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Research Article

A Study of Caputo Sequential Fractional Differential Equations with Mixed Boundary Conditions

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Abstract

In this paper, we investigate the existence of solutions for a sequential fractional differential equation involving Caputo-type derivative subject to mixed boundary conditions. The core results are derived by employing Krasnoselskii's fixed point theorem and the Leray-Schauder fixed point theorem. We end this study by two illustrative numerical examples, which validate the applicability of our obtained results.

1. Introduction

Fractional calculus has emerged as a critical field in mathematics, generalizing traditional differentiation and integration to non-integer orders. This extension provides a powerful framework for modeling complex phenomena and systems across diverse disciplines such as physics, engineering, biology, engineering, mechanics, economics, and other fields [1–5].
Boundary value problems (BVPs) for fractional differential equations (FDEs) that emerge and describe linear and nonlinear phenomena have obtained much attention in the scientific community and specially in engineering. Recently, there are several researchers [6–10] that have used FDEs to model natural phenomena. In this regard, due to this exponential growth of the fractional calculus added to differential equations, many researchers have focused their attention on the investigation of existence, uniqueness and stability of solutions of FDEs under different types of boundary conditions by using a set of fixed point theories, such as Banach's, the Leray- Schauder alternative, Darbo's theorem and Mönch's fixed point theorem [11–15] and the references therein. Sequential FDEs have also received considerable attention for instance see [16–22].
It is worth mentioning that Mahmudov et al. [23] establish the existence of solutions for the following nonlinear sequential fractional differential equation subject to nonlocal fractional integral conditions:

$$\begin{cases} \left({}^{C}D^{\nu}+\omega^{C}D^{\nu-1}\right)\kappa(\tau)=f\left(\tau,\kappa(\tau),{}^{C}D^{\nu-1}\kappa(\tau)\right),\ 1<\nu<2, 0\leq\tau\leq T,\\ \alpha_1\kappa(\eta)+\beta_1\kappa(T)=\gamma_1\int_0^{\xi}\kappa(s)\,ds+\varepsilon_1,\\ \alpha_2{}^{C}D^{\nu-1}\kappa(\eta)+\beta_2{}^{C}D^{\nu-1}\kappa(T)=\gamma_2\int_{\zeta}^{\xi}\kappa(s)\,ds+\varepsilon_2,\end{cases}$$

where ${}^{C}D^{\nu}$ is the standard Caputo fractional derivative of order ν, $0\leq\eta\leq T$, $0<\xi<\zeta<T$, $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,\varepsilon_1,\varepsilon_2\in\mathbb{R}$.
In [24], Awadalla et al. studied the following nonlinear sequential FDE to nonseparated nonlocal integral fractional boundary conditions:

$$\begin{cases} \left({}^{C}D^{\nu}+\omega^{C}D^{\nu-1}\right)\kappa(\tau)=f(\tau,\kappa(\tau)),\ 1\leq\nu\leq2, 0\leq\tau\leq T,\\ \omega_1\kappa(\sigma)+\rho_1\kappa(T)=\upsilon_1\int_0^{\eta}\kappa(\zeta)\,d\zeta,\\ \omega_2{}^{C}D^{\nu-1}\kappa(\sigma)+\rho_2{}^{C}D^{\nu-1}\kappa(T)=\upsilon_2\int_{\varsigma}^{T}\kappa(\zeta)\,d\zeta,\end{cases}$$

where $0\leq\sigma\leq T$, $0<\eta<\varsigma<T$, $\omega\in\mathbb{R}_+$, $\omega_1,\omega_2,\rho_1,\rho_2,\upsilon_1,\upsilon_2\in\mathbb{R}$.

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In [25], Yan investigated the existence and uniqueness of solutions to the boundary value problem of a nonlinear FDEs:

$$\begin{cases} {}^{C}D^{\nu}\mu(\tau)+{}^{C}D^{\nu-1}[p(\tau)\mu(\tau)]=h(\tau,\mu(\tau)), 0<\tau<1, \\ \mu(0)=\mu'(0)=\mu'(1)=0, \end{cases}$$

where $2<\mu\leq 3$, $h\in C[0,1]$ and $p\in C^{3}([0,1],\mathbb{R})$.

Inspired by the works mentioned above, we investigate the existence results for a sequential FDEs of the form

$$\begin{cases} ({}^{c}D^{\nu+1}+\omega{}^{c}D^{\nu})\kappa(\tau)=f(\tau,\kappa(\tau),{}^{c}D^{\nu-1}\kappa(\tau)), \quad 1<\nu\leq 2, \omega>0, \tau\in[0,1], \\ \kappa(0)+\beta\kappa(1)=I^{\nu-1}\kappa(\mu)+I^{\nu}\kappa(\mu), \quad 0<\mu<1, \\ \kappa'(0)+\gamma\kappa'(1)={}^{c}D^{\nu-1}\kappa(\mu)+{}^{c}D^{\nu}\kappa(\mu), \quad \beta,\gamma\in\mathbb{R}, \\ \kappa''(0)=0. \end{cases} \tag{1.1}$$

Here ${}^{c}D^{\nu+1}$, ${}^{c}D^{\nu}$, ${}^{c}D^{\nu-1}$ are the Caputo fractional derivatives of order $\nu+1$, ν, and $\nu-1$ respectively, $f:[0,1]\times\mathbb{R}^{2}\longrightarrow\mathbb{R}$ is a continuous function, and $1+\gamma-\frac{\mu^{2-\nu}}{\Gamma(3-\nu)}\neq 0$, $1+\beta-\frac{\nu\mu^{\nu-1}+\mu^{\nu}}{\Gamma(\nu+1)}\neq 0$.

The rest of this paper is organised as follows. Section 2 presents definitions and preliminary concepts. Section 3 investigates the existence of solutions for the problem (1.1) by using Krasnoselskii's fixed point theorem and the Leray-Schauder fixed point theorem. Section 4 gives examples, and conclusion section is deticated to summarizing our obtained results.

2. Preliminaries

In this section, it is essential to present some basic concepts and important lemmas. For more details, the interested readers can consult [3].

Definition 2.1. *The Riemann-Liouville fractional integral of order $\nu>0$ for a function $f:(0,+\infty)\to\mathbb{R}$ is defined as*

$$I^{\nu}f(\tau)=\frac{1}{\Gamma(\nu)}\int_{0}^{\tau}(\tau-s)^{\nu-1}f(s)ds,$$

provided the right side is pointwise defined on $(0,+\infty)$ where $\Gamma(.)$ is the Gamma function.

Definition 2.2. *Let a function $f:[0,+\infty)\to\mathbb{R}$, the Caputo derivative of fractional order $\nu>0$ is defined as*

$${}^{c}D^{\nu}f(\tau)=\frac{1}{\Gamma(n-\nu)}\int_{0}^{\tau}(\tau-s)^{n-\nu-1}f^{(n)}(s)ds,\ n=[\nu]+1,$$

where $[\nu]$ denotes the integer part of the real number ν, provided the right side is pointwise defined on $(0,+\infty)$.

Lemma 2.3. *Let $\nu>0$ and $f\in AC^{N}[0,1]$. Then the equation*

$${}^{c}D^{\nu}f(\tau)=0,$$

has a unique solution

$$f(\tau)=\sum_{i=0}^{N-1}a_{i}\tau^{i},$$

and

$$I^{\nu c}D^{\nu}f(\tau)=f(\tau)+\sum_{i=0}^{N-1}a_{i}\tau^{i}$$

for some $a_{i}\in\mathbb{R}$, $i=0,1,2,...,N-1$, $N=[\nu]+1$.

Lemma 2.4. *Let $\nu>\sigma>0$ and $f\in L^{p}(0,1)\subset L^{1}(0,1), 0\leq p\leq+\infty$. Then the next formulas hold.*

(i) $({}^{c}D^{\sigma}I^{\nu}f)(\tau)=I^{\nu-\sigma}f(\tau)$,
(ii) $({}^{c}D^{\nu}I^{\nu}f)(\tau)=f(\tau)$.

Definition 2.5. *([26]) The sequential fractional derivative for a function f can be written as*

$$D^{\nu}f(\tau)=D^{\nu_{1}}D^{\nu_{2}}...D^{\nu_{m}}f(\tau),$$

where $\nu=(\nu_{1},\nu_{2},...,\nu_{m})$ is a multi-index.

Lemma 2.6. *For a given $\xi\in C([0,1],\mathbb{R})$, the unique solution of the problem*

$$\begin{cases} ({}^{c}D^{\nu+1}+\omega{}^{c}D^{\nu})\kappa(\tau)=\xi(\tau), \quad \tau\in[0,1], \\ \kappa(0)+\beta\kappa(1)=I^{\nu-1}\kappa(\mu)+I^{\nu}\kappa(\mu), \\ \kappa'(0)+\gamma\kappa'(1)={}^{c}D^{\nu-1}\kappa(\mu)+{}^{c}D^{\nu}\kappa(\mu), \\ \kappa''(0)=0, \end{cases}$$

is expressed as

$$\begin{aligned}\kappa(\tau)=&\int_0^\tau e^{-\omega(\tau-\sigma)}I^\nu\xi(\sigma)d\sigma+\frac{\tau}{\Theta_1}\left[(\omega^2-\omega)\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\times\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu\xi(\vartheta)d\vartheta\right)d\sigma\right.\\&\left.+\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu\xi(\sigma)+I^{\nu-1}\xi(\sigma)]d\sigma+\gamma\omega\int_0^1 e^{-\omega(1-\sigma)}I^\nu\xi(\sigma)d\sigma-\gamma I^\nu\xi(1)\right]\\&+\frac{1}{\Theta_2}\int_0^\mu\left[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right]\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu\xi(\vartheta)d\vartheta\right)d\sigma\\&+\frac{\Theta_3}{\Theta_1\Theta_2}\left[(\omega^2-\omega)\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu\xi(\vartheta)d\vartheta\right)d\sigma\right.\\&\left.+\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu\xi(\sigma)+I^{\nu-1}\xi(\sigma)]d\sigma+\gamma\omega\int_0^1 e^{-\omega(1-\sigma)}I^\nu\xi(\sigma)d\sigma-\gamma I^\nu\xi(1)\right],\end{aligned}\tag{2.1}$$

where

$$\Theta_1=1+\gamma-\frac{\mu^{2-\nu}}{\Gamma(3-\nu)},\quad \Theta_2=1+\beta-\frac{\nu\mu^{\nu-1}+\mu^\nu}{\Gamma(\nu+1)},\quad \Theta_3=\frac{\mu^\nu(\mu+\nu+1)}{\Gamma(\nu+2)}-\beta.\tag{2.2}$$

