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ON STAKHOV FUNCTIONS AND NEW HYPERBOLOID SURFACES

Abstract. This paper presents an investigation into the generalization of hyperbolic Fibonacci sine and cosine functions, as well as Fibonacci spirals. Initially, we establish the main definitions and theoretically model them, listing several special cases. We then uncover fundamental results, including the De Moivre and Pythagorean formulas. Based on these new definitions, we introduce new classes of three-dimensional hyperboloid surfaces and compute their Gauss and mean curvatures. Notably, we demonstrate that these surfaces are geodesic.

1. Introduction

The usual Fibonacci numbers are defined by the following recurrence relation: for $n \geqslant 0$

$$F_{n+2} = F_{n+1} + F_n, \tag{1.1}$$

where $F_0 = 0$ and $F_1 = 1$. These numbers can also be produced by using the Binet's formula in the form of

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \tag{1.2}$$

where α and β are the positive and negative roots of the equation $x^2 - x - 1 = 0$, respectively.

In the literature, many interesting properties and applications of the recurrence sequences have been studied by many authors; see for example, [1], [2], [3]. In 1993, the Ukrainian mathematicians Stakhov and Tkachenko put forth a new idea to describe hyperbolic geometry [4]. Inspired by the Binet's formula in Eq. (1.2), the authors introduced a new class of hyperbolic functions, which are called the Hyperbolic Fibonacci and Lucas functions. In [5], Stakhov provided detailed information with applications to the available literature. In [6], Stakhov and Rozin further developed the ideas of the hyperbolic Fibonacci and Lucas functions, and

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defined the symmetric hyperbolic Fibonacci and Lucas functions as follows:

$$sFs\left(x\right)=\frac{\alpha^{x}-\alpha^{-x}}{\sqrt{5}},\ cFs\left(x\right)=\frac{\alpha^{x}+\alpha^{-x}}{\sqrt{5}},\ sLs\left(x\right)=\alpha^{x}-\alpha^{-x},\\ \text{and}\, cLs\left(x\right)=\alpha^{x}+\alpha^{-x}, \tag{1.3}$$

where x is any real numbers. To be clear, for $k\in Z$, $sFs\left(2k\right)=F_{2k}$, $cLs\left(2k\right)=L_{2k}$, $cFs\left(2k+1\right)=F_{2k+1}$, and $sLs\left(2k+1\right)=L_{2k+1}$. In [7], Stakhov and Rozin defined the quasi-sine Fibonacci functions and Fibonacci spirals to eliminate the discrete case in Eq. (1.3) as follows:

$$FF\left(x\right)=\frac{\alpha^{x}-\cos\left(\pi x\right)\alpha^{-x}}{\sqrt{5}}\ \text{and}\, CFF\left(x\right)=\frac{\alpha^{x}-\cos\left(\pi x\right)\alpha^{-x}}{\sqrt{5}}+i\frac{\sin\left(\pi x\right)\alpha^{-x}}{\sqrt{5}}, \tag{1.4}$$

where i is the complex unit. Note that in [8], Stakhov and Rozin presented a brief description of these hyperbolic phenomenons in the world.

According to these developments, in [9], Falcón and Plaza defined a new class of hyperbolic sine&cosine, quasi-sine, and spiral-like functions using the k-Fibonacci sequence as follows:

$$sF_{k}h\left(x\right)=\frac{\sigma^{x}-\sigma^{-x}}{\sigma+\sigma^{-1}},cF_{k}h\left(x\right)=\frac{\sigma^{x}+\sigma^{-x}}{\sigma+\sigma^{-1}},FF_{k}h\left(x\right)=\frac{\sigma^{x}-\cos\left(\pi x\right)\sigma^{-x}}{\sigma+\sigma^{-1}},\\ \text{and}\, CFF_{k}\left(x\right)=\frac{\sigma^{x}-\cos\left(\pi x\right)\sigma^{-x}}{\sigma+\sigma^{-1}}+i\frac{\sin\left(\pi x\right)\sigma^{-x}}{\sigma+\sigma^{-1}}, \tag{1.5}$$

