n-2)\mathcal{V}_{2n-2}(x)\mathcal{V}_{2n-2}(x) - \mathcal{L}^2(2n-2)(\mathcal{V}_{2n-2}(x))^2 \\ &= \mathcal{L}(2n-2)\left(\frac{x+1}{2} - \mathcal{L}(2n-2)\right)(\mathcal{V}_{2n-2}(x))^2 \\ &= \mathcal{L}(s-1)\left(\frac{x+1}{2} - \mathcal{L}(s-1)\right)(\mathcal{V}_{s-1}(x))^2 \end{aligned}$$

□

Proposition 3.19 (d'Ocagne-like Identity)**.** Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence. Then,

$$\mathcal{V}_s(x)\mathcal{V}_{t+1}(x) - \mathcal{V}_{s+1}(x)\mathcal{V}_t(x) = \begin{cases} (\mathcal{L}(t) - \mathcal{L}(s))\,\mathcal{V}_s(x)\mathcal{V}_t(x), & s = 2n \text{ and } t = 2m \\ \left(1 - \frac{2\mathcal{L}(s)}{x+1}\right)\mathcal{V}_s(x)\mathcal{V}_{t+1}(x), & s = 2n \text{ and } t = 2m-1 \\ \left(\frac{2\mathcal{L}(t)}{x+1} - 1\right)\mathcal{V}_{s+1}(x)\mathcal{V}_t(x), & s = 2n-1 \text{ and } t = 2m \\ 0, & s = 2n-1 \text{ and } t = 2m-1 \end{cases} \tag{3.23}$$

where $n, m \geq 1$.

Proof. Consider (3.3). For $s = 2n$ and $t = 2m$,

$$\begin{aligned}\mathcal{V}_s(x)\mathcal{V}_{t+1}(x) - \mathcal{V}_{s+1}(x)\mathcal{V}_t(x) &= \mathcal{V}_{2n}(x)\mathcal{V}_{2m+1}(x) - \mathcal{V}_{2n+1}(x)\mathcal{V}_{2m}(x)\\ &= \mathcal{V}_{2n}(x)\mathcal{L}(2m)\mathcal{V}_{2m}(x) - \mathcal{L}(2n)\mathcal{V}_{2n}(x)\mathcal{V}_{2m}(x)\\ &= \left(\mathcal{L}(2m) - \mathcal{L}(2n)\right)\mathcal{V}_{2n}(x)\mathcal{V}_{2m}(x)\\ &= \left(\mathcal{L}(t) - \mathcal{L}(s)\right)\mathcal{V}_s(x)\mathcal{V}_t(x)\end{aligned}$$

Additionally, by (2.2),

$$\mathcal{V}_s(x)\mathcal{V}_{t+1}(x) - \mathcal{V}_{s+1}(x)\mathcal{V}_t(x) = \left(\frac{t-s}{(s+2)(t+2)}\right)\mathcal{V}_s(x)\mathcal{V}_t(x)$$

Then, for $s = 2n$ and $t = 2m-1$,

$$\begin{aligned}\mathcal{V}_s(x)\mathcal{V}_{t+1}(x) - \mathcal{V}_{s+1}(x)\mathcal{V}_t(x) &= \mathcal{V}_{2n}(x)\mathcal{V}_{2m}(x) - \mathcal{V}_{2n+1}(x)\mathcal{V}_{2m-1}(x)\\ &= \mathcal{V}_{2n}(x)\mathcal{V}_{2m}(x) - \mathcal{L}(2n)\mathcal{V}_{2n}(x)\frac{2\mathcal{V}_{2m}(x)}{x+1}\\ &= \left(1 - \frac{2\mathcal{L}(2n)}{x+1}\right)\mathcal{V}_{2n}(x)\mathcal{V}_{2m}(x)\\ &= \left(1 - \frac{2\mathcal{L}(s)}{x+1}\right)\mathcal{V}_s(x)\mathcal{V}_{t+1}(x)\end{aligned}$$