Proof. By Lemma 2.3, we find

$$(D+\omega)\kappa(\tau)=I^\nu\xi(\tau)+a_0+a_1\tau,\tag{2.3}$$

where $a_0,a_1\in\mathbb{R}$. Then, (2.3) is equivalent to

$$D(e^{\omega\tau}\kappa(\tau))=e^{\omega\tau}(I^\nu\xi(\tau)+a_0+a_1\tau),$$

and integrating this expression from 0 to τ, we have

$$e^{\omega\tau}\kappa(\tau)=\int_0^\tau e^{\omega\sigma}I^\nu\xi(\sigma)d\sigma+\left(\frac{a_0}{\omega}-\frac{a_1}{\omega^2}\right)e^{\omega\tau}+\frac{a_1\tau}{\omega}e^{\omega\tau}+\left(\frac{a_1}{\omega^2}-\frac{a_0}{\omega}+\kappa(0)\right).$$

Therefore we deduce that

$$\kappa(\tau)=\mathscr{A}+\mathscr{B}\tau+\mathscr{C}e^{-\omega\tau}+\int_0^\tau e^{-\omega(\tau-\sigma)}I^\nu\xi(\sigma)d\sigma,\tag{2.4}$$

where $\mathscr{A}=\frac{a_0}{\omega}-\frac{a_1}{\omega^2}$, $\mathscr{B}=\frac{a_1}{\omega}$ and $\mathscr{C}=\frac{a_1}{\omega^2}-\frac{a_0}{\omega}+\kappa(0)$.
Then, the second derivative of function κ with respect to τ is given by

$$\kappa''(\tau)=\mathscr{C}\omega^2e^{-\omega\tau}+\omega^2\int_0^\tau e^{-\omega(\tau-\sigma)}I^\nu\xi(\sigma)d\sigma-\omega I^\nu\xi(\tau)+I^{\nu-1}\xi(\tau).$$

By condition $\kappa''(0)=0$, we have $\mathscr{C}=0$.
From (2.4), we get

$${}^cD^{\nu-1}\kappa(\mu)+{}^cD^\nu\kappa(\mu)=\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left[(\omega^2-\omega)\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu\xi(\vartheta)d\vartheta+(1-\omega)I^\nu\xi(\sigma)+I^{\nu-1}\xi(\sigma)+\mathscr{B}\right]d\sigma,$$

and

$$\begin{aligned}I^{\nu-1}\kappa(\mu)+I^\nu\kappa(\mu)=&\mathscr{A}\int_0^\mu\left(\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right)d\sigma+\mathscr{B}\int_0^\mu\left(\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right)\sigma d\sigma\\&+\int_0^\mu\left(\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right)\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu\xi(\vartheta)d\vartheta\right)d\sigma.\end{aligned}$$

The condition $\kappa(0)+\beta\kappa(1)=I^{\nu-1}\kappa(\mu)+I^\nu\kappa(\mu)$ gives

$$\begin{aligned}&\mathscr{A}\left(1+\beta-\int_0^\mu\left(\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right)d\sigma\right)+\mathscr{B}\left(\beta-\int_0^\mu\left(\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right)\sigma d\sigma\right)\\&=\int_0^\mu\left(\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right)\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu\xi(\vartheta)d\vartheta\right)d\sigma,\end{aligned}\tag{2.5}$$

and the condition $\kappa'(0)+\gamma\kappa'(1)={}^cD^{\nu-1}\kappa(\mu)+{}^cD^\nu\kappa(\mu)$ gives

$$\begin{aligned}\mathscr{B}\left(1+\gamma-\frac{\mu^{2-\nu}}{\Gamma(3-\nu)}\right)=&(\omega^2-\omega)\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu\xi(\vartheta)d\vartheta\right)d\sigma\\&+\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu\xi(\sigma)+I^{\nu-1}\xi(\sigma)]d\sigma+\gamma\omega\int_0^1 e^{-\omega(1-\sigma)}I^\nu\xi(\sigma)d\sigma-\gamma I^\nu\xi(1).\end{aligned}\tag{2.6}$$

A simultaneous solution of (2.5) and (2.6) yields to

$$\begin{aligned}\mathscr{A} &= \frac{1}{\Theta_2}\int_0^{\mu}\left[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right]\left(\int_0^{\sigma} e^{-\omega(\sigma-\vartheta)}I^{\nu}\xi(\vartheta)d\vartheta\right)d\sigma \\ &+\frac{\Theta_3}{\Theta_1\Theta_2}\left[(\omega^2-\omega)\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^{\sigma} e^{-\omega(\sigma-\vartheta)}I^{\nu}\xi(\vartheta)d\vartheta\right)d\sigma\right. \\ &\left.+\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^{\nu}\xi(\sigma)+I^{\nu-1}\xi(\sigma)]d\sigma+\gamma\omega\int_0^1 e^{-\omega(1-\sigma)}I^{\nu}\xi(\sigma)d\sigma-\gamma I^{\nu}\xi(1)\right],\end{aligned}$$

$$\begin{aligned}\mathscr{B} &= \frac{1}{\Theta_1}\left[(\omega^2-\omega)\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^{\sigma} e^{-\omega(\sigma-\vartheta)}I^{\nu}\xi(\vartheta)d\vartheta\right)d\sigma\right. \\ &\left.+\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^{\nu}\xi(\sigma)+I^{\nu-1}\xi(\sigma)]d\sigma+\gamma\omega\int_0^1 e^{-\omega(1-\sigma)}I^{\nu}\xi(\sigma)d\sigma-\gamma I^{\nu}\xi(1)\right].\end{aligned}$$

Inserting the values of $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ into (2.4), we get (2.1). □

Lemma 2.7. *For $\xi\in C([0,1],\mathbb{R})$ with $\|\xi\|=\sup_{\kappa\in[0,1]}|\xi(\tau)|$, we have*

i) $|I^{\nu}\xi(\tau)|\le\frac{1}{\Gamma(\nu+1)}\|\xi\|$.

ii) $\left|\int_0^{\tau} e^{-\omega(\tau-\sigma)}I^{\nu}\xi(\sigma)d\sigma\right|\le\frac{1-e^{-\omega}}{\omega\Gamma(\nu+1)}\|\xi\|$.

iii) $\left|\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^{\sigma} e^{-\omega(\sigma-\vartheta)}I^{\nu}\xi(\vartheta)d\vartheta\right)d\sigma\right|\le\frac{\mu(\omega\mu+e^{-\omega\mu}-1)}{\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}\|\xi\|$.

iv) $\left|\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^{\nu}\xi(\sigma)+I^{\nu-1}\xi(\sigma)]d\sigma\right|\le\frac{|1-\omega|\mu^2+\nu\mu}{\Gamma(\nu+1)\Gamma(3-\nu)}\|\xi\|$.

v) $\left|\int_0^{\mu}\left[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right]\left(\int_0^{\sigma} e^{-\omega(\sigma-\vartheta)}I^{\nu}\xi(\vartheta)d\vartheta\right)d\sigma\right|\le\frac{(1-e^{-\omega\mu})(\nu\mu^{2\nu-1}+\mu^{2\nu})}{\omega[\Gamma(\nu+1)]^2}\|\xi\|$,

where $I^{\nu}\xi(\tau)=\int_0^{\tau}\frac{(\tau-\eta)^{\nu-1}}{\Gamma(\nu)}\xi(\eta)d\eta,\quad I^{\nu-1}\xi(\tau)=\int_0^{\tau}\frac{(\tau-\eta)^{\nu-2}}{\Gamma(\nu-1)}\xi(\eta)d\eta$.

Proof. For $\xi\in C([0,1],\mathbb{R})$ with $\|\xi\|=\sup_{\kappa\in[0,1]}|\xi(\tau)|$, we have

i)

$$\begin{aligned}|I^{\nu}\xi(\tau)| &= \left|\int_0^{\tau}\frac{(\tau-\sigma)^{\nu-1}}{\Gamma(\nu)}\xi(\sigma)d\sigma\right| \\ &\le\int_0^{\tau}\frac{(\tau-\sigma)^{\nu-1}}{\Gamma(\nu)}|\xi(\sigma)|d\sigma \\ &\le\frac{\tau^{\nu}}{\Gamma(\nu+1)}\|\xi\| \\ &\le\frac{1}{\Gamma(\nu+1)}\|\xi\|.\end{aligned}$$

ii)

$$\begin{aligned}\left|\int_0^{\tau} e^{-\omega(\tau-\sigma)}I^{\nu}\xi(\sigma)d\sigma\right| &\le\int_0^{\tau} e^{-\omega(\tau-\sigma)}\left|I^{\nu}\xi(\sigma)\right|d\sigma \\ &\le\frac{\|\xi\|}{\Gamma(\nu+1)}\int_0^{\tau} e^{-\omega(\tau-\sigma)}d\sigma \\ &\le\frac{1-e^{-\omega}}{\omega\Gamma(\nu+1)}\|\xi\|.\end{aligned}$$

iii)

$$\int_0^{\vartheta}\frac{(\vartheta-\eta)^{\nu-1}}{\Gamma(\nu)}d\eta=\frac{\vartheta^{\nu}}{\Gamma(\nu+1)}$$

and

$$\int_0^{\sigma} e^{-\omega(\sigma-\vartheta)}\frac{\vartheta^{\nu}}{\Gamma(\nu+1)}d\vartheta\le\frac{\sigma^{\nu}}{\Gamma(\nu+1)}\int_0^{\sigma} e^{-\omega(\sigma-\vartheta)}d\vartheta=\frac{\sigma^{\nu}(1-e^{-\omega\sigma})}{\omega\Gamma(\nu+1)}.$$