where σ is the positive root of $\sigma^{2}=k\sigma+1$ and k is any positive real number. Motivated by the definitions of Stakhov and Rozin [6,7], and Falcón and Plaza [9], Daşdemir et al. gave a generalized version of the functions in Eqs. (1.3)-(1.4) as follows [10]:

$$\mathcal{H}_{s}\left(x\right)=\frac{A\alpha^{x}-B\alpha^{-x}}{\Delta},\ \mathcal{H}_{c}\left(x\right)=\frac{A\alpha^{x}+B\alpha^{-x}}{\Delta},\ \mathcal{H}\left(x\right)=\frac{A\alpha^{x}-\cos\left(\pi x\right)B\alpha^{-x}}{\Delta},\\ \text{and}\, \mathcal{CH}\left(x\right)=\frac{A\alpha^{x}-\cos\left(\pi x\right)B\alpha^{-x}}{\Delta}+i\frac{\sin\left(\pi x\right)B\alpha^{-x}}{\Delta}, \tag{1.6}$$

which are called the Horadam hyperbolic sine function, the Horadam hyperbolic cosine function, the quasi-sine Horadam function, and Horadam spiral, respectively. Here, α is the positive root of $\lambda^{2}=f\left(x\right)\lambda+1$, $\Delta=\sqrt{f^{2}\left(x\right)+4}$, $A=b\left(x\right)+a\left(x\right)\alpha^{-1}$, $B=b\left(x\right)-a\left(x\right)\alpha$, and $a\left(x\right)$ and $b\left(x\right)$ are any continue real-valued function.

As the above literature survey reveals, the functions in (1.3)-(1.5) only vary on the real variable x, while other parameters are constant. In Eq. (1.6), the parameters α and β depend on a continuous function of x. Consequently, it would be interesting to consider the mentioned functions in a more general form such that the roots of the algebraic equation depend on two real-valued functions. To be clear, this consideration is due to the generalized second-order sequence designated by Horadam [11]. This motivates us to revise the mentioned functions. For this purpose, presented herein is to generalize the definitions introduced by Stakhov and Tkachenko [4], Stakhov and Rozin [5], Falcón and Plaza [9], and Daşdemir et. al [10]. This is the main focus of the present paper, and a particular concern will be paid to some elementary results and geometrical considerations.

2. Main Results

In this section, we will present the outcomes of the paper, including some definitions, fundamental considerations, and elementary properties.

2.1. **Fundamental definitions.** Let $f(x)$ and $g(x)$ be an arbitrary non-zero continuous functions of real number x. Consider the second-order equation

$$\lambda^2 - f(x)\lambda - g(x) = 0. \tag{2.1}$$

Hence, Eq. (2.1) has the following distinct two roots

$$\lambda_1 = \alpha(x) = \frac{f(x) + \sqrt{f^2(x) + 4g(x)}}{2} \text{ and } \lambda_2 = \beta(x) = \frac{f(x) - \sqrt{f^2(x) + 4g(x)}}{2}.$$

To ensure that the solution is real, we assume that condition $f^2(x) + 4g(x) > 0$ is met. Here, we can write

$$\alpha(x) + \beta(x) = f(x), \ \alpha(x)\beta(x) = -g(x), \alpha(x) - \beta(x) = \Delta(x), \tag{2.2}$$

where $= \Delta(x) = \sqrt{f^2(x) + 4g(x)}$. In consequence, we obtained two distinct solutions. Therefore, their linear combination, i.e., $c_1\{\alpha(x)\}^x + c_2\{\beta(x)\}^x$, is also a solution of Eq. (2.1). Solving the system of equations for $x = 0$ and $x = 1$, we find

$$c_1 = \frac{b(x) - a(x)\beta(x)}{\alpha(x) - \beta(x)} \text{ and } c_2 = -\frac{b(x) - a(x)\alpha(x)}{\alpha(x) - \beta(x)}.$$

As a result, we can give the following definition.

Definition 2.1. *Let $a(x)$ and $b(x)$ be an arbitrary continuous function. Then, the Horadam functions are defined as*

$$\mathcal{H}(a, b, f, g, x) = \mathcal{H}(x) = \frac{\tilde{A}(x)[\alpha(x)]^x - \tilde{B}(x)[\beta(x)]^x}{\alpha(x) - \beta(x)}, \tag{2.3}$$

where $\tilde{A}(x) = b(x) + a(x)[\alpha(x)]^{-1}$ and $\tilde{B}(x) = b(x) - a(x)\alpha(x)$.