For $s = 2n-1$ and $t = 2m$,

$$\begin{aligned}\mathcal{V}_s(x)\mathcal{V}_{t+1}(x) - \mathcal{V}_{s+1}(x)\mathcal{V}_t(x) &= \mathcal{V}_{2n-1}(x)\mathcal{V}_{2m+1}(x) - \mathcal{V}_{2n}(x)\mathcal{V}_{2m}(x)\\ &= \frac{2\mathcal{V}_{2n}(x)}{x+1}\mathcal{L}(2m)\mathcal{V}_{2m}(x) - \mathcal{V}_{2n}(x)\mathcal{V}_{2m}(x)\\ &= \left(\frac{2\mathcal{L}(2m)}{x+1} - 1\right)\mathcal{V}_{2n}(x)\mathcal{V}_{2m}(x)\\ &= \left(\frac{2\mathcal{L}(t)}{x+1} - 1\right)\mathcal{V}_{s+1}(x)\mathcal{V}_t(x)\end{aligned}$$

For $s = 2n-1$ and $t = 2m-1$,

$$\begin{aligned}\mathcal{V}_s(x)\mathcal{V}_{t+1}(x) - \mathcal{V}_{s+1}(x)\mathcal{V}_t(x) &= \mathcal{V}_{2n-1}(x)\mathcal{V}_{2m}(x) - \mathcal{V}_{2n}(x)\mathcal{V}_{2m-1}(x)\\ &= \frac{2\mathcal{V}_{2n}(x)}{x+1}\mathcal{V}_{2m}(x) - \mathcal{V}_{2n}(x)\frac{2\mathcal{V}_{2m}(x)}{x+1}\\ &= 0\end{aligned}$$

□

Remark 3.20. For $x = 1$, (3.23) becomes the following formula as in [18]:

$$v_s v_{t+1} - v_{s+1}v_t = \begin{cases} \dfrac{t-s}{(s+2)(t+2)}v_s v_t, & s = 2n \text{ and } t = 2m\\ \dfrac{1}{s+2}v_s v_{t+1}, & s = 2n \text{ and } t = 2m-1\\ -\dfrac{1}{t+2}v_{s+1}v_t, & s = 2n-1 \text{ and } t = 2m\\ 0, & s = 2n-1 \text{ and } t = 2m-1\end{cases}$$

Proposition 3.21. Let $\{\mathcal{V}_s(x)\}_{s\geq 0}$ be Vietoris-like polynomial sequence and $n \geq 2$. Then,

$$\mathcal{V}_{2n+2}(x) = \left(\frac{x+1}{2}\right)^2 \frac{2n-1}{2^2(n+1)}\mathcal{V}_{2n-2}(x) + \left(\frac{x+1}{4}\right)\left(\frac{1}{2}\mathcal{V}_{2n}(x) + \frac{x+1}{4}\mathcal{V}_{2n-1}(x)\right) \tag{3.24}$$

Proof. From (3.4) and the Pascal's identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$,

$$\begin{aligned}
\mathcal{V}_{2n+2}(x) &= \left(\frac{x+1}{2}\right)^{n+1} \frac{1}{2^{2n+2}}\binom{2n+2}{n+1} \\
&= \left(\frac{x+1}{2}\right)^{n+1} \frac{1}{2^{2n+2}}\left(\binom{2n+1}{n} + \binom{2n+1}{n+1}\right) \\
&= \left(\frac{x+1}{2}\right)^{n+1} \frac{1}{2^{2n+2}}\left(\binom{2n}{n-1} + \binom{2n}{n} + \binom{2n}{n} + \binom{2n}{n+1}\right) \\
&= \left(\frac{x+1}{2}\right)^{n+1} \left(\frac{1}{2^{2n+2}}\left(\binom{2n}{n-1} + \binom{2n}{n+1}\right) + \frac{1}{2^{2n+1}}\binom{2n}{n}\right)
\end{aligned}$$