Hence

$$\left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu\xi(\vartheta)d\vartheta\right)d\sigma\right|$$
$$=\left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}\left(\int_0^\vartheta \frac{(\vartheta-\eta)^{\nu-1}}{\Gamma(\nu)}\xi(\eta)d\eta\right)d\vartheta\right)d\sigma\right|$$
$$\leq \|\xi\|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\frac{\sigma^\nu(1-e^{-\omega\sigma})}{\omega\Gamma(\nu+1)}d\sigma$$
$$\leq \|\xi\|\frac{\mu^{1-\nu}}{\Gamma(2-\nu)}\frac{\mu^\nu}{\omega\Gamma(\nu+1)}\int_0^\mu(1-e^{-\omega\sigma})d\sigma$$
$$\leq \frac{\mu(\omega\mu+e^{-\omega\mu}-1)}{\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}\|\xi\|.$$

iv)

$$\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}d\sigma \leq \int_0^\mu \frac{\mu^{1-\nu}}{\Gamma(2-\nu)}d\sigma,$$

and

$$\left|I^{\nu-1}\xi(\sigma)\right|=\left|\int_0^\sigma \frac{(\sigma-\eta)^{\nu-2}}{\Gamma(\nu-1)}\xi(\eta d\eta\right|\leq \frac{\sigma^\nu}{\Gamma(\nu+1)}\|\xi\|.$$

Hence

$$\left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu\xi(\sigma)+I^{\nu-1}\xi(\sigma)]d\sigma\right|\leq \int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left[|1-\omega|\left|I^\nu\xi(\sigma)\right|+\left|I^{\nu-1}\xi(\sigma)\right|\right]d\sigma$$
$$\leq \int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left[\frac{|1-\omega|\sigma^\nu+\nu\sigma^{\nu-1}}{\Gamma(\nu+1)}\right]\|\xi\|d\sigma$$
$$\leq \frac{|1-\omega|\mu^\nu+\nu\mu^{\nu-1}}{\Gamma(\nu+1)}\|\xi\|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}d\sigma$$
$$\leq \frac{|1-\omega|\mu^2+\nu\mu}{\Gamma(\nu+1)\Gamma(3-\nu)}\|\xi\|.$$

v)

$$\int_0^\mu \frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}d\sigma=\frac{\mu^{\nu-1}}{\Gamma(\nu)},\quad \int_0^\mu \frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}d\sigma=\frac{\mu^\nu}{\Gamma(\nu+1)}$$

and

$$\left|\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu\xi(\vartheta)d\vartheta\right|\leq \frac{\sigma^\nu(1-e^{-\omega\sigma})}{\omega\Gamma(\nu+1)}\|\xi\|.$$

Hence

$$\left|\int_0^\mu\left[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right]\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu\xi(\vartheta)d\vartheta\right)d\sigma\right|$$
$$\leq \int_0^\mu\left[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right]\left|\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu\xi(\vartheta)d\vartheta\right|d\sigma$$
$$\leq \frac{\mu^\nu(1-e^{-\omega})}{\omega\Gamma(\nu+1)}\|\xi\|\int_0^\mu\left[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right]d\sigma\ \leq\ \frac{(1-e^{-\omega\mu})(\nu\mu^{2\nu-1}+\mu^{2\nu})}{\omega[\Gamma(\nu+1)]^2}\|\xi\|.$$

□

Lemma 2.8. *(Krasnoselskii's fixed point theorem [27]). Let $\mathfrak{X}$ be a Banach space, $Y\subseteq\mathfrak{X}$ be nonempty, bounded, closed and convex. Let $\mathfrak{T}_1,\mathfrak{T}_2$ be two maps and satisfy:*

(i) $\mathfrak{T}_1y_1+\mathfrak{T}_2y_2\in Y$, $\forall y_1,y_2\in Y$;
(ii) $\mathfrak{T}_1$ is compact and continuous;
(iii) $\mathfrak{T}_2$ is a contraction mapping.

Then there exists $y_3\in Y$ such that $y_3=\mathfrak{T}_1y_3+\mathfrak{T}_2y_3$.

Lemma 2.9. *(Leray-Schauder fixed point theorem [28]) Let $\mathfrak{X}$ be a Banach space, $Y\subseteq\mathfrak{X}$ be nonempty, bounded and convex, H be an open subset of Y with $0\in H$. Let map $\mathfrak{G}:\overline{H}\longrightarrow Y$ be continuous and compact (that is, $\mathfrak{G}(\overline{H})$ is a relatively compact subset of Y). Then, one of the following representations is true:*

(i) there exist $z\in\partial H$ and $\varepsilon\in(0,1)$ such that $z=\varepsilon\mathfrak{G}(z)$;
(ii) $\mathfrak{G}$ has a fixed point $z\in H$.

3. Existence Results

This section deals with the existence results for problem (1.1).
Let $\mathfrak{X}=\left\{\kappa \mid \kappa\in C([0,1],\mathbb{R}) \textit{ and } {}^{c}D^{\nu-1}\kappa\in C([0,1],\mathbb{R})\right\}$ denote the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ endowed with the usual norm defined by

$$\|\kappa\|_{\mathfrak{X}}=\|\kappa\|+\left\|{}^{c}D^{\nu-1}\kappa\right\|=\sup_{\tau\in[0,1]}|\kappa(\tau)|+\sup_{\tau\in[0,1]}\left|{}^{c}D^{\nu-1}\kappa(\tau)\right|,$$

where $1<\nu\leq 2$.
In view of Lemma 2.6, we transform problem (1.1) to an equivalent fixed point problem as

$$\kappa=\mathfrak{G}\kappa,$$

$$\begin{aligned}
(\mathfrak{G}\kappa)(\tau)=&\int_0^{\tau}e^{-\omega(\tau-\sigma)}I^{\nu}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))d\sigma\\
&+\frac{\tau}{\Theta_1}\left[(\omega^2-\omega)\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^{\sigma}e^{-\omega(\sigma-\vartheta)}I^{\nu}f(\vartheta,\kappa(\vartheta),{}^{c}D^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\right.\\
&+\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^{\nu}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))+I^{\nu-1}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))]d\sigma\\
&\left.+\gamma\omega\int_0^{1}e^{-\omega(1-\sigma)}I^{\nu}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))d\sigma-\gamma I^{\nu}f(1,\kappa(1),{}^{c}D^{\nu-1}\kappa(1))\right]\\
&+\frac{1}{\Theta_2}\int_0^{\mu}\left[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right]\left(\int_0^{\sigma}e^{-\omega(\sigma-\vartheta)}I^{\nu}f(\vartheta,\kappa(\vartheta),{}^{c}D^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\\
&+\frac{\Theta_3}{\Theta_1\Theta_2}\left[(\omega^2-\omega)\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^{\sigma}e^{-\omega(\sigma-\vartheta)}I^{\nu}f(\vartheta,\kappa(\vartheta),{}^{c}D^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\right.\\
&+\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^{\nu}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))+I^{\nu-1}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))]d\sigma\\
&\left.+\gamma\omega\int_0^{1}e^{-\omega(1-\sigma)}I^{\nu}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))d\sigma-\gamma I^{\nu}f(1,\kappa(1),{}^{c}D^{\nu-1}\kappa(1))\right].
\end{aligned}\tag{3.1}$$

For convenience, we let

$$\begin{aligned}
\Pi_1=&\frac{1-e^{-\omega}}{\omega\Gamma(\nu+1)}+\frac{\mu(|\Theta_2|+|\Theta_3|)(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{|\Theta_1|\,|\Theta_2|\,\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}\\
&+\frac{\gamma(|\Theta_2|+|\Theta_3|)(2-e^{-\omega})}{|\Theta_1|\,|\Theta_2|\,\Gamma(\nu+1)}+\frac{(|\Theta_2|+|\Theta_3|)(|1-\omega|\,\mu^2+\nu\mu)}{|\Theta_1|\,|\Theta_2|\,\Gamma(\nu+1)\Gamma(3-\nu)}+\frac{(1-e^{-\omega\mu})(\nu\mu^{2\nu-1}+\mu^{2\nu})}{\omega\,|\Theta_2|\,[\Gamma(\nu+1)]^2}.
\end{aligned}\tag{3.2}$$

$$\Pi_2=\frac{(|\Theta_1|+|\gamma|)(2-e^{-\omega})}{|\Theta_1|\,\Gamma(\nu+1)}+\frac{\mu(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{|\Theta_1|\,\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}+\frac{|1-\omega|\,\mu^2+\nu\mu}{|\Theta_1|\,|\Gamma(\nu+1)\Gamma(3-\nu)}.$$

$$\begin{aligned}
\widetilde{\Pi}_1&=\Pi_1-\frac{1-e^{-\omega}}{\omega\Gamma(\nu+1)}.\\
\widetilde{\Pi}_2&=\Pi_2-\frac{2-e^{-\omega}}{\Gamma(\nu+1)}.
\end{aligned}\tag{3.3}$$

Theorem 3.1. *Assume $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ be a continuous function, which satisfies the following conditions:*
$(\mathfrak{B}_1)$ *There exists a constant $q>0$ such that*

$$|f(\tau,\kappa_1,\kappa_2)-f(\tau,\widetilde{\kappa}_1,\widetilde{\kappa}_2)|\leq q\left(\|\kappa_1-\widetilde{\kappa}_1\|+\|\kappa_2-\widetilde{\kappa}_2\|\right),$$

$\forall\tau\in[0,1],\ \kappa_i,\widetilde{\kappa}_i\in\mathbb{R}, i=1,2.$
$(\mathfrak{B}_2)$ $\forall\tau\in[0,1],\ \forall\,\kappa_1,\kappa_2\in\mathbb{R},\exists\theta\in C([0,1],\mathbb{R}^+):\ |f(\tau,\kappa_1,\kappa_2)|\leq\theta(\tau).$
Then the problem (1.1) has at least one solution on $[0,1]$ if

$$q\left(\widetilde{\Pi}_1+\frac{\widetilde{\Pi}_2}{\Gamma(3-\nu)}\right)<1,$$

where $\widetilde{\Pi}_1,\widetilde{\Pi}_2$ are given by (3.3).