This is a similar form to the generalized second-order sequence given by Horadam [11]. We can, therefore, call Eq. (2.3) the *Horadam functions* due to Australian mathematician Alwyn Francis Horadam's great contributions to the available literature. Note that, for the sake of presentation simplicity, all the functions will be represented in the non-parentheses form.

Substituting $\alpha\beta = -g$ into Eq. (2.3), we can write

$$\mathcal{H}(x) = \frac{\tilde{A}\alpha^x - \tilde{B}\left(-g\alpha^{-1}\right)^x}{\Delta} = \frac{\tilde{A}\alpha^x - (-1)^x\tilde{B}g^x\alpha^{-x}}{\Delta},$$

Here, we run into the problem of what the real power of -1 will be. To address this issue, from the famous Euler's formula, we can write $e^{\mp i\pi} = \cos\pi \mp i\sin\pi = -1$, where e is Euler's constant and i is the imaginary unit. As a result, we can give the following definition.

Definition 2.2. *Let a and b be any continuous function. Then, the Stakhov spiral is defined as*

$$\mathcal{SR}(a, b, f, g, x) = \mathcal{SR}(x) = \frac{A\alpha^x - \cos(\pi x)B\alpha^{-x}}{\Delta} + i\frac{\sin(\pi x)B\alpha^{-x}}{\Delta}, \tag{2.4}$$

where $A = b + a\alpha^{-1}$ and $B = g^x(b - a\alpha)$.

Table 1. Special cases for the Stakhov spiral functions

Stakhov spiral functions	Symbols $\mathcal{SR}(x)$	a	b	f	g
Horadam spiral [10]	$\mathcal{H}(x)$				1
Fibonacci spiral [6]	$\mathrm{CFF}(x)$	0	1	1	1
Lucas spiral	$\mathrm{CLL}(x)$	2	1	1	1
k-Fibonacci spiral [9]	$\mathrm{CFF}_k(x)$	0	1	k	1
Pell spiral	$\mathrm{CPP}(x)$	0	1	2	1
Modified Pell spiral	$\mathrm{CRR}(x)$	1	1	2	1
Pell-Lucas spiral	$\mathrm{CQQ}(x)$	2	2	2	1
Jacobsthal spiral	$\mathrm{CJJ}(x)$	0	1	1	2
Jacobsthal-Lucas spiral	$\mathrm{CJJL}(x)$	2	1	1	2
Fermat spiral	$\mathrm{CFFR}(x)$	1	3	3	2

This looks like a three-dimensional spiral-like curve and is the most general form of Eq. (1.4) given by Stakhov and Rozin [7]. Table 1 indicates the special cases that can be obtained depending on the particular choice of a, b, f, and g. For integer values of x, the imaginary part of the function $\mathcal{SR}(x)$ vanishes. The reason for the name "Stakhov spiral function" is that the great Ukrainian mathematician Stakhov has attributed an indescribable contribution both to the subject of this paper and to the literature on Fibonacci numbers.

For the concrete examples, we consider the following cases:

$$\text{Case I}: a(x) = \sin x,\ b(x) = \cos x,\ f(x) = \ln\left(1+x^2\right),\ \text{and } g(x) = \cosh(x)$$

$$\text{Case II}: a(x) = \sqrt[3]{x},\ b(x) = x,\ f(x) = \operatorname{arcsinh}\left(1+x^2\right),\ \text{and } g(x) = e^{-x}$$

Fig. 1 displays the three-dimensional graphs of the Stakhov spirals for Case I (Fig. 1.a) and Case II (Fig. 1.b), respectively. As seen, the distributions are a spiral-like curve.