Using $\binom{n}{k} = \binom{n}{n-k}$,

$$\begin{aligned}
\mathcal{V}_{2n+2}(x) &= \left(\frac{x+1}{2}\right)^{n+1} \left(\frac{1}{2^{2n+2}}\left(\binom{2n}{n-1} + \binom{2n}{2n-n-1}\right) + \frac{1}{2^{2n+1}}\binom{2n}{n}\right) \\
&= \left(\frac{x+1}{2}\right)^{n+1} \left(\frac{1}{2^{2n+1}}\binom{2n}{n-1} + \frac{1}{2^{2n+1}}\binom{2n}{n}\right)
\end{aligned}$$

Using $\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}$ and (3.4),

$$\begin{aligned}
\mathcal{V}_{2n+2}(x) &= \left(\frac{x+1}{2}\right)^{n+1} \frac{1}{2^{2n+1}}\frac{2n}{(n+1)}\binom{2n-1}{n-1} + \left(\frac{x+1}{4}\right)\left(\frac{x+1}{2}\right)^{n}\frac{1}{2^{2n}}\binom{2n}{n} \\
&= \left(\frac{x+1}{2}\right)^{2} \frac{2n-1}{2^2(n+1)}\left(\frac{x+1}{2}\right)^{n-1}\frac{1}{2^{2n-2}}\binom{2n-2}{n-1} + \left(\frac{x+1}{4}\right)\mathcal{V}_{2n}(x) \\
&= \left(\frac{x+1}{2}\right)^{2} \frac{2n-1}{2^2(n+1)}\mathcal{V}_{2n-2}(x) + \left(\frac{x+1}{4}\right)\mathcal{V}_{2n}(x)
\end{aligned}$$

From (3.1) and (3.3),

$$\mathcal{V}_{2n+2}(x) = \left(\frac{x+1}{2}\right)^2 \frac{2n-1}{2^2(n+1)}\mathcal{V}_{2n-2}(x) + \left(\frac{x+1}{4}\right)\left(\frac{1}{2}\mathcal{V}_{2n}(x) + \frac{x+1}{4}\mathcal{V}_{2n-1}(x)\right)$$

□

Remark 3.22. For $x = 1$, (3.24) becomes the following equality as in [18]:

$$v_{2n+2} = \left(\frac{1}{4}v_{2n} + \frac{1}{4}v_{2n-1}\right) + \frac{2n-1}{4(n+1)}v_{2n-2}, \quad n \geq 2$$

4. Conclusion

Many researchers have studied number sequences and their properties, which play an essential role in mathematics. Hence, the polynomial forms of these number sequences for any variable quantity x have also become an area of significant interest. The Fibonacci polynomials were among the first polynomial forms considered. Since Fibonacci-type polynomials have significant applications in geometry and algebra, various researchers have extensively studied them in number theory. In this paper, we provided an affirmative answer to a question related to the existence of special Vietoris-like polynomials by using the properties of Vietoris' numbers. Hence, we derived special Vietoris-like polynomials and investigated their basic properties, recurrence relations, and special equalities. We also constructed an analogy with the studies [10–12, 15–18] using Vietoris-like polynomial approach and established some conditions for obtaining interesting results inspired by studies [2–9]. We determined Catalan-like, Cassini-like and d'Ocagne-like identities. We also presented their special cases corresponding to the existing identities in Vietoris' number sequence. We believe that the calculations of this work contribute to the broader understanding of polynomial structures and their connections with well-known number sequences and enable new studies. Specifically, the results of Vietoris-like polynomials and the properties of Vietoris' hybrid numbers (for more details on hybrid numbers, see [21]) of the form $\mathcal{VH}s = v_s + v_{s+1}i + \varepsilon v_{s+2} + hv_{s+3}$ where $i^2 = -1$, $\varepsilon^2 = 0$, $h^2 = 1$, and $ih = -hi = \varepsilon + i$ [22], Vietoris-like hybrid binomial sequence and its remarkable features represent key areas for future research.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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