Proof. Set $\sup_{\kappa\in[0,1]}|\kappa(\tau)| = \|\kappa\|$, we fix

$$\rho \geq \left(\Pi_1 + \frac{\Pi_2}{\Gamma(3-\nu)}\right)\|\kappa\|,$$

where Π_1, Π_2 given by (3.2) and define the ball $\mathscr{S}_\rho = \{\kappa \in \mathfrak{X} : \|\kappa\|_{\mathfrak{X}} \leq \rho\}$. Consider the operators $\mathfrak{G}_1$ and $\mathfrak{G}_2$ on $\mathscr{S}_\rho$

$$(\mathfrak{G}_1\kappa)(\tau) = \int_0^\tau e^{-\omega(\tau-\sigma)} I^\nu f\big(\sigma, \kappa(\sigma), {}^cD^{\nu-1}\kappa(\sigma)\big)\, d\sigma,$$

$$\begin{aligned}(\mathfrak{G}_2\kappa)(\tau) = \frac{\tau}{\Theta_1}\Big[&(\omega^2-\omega)\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\Big(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta, \kappa(\vartheta), {}^cD^{\nu-1}\kappa(\vartheta))\,d\vartheta\Big)\,d\sigma\\
&+\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\big[(1-\omega)I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma)) + I^{\nu-1} f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))\big]\,d\sigma\\
&+\gamma\omega\int_0^1 e^{-\omega(1-\sigma)} I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))\,d\sigma - \gamma I^\nu f\big(1,\kappa(1),{}^cD^{\nu-1}\kappa(1)\big)\Big]\\
&+\frac{1}{\Theta_2}\int_0^\mu\Big[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)} + \frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\Big]\Big(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta,\kappa(\vartheta),{}^cD^{\nu-1}\kappa(\vartheta))\,d\vartheta\Big)\,d\sigma\\
&+\frac{\Theta_3}{\Theta_1\Theta_2}\Big[(\omega^2-\omega)\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\Big(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta,\kappa(\vartheta),{}^cD^{\nu-1}\kappa(\vartheta))\,d\vartheta\Big)\,d\sigma\\
&+\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\big[(1-\omega)I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma)) + I^{\nu-1} f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))\big]\,d\sigma\\
&+\gamma\omega\int_0^1 e^{-\omega(1-\sigma)} I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))\,d\sigma - \gamma I^\nu f\big(1,\kappa(1),{}^cD^{\nu-1}\kappa(1)\big)\Big].\end{aligned}$$

In what follows, we use three steps to complete the proof of the theorem.

Step 1. $\forall \kappa_1, \kappa_2 \in \mathscr{S}_\rho$, $(\mathfrak{G}_1\kappa_1)(\tau) + (\mathfrak{G}_2\kappa_2)(\tau) \in \mathscr{S}_\rho$.
From Lemma 2.7 and by the use of condition $(\mathfrak{B}_2)$, for each $\kappa_1, \kappa_2 \in \mathscr{S}_\rho$

$$\begin{aligned}&|(\mathfrak{G}_1\kappa_1)(\tau) + (\mathfrak{G}_2\kappa_2)(\tau)|\\
&\leq \left|\int_0^\tau e^{-\omega(\tau-\sigma)} I^\nu f(\sigma,\kappa_1(\sigma),{}^cD^{\nu-1}\kappa_1(\sigma))d\sigma\right|\\
&\quad+\left|\frac{\tau}{\Theta_1}\right|\Big[\left|\omega^2-\omega\right|\left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta,\kappa_2(\vartheta),{}^cD^{\nu-1}\kappa_2(\vartheta))d\vartheta\right)d\sigma\right|\\
&\quad+\left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma)) + I^{\nu-1} f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))]d\sigma\right|\\
&\quad+|\gamma\omega|\left|\int_0^1 e^{-\omega(1-\sigma)} I^\nu f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))d\sigma\right| + |\gamma|\left|I^\nu f(1,\kappa_2(1),{}^cD^{\nu-1}\kappa_2(1))\right|\Big]\\
&\quad+\frac{1}{|\Theta_2|}\left|\int_0^\mu\left[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)} + \frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right]\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta,\kappa_2(\vartheta),{}^cD^{\nu-1}\kappa_2(\vartheta))d\vartheta\right)d\sigma\right|\\
&\quad+\left|\frac{\Theta_3}{\Theta_1\Theta_2}\right|\Big[\left|\omega^2-\omega\right|\left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta,\kappa_2(\vartheta),{}^cD^{\nu-1}\kappa_2(\vartheta))d\vartheta\right)d\sigma\right|\\
&\quad+\left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma)) + I^{\nu-1} f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))]d\sigma\right|\\
&\quad+|\gamma\omega|\left|\int_0^1 e^{-\omega(1-\sigma)} I^\nu f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))d\sigma\right| + |\gamma|\left|I^\nu f(1,\kappa_2(1),{}^cD^{\nu-1}\kappa_2(1))\right|\Big]\\
&\leq \frac{1-e^{-\omega}}{\omega\Gamma(\nu+1)}\|\theta\| + \frac{1}{|\Theta_1|}\left(\frac{\mu(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}\|\theta\| + \frac{|1-\omega|\mu^2+\nu\mu}{\Gamma(\nu+1)\Gamma(3-\nu)}\|\theta\|\right.\\
&\quad\left.+\frac{|\gamma|(1-e^{-\omega})}{\Gamma(\nu+1)}\|\theta\| + \frac{|\gamma|}{\Gamma(\nu+1)}\|\theta\|\right) + \frac{(1-e^{-\omega\mu})(\nu\mu^{2\nu-1}+\mu^{2\nu})}{\omega|\Theta_2|[\Gamma(\nu+1)]^2}\|\theta\|\\
&\quad+\frac{|\Theta_3|}{|\Theta_1||\Theta_2|}\left(\frac{\mu(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}\|\theta\| + \frac{|1-\omega|\mu^2+\nu\mu}{\Gamma(\nu+1)\Gamma(3-\nu)}\|\theta\|\right.\\
&\quad\left.+\frac{|\gamma|(1-e^{-\omega})}{\Gamma(\nu+1)}\|\theta\| + \frac{|\gamma|}{\Gamma(\nu+1)}\|\theta\|\right)\\
&\leq \left(\frac{1-e^{-\omega}}{\omega\Gamma(\nu+1)} + \frac{\mu(|\Theta_2|+|\Theta_3|)(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{|\Theta_1||\Theta_2|\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}\right.\\
&\quad+\frac{\gamma(|\Theta_2|+|\Theta_3|)(2-e^{-\omega})}{|\Theta_1||\Theta_2|\Gamma(\nu+1)} + \frac{(|\Theta_2|+|\Theta_3|)(|1-\omega|\mu^2+\nu\mu)}{|\Theta_1||\Theta_2|\Gamma(\nu+1)\Gamma(3-\nu)}\\
&\quad\left.+\frac{(1-e^{-\omega\mu})(\nu\mu^{2\nu-1}+\mu^{2\nu})}{\omega|\Theta_2|[\Gamma(\nu+1)]^2}\right)\|\theta\|\end{aligned}$$

$$\leq \Pi_1 \|\theta\|.$$

Thus

$$\|(\mathfrak{G}_1\kappa_1)+(\mathfrak{G}_2\kappa_2)\| \leq \Pi_1 \|\theta\|.$$

Also we have

$$(\mathfrak{G}'_1\kappa_1)(\tau) = -\omega\int_0^\tau e^{-\omega(\tau-\sigma)} I^\nu f(\sigma,\kappa_1(\sigma),{}^cD^{\nu-1}\kappa_1(\sigma))d\sigma + I^\nu f(\tau,\kappa_1(\tau),{}^cD^{\nu-1}\kappa_1(\tau)).$$

$$\begin{aligned}(\mathfrak{G}'_2\kappa_2)(\tau) = \frac{1}{\Theta_1}\Bigg[&(\omega^2-\omega)\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta,\kappa_2(\vartheta),{}^cD^{\nu-1}\kappa_2(\vartheta))d\vartheta\right)d\sigma \\ &+ \int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma)) + I^{\nu-1}f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))]d\sigma \\ &+ \gamma\omega\int_0^1 e^{-\omega(1-\sigma)} I^{\nu-1} f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))d\sigma - \gamma I^\nu f(1,\kappa_2(1),{}^cD^{\nu-1}\kappa_2(1))\Bigg].\end{aligned}$$