Under the assumption that the $Oy-$ and $Oz-$axes are real and imaginary directions, respectively, we can build up the following system of equations:

$$\begin{cases} y - \frac{A\alpha^x}{\Delta} = -\frac{\cos(\pi x)B\alpha^{-x}}{\Delta} \\ z = \frac{\sin(\pi x)B\alpha^{-x}}{\Delta} \end{cases}$$

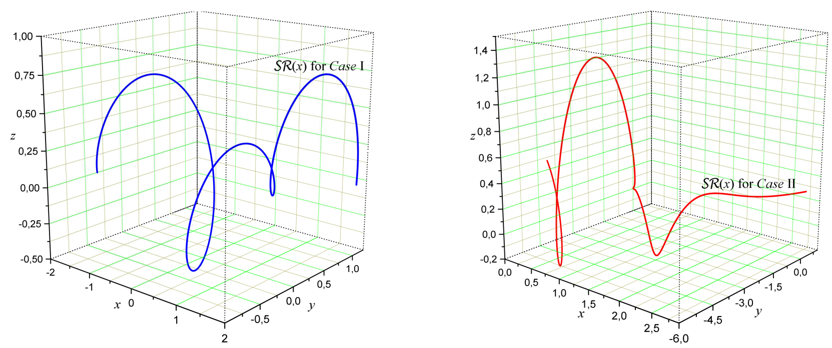
Thus, after some operations, we get the following equation:

$$\left(y - \frac{A\alpha^x}{\Delta}\right)^2 + z^2 = \left(\frac{B\alpha^{-x}}{\Delta}\right)^2 \tag{2.5}$$

or in the re-organized form

$$z^2 = \left(\frac{A\alpha^x + B\alpha^{-x}}{\Delta} - y\right)\left(y - \frac{A\alpha^x - B\alpha^{-x}}{\Delta}\right). \tag{2.6}$$

Note that Eq. (2.4) is a complex-valued function. However, we are usually not concerned with what is going on in the imaginary axis, as we generally work in real space. This idea coincides also with the approaches by the references [6], [9], and [10]. Thus, considering the real part of the Stakhov spiral functions, we can express the following definition.



(a) Case I **(b)** Case II

Figure 1. Distributions of the Stakhov spirals for Cases I and II

Definition 2.3. *For the functions a and b, the quasi-sine Stakhov function is defined by*

$$\mathcal{S}_q\left(a,b,f,g,x\right)=\mathcal{QS}\left(x\right)=\frac{A\alpha^x-\cos\left(\pi x\right)B\alpha^{-x}}{\Delta}. \tag{2.7}$$

Meanwhile, Eqs. (2.6) and (2.7) are whispering some new definitions. In Eq. (2.6), each function in parentheses has a hyperbolic structure. Besides, combining these functions utilizing the character of $\cos\left(\pi x\right)$ yields Eq. (2.7). Based on this idea, we can then discretize the function $\mathcal{S}_q\left(x\right)$ as follows.

Definition 2.4. *Let a and b be any continuous function. Then, the hyperbolic Stakhov sine and cosine functions are defined by*

$$\mathcal{S}_s\left(a,b,f,g,x\right)=\mathcal{S}_s\left(x\right)=\frac{A\alpha^x-B\alpha^{-x}}{\Delta} \tag{2.8}$$

and

$$\mathcal{S}_c\left(a,b,f,g,x\right)=\mathcal{S}_c\left(x\right)=\frac{A\alpha^x+B\alpha^{-x}}{\Delta}, \tag{2.9}$$

respectively.

From the last definitions, we can give the following additional definitions.

Definition 2.5. *Let a and b be continuous. Then, the hyperbolic Stakhov tangent and cotangent functions are defined by*

$$\mathcal{S}_t\left(a,b,f,g,x\right)=\frac{\mathcal{S}_s\left(x\right)}{\mathcal{S}_c\left(x\right)}=\frac{A\alpha^x-B\alpha^{-x}}{A\alpha^x+B\alpha^{-x}}=1-\frac{2B}{B+A\alpha^{2x}}, \tag{2.10}$$

and

$$\mathcal{S}_{ct}\left(a,b,f,g,x\right)=\frac{\mathcal{S}_c\left(x\right)}{\mathcal{S}_s\left(x\right)}=\frac{A\alpha^x+B\alpha^{-x}}{A\alpha^x-B\alpha^{-x}}=1-\frac{2B}{B-A\alpha^{2x}}, \tag{2.11}$$

respectively.