Hence

$$\begin{aligned}&\left|(\mathfrak{G}'_1\kappa_1)(\tau)+(\mathfrak{G}'_2\kappa_2)(\tau)\right| \\ &\leq \omega\left|\int_0^\tau e^{-\omega(\tau-\sigma)} I^\nu f(\sigma,\kappa_1(\sigma),{}^cD^{\nu-1}\kappa_1(\sigma))d\sigma\right| + \left|I^\nu f(\tau,\kappa_1(\tau),{}^cD^{\nu-1}\kappa_1(\tau))\right| \\ &+ \frac{1}{|\Theta_1|}\left[\left|\omega^2-\omega\right|\left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)} I^\nu f(\vartheta,\kappa_2(\vartheta),{}^cD^{\nu-1}\kappa_2(\vartheta))d\vartheta\right)d\sigma\right|\right. \\ &+ \left|\int_0^\mu \frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma)) + I^{\nu-1}f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))]d\sigma\right| \\ &\left.+ |\gamma|\,\omega\left|\int_0^1 e^{-\omega(1-\sigma)} I^{\nu-1} f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))d\sigma\right| + |\gamma|\left|I^\nu f(1,\kappa_2(1),{}^cD^{\nu-1}\kappa_2(1))\right|\right] \\ &\leq \frac{1-e^{-\omega}}{\Gamma(\nu+1)}\|\theta\| + \frac{1}{\Gamma(\nu+1)}\|\theta\| + \frac{\mu(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{|\Theta_1|\,\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}\|\theta\| \\ &+ \frac{|1-\omega|\,\mu^2+\nu\mu}{|\Theta_1|\,|\Gamma(\nu+1)\Gamma(3-\nu)}\|\theta\| + \frac{|\gamma|\,(1-e^{-\omega})}{|\Theta_1|\,\Gamma(\nu+1)}\|\theta\| + \frac{|\gamma|}{|\Theta_1|\,\Gamma(\nu+1)}\|\theta\| \\ &\leq \left(\frac{(|\Theta_1|+|\gamma|)(2-e^{-\omega})}{|\Theta_1|\,\Gamma(\nu+1)} + \frac{\mu(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{|\Theta_1|\,\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}\right. \\ &\quad\left.+ \frac{|1-\omega|\,\mu^2+\nu\mu}{|\Theta_1|\,|\Gamma(\nu+1)\Gamma(3-\nu)}\right)\|\theta\| \\ &\leq \Pi_2\|\theta\|.\end{aligned}$$

From Definition 2.2 with $1<\nu\leq 2$, we get

$$\begin{aligned}\left|{}^cD^{\nu-1}(\mathfrak{G}_1\kappa_1+\mathfrak{G}_2\kappa_2)(\tau)\right| &\leq \int_0^\tau \frac{(\tau-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left|(\mathfrak{G}'_1\kappa_1+\mathfrak{G}'_2\kappa_2)(\sigma)\right|d\sigma \\ &\leq \Pi_2\|\theta\|\int_0^\tau \frac{(\tau-\sigma)^{1-\nu}}{\Gamma(2-\nu)}d\sigma \\ &\leq \Pi_2\|\theta\|\frac{\tau^{2-\nu}}{\Gamma(3-\nu)} \\ &\leq \frac{\Pi_2}{\Gamma(3-\nu)}\|\theta\|.\end{aligned}$$

From the above inequalities, we get

$$\begin{aligned}\|\mathfrak{G}_1\kappa_1+\mathfrak{G}_2\kappa_2\|_{\mathfrak{X}} &= \|\mathfrak{G}_1\kappa_1+\mathfrak{G}_2\kappa_2\| + \left\|{}^cD^{\nu-1}(\mathfrak{G}_1\kappa_1+\mathfrak{G}_2\kappa_2)\right\| \\ &\leq \left(\Pi_1+\frac{\Pi_2}{\Gamma(3-\nu)}\right)\|\theta\| \\ &\leq \rho.\end{aligned}$$

Thus, $\mathfrak{G}_1\kappa_1+\mathfrak{G}_2\kappa_2 \in \mathscr{S}_\rho$.

Step 2. $\mathfrak{G}_1 : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is continuous and compact.
Let $\tau_1,\tau_2 \in [0,1]$ with $\tau_1 < \tau_2$ and $\kappa \in \mathscr{S}_\rho$. By using Lemma 2.7 and condition $(\mathfrak{B}_2)$, one can find

$$\begin{aligned}
\left|(\mathfrak{G}_1\kappa)(\tau_2)-(\mathfrak{G}_1\kappa)(\tau_1)\right| &= \Big|\int_0^{\tau_2} e^{-\omega(\tau_2-\sigma)} I^\nu f\big(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma)\big)\,d\sigma - \int_0^{\tau_1} e^{-\omega(\tau_1-\sigma)} I^\nu f\big(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma)\big)\,d\sigma\Big| \\
&= \Big|\int_0^{\tau_1} e^{\omega\sigma}(e^{-\omega\tau_2}-e^{-\omega\tau_1}) I^\nu f\big(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma)\big)\,d\sigma + \int_{\tau_1}^{\tau_2} e^{-\omega(\tau_2-\sigma)} I^\nu f\big(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma)\big)\,d\sigma\Big| \\
&\le \int_0^{\tau_1} e^{\omega\sigma}\left|e^{-\omega\tau_2}-e^{-\omega\tau_1}\right|\left|I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))\right|d\sigma + \int_{\tau_1}^{\tau_2} e^{-\omega(\tau_2-\sigma)}\left|I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))\right|d\sigma \\
&\le \frac{1}{\Gamma(\nu+1)}\Big(\int_0^{\tau_1} e^{\omega\sigma}\left|e^{-\omega\tau_2}-e^{-\omega\tau_1}\right|d\sigma + \int_{\tau_1}^{\tau_2} e^{-\omega(\tau_2-\sigma)}\,d\sigma\Big)\|\theta\|.
\end{aligned}$$

and

$$\begin{aligned}
\left|{}^cD^{\nu-1}(\mathfrak{G}_1\kappa)(\tau_2)-{}^cD^{\nu-1}(\mathfrak{G}_1\kappa)(\tau_1)\right| &= \Big|\int_0^{\tau_2}\frac{(\tau_2-\sigma)^{1-\nu}}{\Gamma(2-\nu)}(\mathfrak{G}_1'\kappa)(\sigma)\,d\sigma - \int_0^{\tau_1}\frac{(\tau_1-\sigma)^{1-\nu}}{\Gamma(2-\nu)}(\mathfrak{G}_1'\kappa)(\sigma)\,d\sigma\Big| \\
&= \Big|\int_0^{\tau_1}\frac{(\tau_2-\sigma)^{1-\nu}-(\tau_1-\sigma)^{1-\nu}}{\Gamma(2-\nu)}(\mathfrak{G}_1'\kappa)(\sigma)\,d\sigma + \int_{\tau_1}^{\tau_2}\frac{(\tau_2-\sigma)^{1-\nu}}{\Gamma(2-\nu)}(\mathfrak{G}_1'\kappa)(\sigma)\,d\sigma\Big| \\
&\le \frac{1}{\Gamma(2-\nu)}\Big(\int_0^{\tau_1}\left|(\tau_2-\sigma)^{1-\nu}-(\tau_1-\sigma)^{1-\nu}\right|\left|(\mathfrak{G}_1'\kappa)(\sigma)\right|d\sigma \\
&\quad + \int_{\tau_1}^{\tau_2}(\tau_2-\sigma)^{1-\nu}\left|(\mathfrak{G}_1'\kappa)(\sigma)\right|d\sigma\Big) \\
&\le \frac{2-e^{-\omega}}{\Gamma(2-\nu)\Gamma(\nu+1)}\Big(\int_0^{\tau_1}\left|(\tau_2-\sigma)^{1-\nu}-(\tau_1-\sigma)^{1-\nu}\right|d\sigma \\
&\quad + \int_{\tau_1}^{\tau_2}(\tau_2-\sigma)^{1-\nu}\,d\sigma\Big)\|\theta\|.
\end{aligned}$$

Clearly, $|(\mathfrak{G}_1\kappa)(\tau_2)-(\mathfrak{G}_1\kappa)(\tau_1)| \to 0$ and $\left|{}^cD^{\nu-1}(\mathfrak{G}_1\kappa)(\tau_2)-{}^cD^{\nu-1}(\mathfrak{G}_1\kappa)(\tau_1)\right| \to 0$ independent of κ as $\tau_1 \to \tau_2$. Thus $\mathfrak{G}_1$ is relatively compact on S_ρ. Then, by the Arzelá-Ascoli theorem, $\mathfrak{G}_1$ is compact on $\mathscr{S}_\rho$.

Step 3. $\mathfrak{G}_2 : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is contraction.
From Lemma 2.7 and the use of condition $(\mathfrak{B}_1)$, for $\tau \in [0,1]$, $\kappa_1,\kappa_2 \in \mathscr{S}_\rho$, we can derive

$$\begin{aligned}
&|(\mathfrak{G}_2\kappa_1)(\tau)-(\mathfrak{G}_2\kappa_2)(\tau)| \le \left|\frac{\tau}{\Theta_1}\right|\Big[\left|\omega^2-\omega\right|\Big|\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\Big(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu[f(\vartheta,\kappa_1(\vartheta),{}^cD^{\nu-1}\kappa_1(\vartheta)) \\
&-f(\vartheta,\kappa_2(\vartheta),{}^cD^{\nu-1}\kappa_2(\vartheta))]d\vartheta\Big)d\sigma\Big| + \Big|\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\Big[(1-\omega)I^\nu[f(\sigma,\kappa_1(\sigma),{}^cD^{\nu-1}\kappa_1(\sigma)) - f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))] \\
&+ I^{\nu-1}\Big[f(\sigma,\kappa_1(\sigma),{}^cD^{\nu-1}\kappa_1(\sigma)) - f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))\Big]\Big]d\sigma\Big| \\
&+|\gamma\omega|\Big|\int_0^1 e^{-\omega(1-\sigma)}I^\nu\Big[f(\sigma,\kappa_1(\sigma),{}^cD^{\nu-1}\kappa_1(\sigma)) - f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))\Big]d\sigma\Big| \\
&+|\gamma|\Big|I^\nu[f(1,\kappa_1(1),{}^cD^{\nu-1}\kappa_1(1)) - f(1,\kappa_2(1),{}^cD^{\nu-1}\kappa_2(1))]\Big|\Big] \\
&+\frac{1}{|\Theta_2|}\Big|\int_0^\mu\Big[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\Big]\Big(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu[f(\vartheta,\kappa_1(\vartheta),{}^cD^{\nu-1}\kappa_1(\vartheta)) - f(\vartheta,\kappa_2(\vartheta),{}^cD^{\nu-1}\kappa_2(\vartheta))]d\vartheta\Big)d\sigma\Big| \\
&+\left|\frac{\Theta_3}{\Theta_1\Theta_2}\right|\Big[\left|\omega^2-\omega\right|\Big|\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\Big(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu\Big[f(\vartheta,\kappa_1(\vartheta),{}^cD^{\nu-1}\kappa_1(\vartheta)) - f(\vartheta,\kappa_2(\vartheta),{}^cD^{\nu-1}\kappa_2(\vartheta))\Big]d\vartheta\Big)d\sigma\Big|
\end{aligned}$$