In working with the above-stated functions, it is useful to consider the following special cases:

- Generalized Stakhov-Fibonacci spirals

$$\mathcal{SR}\left(0,1,f,g,x\right)=\mathcal{S}_u\left(x\right)=\frac{\alpha^x-\cos\left(\pi x\right)g^x\alpha^{-x}}{\Delta}+i\frac{\sin\left(\pi x\right)g^x\alpha^{-x}}{\Delta} \tag{2.12}$$

- Generalized Stakhov-Lucas spirals

$$\mathcal{SR}\left(2,f,f,g,x\right)=\mathcal{S}_{v}\left(x\right)=\alpha^{x}+\cos\left(\pi x\right)g^{x}\alpha^{-x}-i\sin\left(\pi x\right)g^{x}\alpha^{-x} \tag{2.13}$$

- Generalized quasi-sine Fibonacci functions

$$\mathcal{S}_{q}\left(0,1,f,g,x\right)=\operatorname{Re}\left(\mathcal{S}_{u}\left(x\right)\right)=\mathcal{U}_{q}\left(x\right)=\frac{\alpha^{x}-\cos\left(\pi x\right)g^{x}\alpha^{-x}}{\Delta} \tag{2.14}$$

- Generalized quasi-sine Lucas functions

$$\mathcal{S}_{q}\left(2,f,f,g,x\right)=\operatorname{Re}\left(\mathcal{S}_{v}\left(x\right)\right)=\mathcal{V}_{q}\left(x\right)=\alpha^{x}+\cos\left(\pi x\right)g^{x}\alpha^{-x} \tag{2.15}$$

- Generalized hyperbolic Fibonacci sine and cosine functions

$$\mathcal{S}_{s}\left(0,1,f,g,x\right)=\mathcal{F}_{s}\left(x\right)=\frac{\alpha^{x}-g^{x}\alpha^{-x}}{\Delta},\ \mathcal{S}_{c}\left(0,1,f,g,x\right)=\mathcal{F}_{c}\left(x\right)=\frac{\alpha^{x}+g^{x}\alpha^{-x}}{\Delta} \tag{2.16}$$

- Generalized hyperbolic Lucas cosine and sine functions

$$\mathcal{S}_{s}\left(2,f,f,g,x\right)=\mathcal{L}_{c}\left(x\right)=\alpha^{x}+g^{x}\alpha^{-x},\ \mathcal{S}_{c}\left(2,f,f,g,x\right)=\mathcal{L}_{s}\left(x\right)=\alpha^{x}-g^{x}\alpha^{-x} \tag{2.17}$$

- Generalized hyperbolic Lucas tangent and cotangent functions

$$\mathcal{S}_{t}\left(0,1,f,g,x\right)=\mathcal{F}_{t}\left(x\right)=\frac{\alpha^{x}-g^{x}\alpha^{-x}}{\alpha^{x}+g^{x}\alpha^{-x}},\ \mathcal{S}_{ct}\left(0,1,f,g,x\right)=\mathcal{F}_{ct}\left(x\right)=\frac{\alpha^{x}+g^{x}\alpha^{-x}}{\alpha^{x}-g^{x}\alpha^{-x}} \tag{2.18}$$

2.2. **Some features.** In this section, some elementary formulas regarding the hyperbolic Stakhov functions will bw developed. We can thus start with the following results.

Theorem 2.1. *The following non-homogeneous recurrence relations hold for any real number x:*

$$\mathcal{S}_{s}\left(x+2\right)=f\mathcal{S}_{c}\left(x+1\right)+g\mathcal{S}_{s}\left(x\right) \tag{2.19}$$

and

$$\mathcal{S}_{c}\left(x+2\right)=f\mathcal{S}_{s}\left(x+1\right)+g\mathcal{S}_{c}\left(x\right). \tag{2.20}$$

Proof. Substituting Eqs. (2.10) and (2.11) into Eq. (2.19) yields

$$\begin{aligned}f\mathcal{S}_{s}\left(x+1\right)+g\mathcal{S}_{c}\left(x\right)&=f\frac{A\alpha^{(x+1)}-B\alpha^{-(x+1)}}{\Delta}+g\frac{A\alpha^{x}+B\alpha^{-x}}{\Delta}\\&=\frac{A\alpha^{x}\left(f\alpha+g\right)+B\alpha^{-x}\left(1-\frac{f}{\alpha}\right)}{\Delta}.\end{aligned}$$