$$+\left|\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left[(1-\omega)I^{\nu}[f(\sigma,\kappa_1(\sigma),{}^cD^{\nu-1}\kappa_1(\sigma))-f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))]\right.\right.$$

$$\left.\left.+I^{\nu-1}[f(\sigma,\kappa_1(\sigma),{}^cD^{\nu-1}\kappa_1(\sigma))-f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))]\right]d\sigma\right|$$

$$+|\gamma\omega|\left|\int_0^1 e^{-\omega(1-\sigma)}I^{\nu}\left[f(\sigma,\kappa_1(\sigma),{}^cD^{\nu-1}\kappa_1(\sigma))-f(\sigma,\kappa_2(\sigma),{}^cD^{\nu-1}\kappa_2(\sigma))\right]d\sigma\right|$$

$$+|\gamma|\left|I^{\nu}[f(1,\kappa_1(1),{}^cD^{\nu-1}\kappa_1(1))-f(1,\kappa_2(1),{}^cD^{\nu-1}\kappa_2(1))]\right|\Big]$$

$$\leq\left(\frac{\mu(|\Theta_2|+|\Theta_3|)(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{|\Theta_1||\Theta_2|\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}+\frac{\gamma(|\Theta_2|+|\Theta_3|)(2-e^{-\omega})}{|\Theta_1||\Theta_2|\Gamma(\nu+1)}+\frac{(|\Theta_2|+|\Theta_3|)(|1-\omega|\mu^2+\nu\mu)}{|\Theta_1||\Theta_2|\Gamma(\nu+1)\Gamma(3-\nu)}+\frac{(1-e^{-\omega\mu})(\nu\mu^{2\nu-1}+\mu^{2\nu})}{\omega|\Theta_2|[\Gamma(\nu+1)]^2}\right)$$

$$.q(\|\kappa_1-\kappa_2\|+\left\|{}^cD^{\nu-1}\kappa_1-{}^cD^{\nu-1}\kappa_2\right\|)$$

$$\leq q\widetilde{\Pi_1}(\|\kappa_1-\kappa_2\|+\left\|{}^cD^{\nu-1}\kappa_1-{}^cD^{\nu-1}\kappa_2\right\|).$$

Also

$$\left|(\mathfrak{G}'_2\kappa_1)(\tau)-(\mathfrak{G}'_2\kappa_2)(\tau)\right|\leq q\widetilde{\Pi_2}(\|\kappa_1-\kappa_2\|+\left\|{}^cD^{\nu-1}\kappa_1-{}^cD^{\nu-1}\kappa_2\right\|).$$

Which implies that

$$\begin{aligned}\left|{}^cD^{\nu-1}(\mathfrak{G}_2\kappa_1)(\tau)-{}^cD^{\nu-1}(\mathfrak{G}_2\kappa_2)(\tau)\right| &\leq \int_0^{\tau}\frac{(\tau-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left|(\mathfrak{G}'_2\kappa_1)(\sigma)-(\mathfrak{G}'_2\kappa_2)(\sigma)\right|d\sigma\\ &\leq q\widetilde{\Pi_2}(\|\kappa_1-\kappa_2\|+\left\|{}^cD^{\nu-1}\kappa_1-{}^cD^{\nu-1}\kappa_2\right\|)\int_0^{\tau}\frac{(\tau-\sigma)^{1-\nu}}{\Gamma(2-\nu)}d\sigma\\ &\leq q\widetilde{\Pi_2}\left(\|\kappa_1-\kappa_2\|+\left\|{}^cD^{\nu-1}\kappa_1-{}^cD^{\nu-1}\kappa_2\right\|\right)\frac{\tau^{2-\nu}}{\Gamma(3-\nu)}\\ &\leq \frac{q\widetilde{\Pi_2}}{\Gamma(3-\nu)}\left(\|\kappa_1-\kappa_2\|+\left\|{}^cD^{\nu-1}\kappa_1-{}^cD^{\nu-1}\kappa_2\right\|\right).\end{aligned}$$

From the above inequalities, we have

$$\begin{aligned}\|\mathfrak{G}_2\kappa_1-\mathfrak{G}_2\kappa_2\|_{\mathfrak{X}} &= \|\mathfrak{G}_2\kappa_1-\mathfrak{G}_2\kappa_2\|+\left\|{}^cD^{\nu-1}(\mathfrak{G}_2\kappa_1)-{}^cD^{\nu-1}(\mathfrak{G}_2\kappa_2)\right\|\\ &\leq q\left(\widetilde{\Pi_1}+\frac{\widetilde{\Pi_2}}{\Gamma(3-\nu)}\right)\left(\|\kappa_1-\kappa_2\|+\left\|{}^cD^{\nu-1}\kappa_1-{}^cD^{\nu-1}\kappa_2\right\|\right)\\ &\leq q\left(\widetilde{\Pi_1}+\frac{\widetilde{\Pi_2}}{\Gamma(3-\nu)}\right)\|\kappa_1-\kappa_2\|_{\mathfrak{X}}.\end{aligned}$$

As $q\left(\widetilde{\Pi_1}+\frac{\widetilde{\Pi_2}}{\Gamma(3-\nu)}\right)<1$, $\mathfrak{G}_2$ is contraction. From Lemma 2.8, there exists $\kappa\in\mathscr{S}_{\rho}$ such that $\kappa(\tau)=(\mathfrak{G}_1\kappa)(\tau)+(\mathfrak{G}_2\kappa)(\tau)=(\mathfrak{G}\kappa)(\tau)$, which means that κ is the solution of problem (1.1). □

The prove of the next result is based on Lemma 2.9.

Theorem 3.2. *Let $f\in C([0,1]\times\mathbb{R}^2,\mathbb{R})$ and assume that*
($\mathfrak{B}_3$) For all $(\tau,\kappa_1,\kappa_2)\in[0,1]\times\mathbb{R}^2$, there exist a function $\mathscr{J}\in C([0,1],\mathbb{R}^+)$, and a nondecreasing continuous function $\mathscr{R}:[0,\infty)\to[0,\infty)$ such that

$$|f(\tau,\kappa_1,\kappa_2)|\leq\mathscr{J}(\tau)\mathscr{R}(\|\kappa_1\|+\|\kappa_2\|);$$

($\mathfrak{B}_4$) there exists a constant $N>0$ such that

$$\frac{N}{\|\mathscr{J}\|\mathscr{R}(N)}>\Pi_1+\frac{\Pi_2}{\Gamma(3-\nu)},$$

where Π_1,Π_2 are given by (3.2).

Then problem (1.1) has at least one solution on $[0,1]$.

Proof. Consider the operator $\mathfrak{G}:\mathfrak{X}\to\mathfrak{X}$ defined in (3.1). At first, we show that $\mathfrak{G}$ maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. For $\rho>0$, let $\mathfrak{D}_\rho=\{\kappa\in C([0,1],\mathbb{R}):\|\kappa\|_{\mathfrak{X}}\le\rho\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then

$$\begin{aligned}
|(\mathfrak{G}\kappa)(\tau)| &\le \left|\int_0^\tau e^{-\omega(\tau-\sigma)}I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))d\sigma\right| \\
&+\left|\frac{\tau}{\Theta_1}\right|\left[\left|\omega^2-\omega\right|\left|\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu f(\vartheta,\kappa(\vartheta),{}^cD^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\right|\right. \\
&+\left|\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))+I^{\nu-1}f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))]d\sigma\right| \\
&\left.+|\gamma\omega|\left|\int_0^1 e^{-\omega(1-\sigma)}I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))d\sigma\right|+|\gamma|\left|I^\nu f(1,\kappa(1),{}^cD^{\nu-1}\kappa(1))\right|\right] \\
&+\frac{1}{|\Theta_2|}\left|\int_0^\mu\left[\frac{(\mu-\sigma)^{\nu-2}}{\Gamma(\nu-1)}+\frac{(\mu-\sigma)^{\nu-1}}{\Gamma(\nu)}\right]\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu f(\vartheta,\kappa(\vartheta),{}^cD^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\right| \\
&+\left|\frac{\Theta_3}{\Theta_1\Theta_2}\right|\left[\left|\omega^2-\omega\right|\left|\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu f(\vartheta,\kappa(\vartheta),{}^cD^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\right|\right. \\
&+\left|\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))\right. \\
&\left.+I^{\nu-1}f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))]d\sigma\right|+|\gamma\omega|\left|\int_0^1 e^{-\omega(1-\sigma)}I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))d\sigma\right| \\
&\left.+|\gamma|\left|I^\nu f(1,\kappa(1),{}^cD^{\nu-1}\kappa(1))\right|\right] \\
&\le\left(\frac{1-e^{-\omega}}{\omega\Gamma(\nu+1)}+\frac{\mu(|\Theta_2|+|\Theta_3|)(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{|\Theta_1|\,|\Theta_2|\,\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}\right. \\
&+\frac{\gamma(|\Theta_2|+|\Theta_3|)(2-e^{-\omega})}{|\Theta_1|\,|\Theta_2|\,\Gamma(\nu+1)}+\frac{(|\Theta_2|+|\Theta_3|)(|1-\omega|\,\mu^2+\nu\mu)}{|\Theta_1|\,|\Theta_2|\,\Gamma(\nu+1)\Gamma(3-\nu)} \\
&\left.+\frac{(1-e^{-\omega\mu})(\nu\mu^{2\nu-1}+\mu^{2\nu})}{\omega\,|\Theta_2|\,[\Gamma(\nu+1)]^2}\right)\|\mathscr{J}\|\mathscr{R}\left(\|\kappa\|+\left\|{}^cD^{\nu-1}\kappa\right\|\right) \\
&\le\Pi_1\|\mathscr{J}\|\mathscr{R}\left(\|\kappa\|+\left\|{}^cD^{\nu-1}\kappa\right\|\right)=\Pi_1\|\mathscr{J}\|\mathscr{R}(\|\kappa\|_{\mathfrak{X}}),
\end{aligned}$$

where Π_1 are given in (3.2).
Hence

$$\|\mathfrak{G}\kappa\|\le\Pi_1\|\mathscr{J}\|\mathscr{R}(\|\kappa\|_{\mathfrak{X}}).$$