Considering Eqs. (2.1) and (2.2), we can write

$$\alpha^{2}=f\alpha+g\text{ and }1-\frac{f}{\alpha}=\frac{g}{\alpha^{2}},$$

which completes the proof. □

Remark. *Theorem 2.1 indicates that symmetric exchange between the functions $\mathcal{S}_{s}\left(x\right)$ and $\mathcal{S}_{c}\left(x\right)$ is possible in all linear relations of the hyperbolic Stakhov functions.*

The recurrence relations in Eqs. (2.19) and (2.20) can be homogenized as follows. Subtracting the resultant equations after multiplying the values of these functions for the real numbers x and $x+2$ with appropriate factors, we obtain

$$\mathcal{S}_s\left(x+2\right)=\left(f^2+2g\right)\mathcal{S}_s\left(x\right)-g^2\mathcal{S}_s\left(x-2\right) \tag{2.21}$$

and

$$\mathcal{S}_c\left(x+2\right)=\left(f^2+2g\right)\mathcal{S}_c\left(x\right)-g^2\mathcal{S}_c\left(x-2\right). \tag{2.22}$$

Note that the new recurrence relations have a forth-order homogeneous form.

The next theorem presents the inverse hyperbolic functions.

Theorem 2.2. *The hyperbolic Stakhov sine and cosine functions have an inverse in the form of*

$$S_s{}^{-1}\left(x\right)=\log_\alpha\left(\tilde{x}\right) \ \ \textit{and}\ S_c{}^{-1}\left(x\right)=\log_\alpha\left(\tilde{x}\right), \tag{2.23}$$

where $\tilde{x}=\frac{\Delta x+\sqrt{\Delta^2x^2+4AB}}{2A}$.

Proof. From the definition of the hyperbolic Stakhov functions, we can write

$$x=\frac{A\alpha^y-B\alpha^{-y}}{\Delta}\Rightarrow\Delta x\alpha^y=A\alpha^{2y}-B\Rightarrow A(\alpha^y)^2-\Delta x\alpha^y-B=0,$$

which is second-order equation. Since $\alpha^y>0$, there is a unique solution, namely $\alpha^y=\frac{\Delta x+\sqrt{\Delta^2x^2+4AB}}{2A}$. As a result, the first equation is obtained. The latter can also be proved after a similar process. □

We give the Pythagorean formula for hyperbolic Stakhov functions.

Theorem 2.3 (Pythagorean formula)**.** *For any real number x, we have*

$$\left[\mathcal{S}_c\left(x\right)\right]^2-\left[\mathcal{S}_s\left(x\right)\right]^2=\frac{4AB}{\Delta^2}. \tag{2.24}$$

Proof. Subtracting the resultant equations after substituting Eqs. (2.8) and (2.9) into the left-hand side of Eq. (2.24) yields the claimed result. □

The next theorem presents a similar result to the famous De Moivre's formula.

Theorem 2.4 (De Moivre-type formula)**.** *Let x be any real number. Then the following identities hold for any positive integer n:*

$$\left[\mathcal{S}_c\left(x\right)+\mathcal{S}_s\left(x\right)\right]^n=\left(\frac{2A}{\Delta}\right)^{n-1}\left[\mathcal{S}_c\left(nx\right)+\mathcal{S}_s\left(nx\right)\right] \tag{2.25}$$

and

$$\left[\mathcal{S}_c\left(x\right)-\mathcal{S}_s\left(x\right)\right]^n=\left(\frac{2B}{\Delta}\right)^{n-1}\left[\mathcal{S}_c\left(nx\right)-\mathcal{S}_s\left(nx\right)\right]. \tag{2.26}$$