Also we have

$$\begin{aligned}
\left|(\mathfrak{G}'\kappa)(\tau)\right| &\le\omega\left|\int_0^\tau e^{-\omega(\tau-\sigma)}I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))d\sigma\right|+\left|I^\nu f(\tau,\kappa(\tau),{}^cD^{\nu-1}\kappa(\tau))\right| \\
&+\frac{1}{|\Theta_1|}\left[\left|\omega^2-\omega\right|\left|\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^\sigma e^{-\omega(\sigma-\vartheta)}I^\nu f(\vartheta,\kappa(\vartheta),{}^cD^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\right|\right. \\
&+\left|\int_0^\mu\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left[(1-\omega)I^\nu f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))+I^{\nu-1}f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))\right]d\sigma\right| \\
&\left.+|\gamma|\,\omega\left|\int_0^1 e^{-\omega(1-\sigma)}I^{\nu-1}f(\sigma,\kappa(\sigma),{}^cD^{\nu-1}\kappa(\sigma))d\sigma\right|+|\gamma|\left|I^\nu f(1,\kappa(1),{}^cD^{\nu-1}\kappa(1))\right|\right] \\
&\le\left(\frac{(|\Theta_1|+|\gamma|)(2-e^{-\omega})}{|\Theta_1|\,\Gamma(\nu+1)}+\frac{\mu(e^{-\omega\mu}+\omega\mu-1)\left|\omega^2-\omega\right|}{|\Theta_1|\,\omega^2\Gamma(2-\nu)\Gamma(\nu+1)}\right. \\
&\left.+\frac{|1-\omega|\,\mu^2+\nu\mu}{|\Theta_1|\,|\Gamma(\nu+1)\Gamma(3-\nu)}\right)\|\mathscr{J}\|\mathscr{R}\left(\|\kappa\|+\left\|{}^cD^{\nu-1}\kappa\right\|\right) \\
&\le\Pi_2\|\mathscr{J}\|\mathscr{R}\left(\|\kappa\|+\left\|{}^cD^{\nu-1}\kappa\right\|\right)=\Pi_2\|\mathscr{J}\|\mathscr{R}(\|\kappa\|_{\mathfrak{X}}),
\end{aligned}$$

where Π_2 given by (3.2).

By Definition 2.2 for $\nu\in(1,2]$, we get

$$\begin{aligned}\left|{}^{c}D^{\nu-1}(\mathfrak{G}\kappa)(\tau)\right| &\leq \int_0^{\tau}\frac{(\tau-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left|(\mathfrak{G}'\kappa)(\sigma)\right|d\sigma\\ &\leq \Pi_2\,\|\mathscr{J}\|\mathscr{R}(\|\kappa\|_{\mathfrak{X}})\int_0^{\tau}\frac{(\tau-\sigma)^{1-\nu}}{\Gamma(2-\nu)}d\sigma\\ &\leq \Pi_2\,\|\mathscr{J}\|\mathscr{R}(\|\kappa\|_{\mathfrak{X}})\frac{\tau^{2-\nu}}{\Gamma(3-\nu)}\\ &\leq \frac{\Pi_2}{\Gamma(3-\nu)}\|\mathscr{J}\|\mathscr{R}(\|\kappa\|_{\mathfrak{X}}).\end{aligned}$$

Hence

$$\begin{aligned}\|\mathfrak{G}\kappa\|_{\mathfrak{X}}=\|\mathfrak{G}\kappa\|+\left\|{}^{c}D^{\nu-1}\mathfrak{G}\kappa\right\| &\leq \left(\Pi_1+\frac{\Pi_2}{\Gamma(3-\nu)}\right)\|\mathscr{J}\|\mathscr{R}(\|\kappa\|_{\mathfrak{X}})\\ &\leq \left(\Pi_1+\frac{\Pi_2}{\Gamma(3-\nu)}\right)\|\mathscr{J}\|\mathscr{R}(\rho).\end{aligned}$$

Next we show that $\mathfrak{G}$ maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $\tau_1,\tau_2\in[0,1]$ with $\tau_1<\tau_2$ and $\kappa\in\mathfrak{D}_\rho$, where $\mathfrak{D}_\rho$ is a bounded set of $C([0,1],\mathbb{R})$. Then we obtain

$$\begin{aligned}|(\mathfrak{G}\kappa)(\tau_2)-(\mathfrak{G}\kappa)(\tau_1)| \leq &\left|\int_0^{\tau_2}e^{-\omega(\tau_2-\sigma)}I^{\nu}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))d\sigma\right.\\ &\left.-\int_0^{\tau_1}e^{-\omega(\tau_1-\sigma)}I^{\nu}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))d\sigma\right|\\ &+\left|\frac{\tau_2-\tau_1}{\Theta_1}\right|\left|(\omega^2-\omega)\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^{\sigma}e^{-\omega(\sigma-\vartheta)}I^{\nu}f(\vartheta,\kappa(\vartheta),{}^{c}D^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\right.\\ &+\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^{\nu}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))+I^{\nu-1}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))]d\sigma\\ &\left.+\gamma\omega\int_0^1e^{-\omega(1-\sigma)}I^{\nu}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))d\sigma-\gamma I^{\nu}f(1,\kappa(1),{}^{c}D^{\nu-1}\kappa(1))\right|\\ \leq &\int_0^{\tau_1}e^{\omega\sigma}\left|e^{-\omega\tau_2}-e^{-\omega\tau_1}\right|\left|I^{\nu}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))\right|d\sigma\\ &+\int_{\tau_1}^{\tau_2}e^{-\omega(\tau_2-\sigma)}\left|I^{\nu}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))\right|d\sigma\\ &+\left|\frac{\tau_2-\tau_1}{\Theta_1}\right|\left|(\omega^2-\omega)\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^{\sigma}e^{-\omega(\sigma-\vartheta)}I^{\nu}f(\vartheta,\kappa(\vartheta),{}^{c}D^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\right.\\ &+\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^{\nu}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))+I^{\nu-1}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))]d\sigma\\ &\left.+\gamma\omega\int_0^1e^{-\omega(1-\sigma)}I^{\nu}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))d\sigma-\gamma I^{\nu}f(1,\kappa(1),{}^{c}D^{\nu-1}\kappa(1))\right|\\ \leq &\left(\int_0^{\tau_1}e^{\omega\sigma}\left|e^{-\omega\tau_2}-e^{-\omega\tau_1}\right|d\sigma+\int_{\tau_1}^{\tau_2}e^{-\omega(\tau_2-\sigma)}d\sigma\right)\frac{\|\mathscr{J}\|\mathscr{R}(\|\kappa\|_{\mathfrak{X}})}{\Gamma(\nu+1)}\\ &+\left|\frac{\tau_2-\tau_1}{\Theta_1}\right|\left|(\omega^2-\omega)\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}\left(\int_0^{\sigma}e^{-\omega(\sigma-\vartheta)}I^{\nu}f(\vartheta,\kappa(\vartheta),{}^{c}D^{\nu-1}\kappa(\vartheta))d\vartheta\right)d\sigma\right.\\ &+\int_0^{\mu}\frac{(\mu-\sigma)^{1-\nu}}{\Gamma(2-\nu)}[(1-\omega)I^{\nu}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))+I^{\nu-1}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))]d\sigma\\ &\left.+\gamma\omega\int_0^1e^{-\omega(1-\sigma)}I^{\nu}f(\sigma,\kappa(\sigma),{}^{c}D^{\nu-1}\kappa(\sigma))d\sigma-\gamma I^{\nu}f(1,\kappa(1),{}^{c}D^{\nu-1}\kappa(1))\right|.\end{aligned}$$

Also

$$\begin{aligned}\left|{}^{c}D^{\nu-1}(\mathfrak{G}\kappa)(\tau_2)-{}^{c}D^{\nu-1}(\mathfrak{G}\kappa)(\tau_1)\right| &=\left|\int_0^{\tau_2}\frac{(\tau_2-\sigma)^{1-\nu}}{\Gamma(2-\nu)}(\mathfrak{G}'\kappa)(\sigma)d\sigma-\int_0^{\tau_1}\frac{(\tau_1-\sigma)^{1-\nu}}{\Gamma(2-\nu)}(\mathfrak{G}'\kappa)(\sigma)d\sigma\right|\\ &\leq\int_0^{\tau_1}\frac{\left|(\tau_2-\sigma)^{1-\nu}-(\tau_1-\sigma)^{1-\nu}\right|}{\Gamma(2-\nu)}\left|(\mathfrak{G}'\kappa)(\sigma)\right|d\sigma+\int_{\tau_1}^{\tau_2}\frac{\left|(\tau_2-\sigma)^{1-\nu}\right|}{\Gamma(2-\nu)}\left|(\mathfrak{G}'\kappa)(\sigma)\right|d\sigma\\ &\leq\left(\int_0^{\tau_1}\frac{\left|(\tau_2-\sigma)^{1-\nu}-(\tau_1-\sigma)^{1-\nu}\right|}{\Gamma(2-\nu)}d\sigma+\int_{\tau_1}^{\tau_2}\frac{\left|(\tau_2-\sigma)^{1-\nu}\right|}{\Gamma(2-\nu)}d\sigma\right)\Pi_2\,\|\mathscr{J}\|\mathscr{R}(\|\kappa\|_{\mathfrak{X}}).\end{aligned}$$

Obviously, the right-hand side of the above inequality tends to zero independently of $\kappa\in\mathfrak{D}_\rho$ as $\tau_2-\tau_1\to0$. As $\mathfrak{G}$ verifies the above assumptions, then by the use of Arzelá-Ascoli theorem, we claim that $\mathfrak{G}:C([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$ is completely continuous.
To achieve the satisfaction of the hypotheses of the Leray-Schauder nonlinear alternative theorem, is to show the boundedness of the set

of all solutions to equation $\kappa = \lambda \mathfrak{G}\kappa$ for $\lambda \in [0,1]$. Assume that κ is a solution, then in the same manner as we show the operator $\mathfrak{G}$ is bounded, we can obtain

$$|\kappa(\tau)| = |\lambda(\mathfrak{G}\kappa)(\tau)| \le \Pi_1 \|\mathscr{J}\| \mathscr{R}(\|\kappa\|_{\mathfrak{X}}).$$

Also we have

$$\left|\kappa^{'}(\tau)\right| = \left|\lambda(\mathfrak{G}^{'}\kappa)(\tau)\right| \le \Pi_2 \|\mathscr{J}\| \mathscr{R}(\|\kappa\|_{\mathfrak{X}}).$$

By Definition 2.2 for $1 < \nu \le 2$, we get

$$\left|{}^cD^{\nu-1}\kappa(\tau)\right| = \left|\lambda\, {}^cD^{\nu-1}(\mathfrak{G}\kappa)(\tau)\right| \le \frac{\Pi_2}{\Gamma(3-\nu)} \|\mathscr{J}\| \mathscr{R}(\|\kappa\|_{\mathfrak{X}}).$$

Hence

$$\|\kappa\|_{\mathfrak{X}} = \|\kappa\| + \left\|{}^cD^{\nu-1}\kappa\right\| \le \left(\Pi_1 + \frac{\Pi_2}{\Gamma(3-\nu)}\right) \|\mathscr{J}\| \mathscr{R}(\|\kappa\|_{\mathfrak{X}}).$$

Consequently, we have

$$\frac{\|\kappa\|_{\mathfrak{X}}}{\|\mathscr{J}\| \mathscr{R}(\|\kappa\|_{\mathfrak{X}})} \le \Pi_1 + \frac{\Pi_2}{\Gamma(3-\nu)}.$$

In view of $(\mathfrak{B}_4)$, there exists N such that $\|\kappa\|_{\mathfrak{X}} \neq N$.
Let us set

$$\mathscr{V} = \{\kappa \in C([0,1],\mathbb{R}) : \|\kappa\|_{\mathfrak{X}} < N\}.$$

Note that the operator $\mathfrak{G} : \overline{\mathscr{V}} \to C([0,1],\mathbb{R})$ is continuous and completely continuous. From the choice of $\mathscr{V}$, there is no $\kappa \in \partial\mathscr{V}$ with $\kappa = \lambda\mathfrak{G}\kappa$ for some $\lambda \in [0,1]$. Consequently, by Lemma 2.9, we conclude that $\mathfrak{G}$ has a fixed point $\kappa \in \overline{\mathscr{V}}$ which is a solution of the problem (1.1). □

4. Examples

Example 4.1. *Consider the following sequential fractional boundary value problem involving Caputo-type derivative :*

$$\begin{cases} ({}^cD^{\frac{5}{2}} + 2\,{}^cD^{\frac{3}{2}})\kappa(\tau) = f(\tau,\kappa(\tau),{}^cD^{\frac{1}{2}}\kappa(\tau)), \quad \tau \in [0,1], \\ \kappa(0)+\kappa(1) = I^{\frac{1}{2}}\kappa(\frac{1}{2}) + I^{\frac{3}{2}}\kappa(\frac{1}{2}), \\ \kappa^{'}(0)+\kappa^{'}(1) = {}^cD^{\frac{1}{2}}\kappa(\frac{1}{2}) + {}^cD^{\frac{3}{2}}\kappa(\frac{1}{2}), \\ \kappa^{''}(0) = 0. \end{cases} \tag{4.1}$$

Here $\nu = \frac{3}{2}$, $\omega = 2$, $\beta = \gamma = 1$, $\mu = \frac{1}{2}$.
With the given values, it is found that
$\Theta_1 \simeq 1.202115439, \Theta_2 \simeq 0.936153918, \Theta_3 \simeq -0.202115439$, *where* Θ_1, Θ_2 *and* Θ_3 *defined by (2.2).*
We take
$f(\tau,\kappa(\tau),{}^cD^{\frac{1}{2}}\kappa(\tau)) = \frac{2}{5(\tau^2+42)}\left(\cos(\kappa(\tau)+1) + \frac{3\left|{}^cD^{\frac{1}{2}}\kappa(\tau)\right|}{3+{}^cD^{\frac{1}{2}}\kappa(\tau)} + e^{-\tau}\sin\tau\right)$ *in (4.1). Then*

$$\begin{aligned} &\left|f(\tau,\kappa_1(\tau),{}^cD^{\frac{1}{2}}\kappa_1(\tau)) - f(\tau,\kappa_2(\tau),{}^cD^{\frac{1}{2}}\kappa_2(\tau))\right| \\ &\le \frac{2}{5(\tau^2+42)}\left(|\cos(\kappa_1(\tau)+1) - \cos(\kappa_2(\tau)+1)| + \left|\frac{3\left|{}^cD^{\frac{1}{2}}\kappa_1(\tau)\right|}{3+{}^cD^{\frac{1}{2}}\kappa_1(\tau)} - \frac{3\left|{}^cD^{\frac{1}{2}}\kappa_2(\tau)\right|}{3+{}^cD^{\frac{1}{2}}\kappa_2(\tau)}\right|\right) \\ &\le \frac{2}{5(\tau^2+42)}\left(|\kappa_1(\tau)-\kappa_2(\tau)| + \left|{}^cD^{\frac{1}{2}}\kappa_1(\tau) - {}^cD^{\frac{1}{2}}\kappa_2(\tau)\right|\right) \\ &\le q\left(\|\kappa_1-\kappa_2\| + \left\|{}^cD^{\frac{1}{2}}\kappa_1 - {}^cD^{\frac{1}{2}}\kappa_2\right\|\right), \end{aligned}$$

with $q = \frac{1}{105}$, *and*

$$\begin{aligned} \left|f(\tau,\kappa(\tau),{}^cD^{\frac{1}{2}}\kappa(\tau))\right| &= \left|\frac{2}{5(\tau^2+42)}\right|\left|\cos(\kappa(\tau)+1) + \frac{3\left|{}^cD^{\frac{1}{2}}\kappa(\tau)\right|}{3+{}^cD^{\frac{1}{2}}\kappa(\tau)} + e^{-\tau}\sin\tau\right| \\ &\le \frac{2(4+e^{-\tau}\sin\tau)}{5(\tau^2+42)} = \theta(\tau). \end{aligned}$$

We found $\widetilde{\Pi_1} \simeq 2.412349768$ *and* $\widetilde{\Pi_2} \simeq 1.905439962$ *(*$\widetilde{\Pi_1}, \widetilde{\Pi_2}$ *defined by (3.3)). Further* $q\left(\widetilde{\Pi_1} + \frac{\widetilde{\Pi_2}}{\Gamma(3-\nu)}\right) \simeq 0.043451509 < 1$. *Thus, all the conditions of Theorem 3.1 are fulfilled. Hence, the problem (4.1) has a solution on* $[0,1]$.

Example 4.2. *Consider the problem (4.1) and*

$$f(\tau,\kappa(\tau),{}^cD^{\frac{1}{2}}\kappa(\tau)) = \frac{1}{2\sqrt{\tau+729}}\left(\tan^{-1}(\kappa(\tau)+1)+\ln(\left|{}^cD^{\frac{1}{2}}\kappa(\tau)\right|+2)\right).$$

Clearly, we get

$$\begin{aligned}\left|f(\tau,\kappa(\tau),{}^cD^{\frac{1}{2}}\kappa(\tau))\right| &\le \left|\frac{1}{2\sqrt{\tau+729}}\right|\left|\tan^{-1}(\kappa(\tau)+1)+\ln(\left|{}^cD^{\frac{1}{2}}\kappa(\tau)\right|+2)\right| \\ &\le \frac{1}{2\sqrt{\tau+729}}\left(|\kappa(\tau)|+\left|{}^cD^{\frac{1}{2}}\kappa(\tau)\right|+3\right) \\ &\le \mathscr{J}(\tau)\mathscr{R}(\|\kappa\|_{\mathfrak{X}}),\end{aligned}$$

where $\mathscr{J}(\tau)=\frac{1}{2\sqrt{\tau+729}}, \mathscr{R}(\|\kappa\|_{\mathfrak{X}})=\|\kappa\|_{\mathfrak{X}}+3.$
With the obove assumption, we can obtain $\Pi_1 \simeq 2.737572985$, $\Pi_2 \simeq 3.308139177$ *(*Π_1,Π_2 *defined by (3.2)),* $\|\mathscr{J}\| = \frac{1}{54}$*.*
By the use of condition ($\mathfrak{B}_4$)*, we find* $N > 0.408402939$*. Hence by Theorem 3.2, the problem (4.1) has a solution on* $[0,1]$.

5. Conclusion

In this paper, we investigate the existence of solutions for a sequential FDEs with integro-differential bounday conditions. Our study is based on Krasnoselskii's fixed point theorem and the Leray-Schauder fixed point theorem under some suitable conditions.

Our research can be extented to the inclusion form of our considerd problem by applying the multivalued fixed point theorems such as the nonlinear alternative of Leray-Schauder type for Kakutani maps, the fixed point theorem contraction multivalued maps due to Covitz and Nadler.

For future works, we plan to investigate the existence results of these equations involving other fractional derivatives, such as Caputo-Hadamard and Hilfer. The stochastic versions of the sequential FDEs will be among the aim of our forthcoming studies. Furthermore, we will study the systems of nonlinear sequential FDEs with deviated arguments by employing numerical methods to approximate their solutions.

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