Proof. We use the induction method to show the validity of theorem. It is clear that Eq. (2.25) holds for $n=1$. Based on the assumption such that this equation is valid for any positive integer n, we can write

$$\begin{aligned}\left[\mathcal{S}_c\left(x\right)+\mathcal{S}_s\left(x\right)\right]^{n+1}&=\left[\mathcal{S}_c\left(x\right)+\mathcal{S}_s\left(x\right)\right]^n\left[\mathcal{S}_c\left(x\right)+\mathcal{S}_s\left(x\right)\right]\\&=\left(\frac{2A}{\Delta}\right)^{n-1}\left[\mathcal{S}_c\left(nx\right)+\mathcal{S}_s\left(nx\right)\right]\left[\frac{A\alpha^x+B\alpha^{-x}}{\Delta}-\frac{A\alpha^x-B\alpha^{-x}}{\Delta}\right]\\&=\left(\frac{2A}{\Delta}\right)^n\left[\frac{A\alpha^{nx}+B\alpha^{-nx}}{\Delta}+\frac{A\alpha^{nx}-B\alpha^{-nx}}{\Delta}\right]\alpha^x\end{aligned}$$

$$= \left(\frac{2A}{\Delta}\right)^n \left[\frac{2A\alpha^{(n+1)x} + B\alpha^{-(n+1)x} - B\alpha^{-(n+1)x}}{\Delta}\right]$$
$$= \left(\frac{2A}{\Delta}\right)^n \left[\mathcal{S}_c\left((n+1)\,x\right) + \mathcal{S}_s\left((n+1)\,x\right)\right],$$

which completes the proof process. Repeating the same procedure, the other can be demonstrated. □

Up to now, we only make our investigation for real values of x. So, what properties will the hyperbolic Stakhov functions have in the complex space? The answer is presented in the following.

Theorem 2.5. *For complex variable $z = x + iy$, we have*

$$\mathcal{S}_s\left(z\right) = \frac{1}{2AB}\left[u\cos\left(y\ln\alpha\right) + iv\sin\left(y\ln\alpha\right)\right] \tag{2.27}$$

and

$$\mathcal{S}_c\left(z\right) = \frac{1}{2AB}\left[v\cos\left(y\ln\alpha\right) + iu\sin\left(y\ln\alpha\right)\right], \tag{2.28}$$

where $u = (A+B)\,\mathcal{S}_s\left(x\right) - (A-B)\,\mathcal{S}_c\left(x\right)$ and $v = (A+B)\,\mathcal{S}_c\left(x\right) - (A-B)\,\mathcal{S}_s\left(x\right)$.

Proof. Considering

$$\mathcal{S}_s\left(z\right) = \frac{A\alpha^z - B\alpha^{-z}}{\Delta} \text{ and } \mathcal{S}_c\left(z\right) = \frac{A\alpha^z + B\alpha^{-z}}{\Delta},$$

we can write

$$\begin{aligned} A\alpha^z = A\alpha^{x+iy} = A\alpha^x\alpha^{iy} &= \frac{\Delta}{2A}\left(\mathcal{S}_c\left(x\right) + \mathcal{S}_s\left(x\right)\right)\alpha^{iy} \\ &= \frac{\Delta}{2A}\left(\mathcal{S}_s\left(x\right) + \mathcal{S}_c\left(x\right)\right)\left[\cos\left(y\ln\alpha\right) + i\sin\left(y\ln\alpha\right)\right] \end{aligned}$$

and

$$\begin{aligned} B\alpha^{-z} = B\alpha^{-x-iy} = B\alpha^{-x}\alpha^{-iy} &= \frac{\Delta}{2B}\left(\mathcal{S}_c\left(x\right) - \mathcal{S}_s\left(x\right)\right)\alpha^{-iy} \\ &= \frac{\Delta}{2B}\left(\mathcal{S}_c\left(x\right) - \mathcal{S}_s\left(x\right)\right)\left[\cos\left(y\ln\alpha\right) - i\sin\left(y\ln\alpha\right)\right]. \end{aligned}$$

Here, we used the well-known Euler's formula. As a result, combining the last two equations, the proof is completed. □

As an example, we give the following special case.

Example 2.1. *Consider $z = \frac{i\pi}{\ln\alpha}$. Let us compute $\mathcal{S}_s\left(z\right)$ and $\mathcal{S}_c\left(z\right)$. In this case, $x = 0$ and $y = \frac{\pi}{\ln\alpha}$. Inserting these values into Eqs.* (2.32) *and* (2.35)*, we can readily obtain $\mathcal{S}_s\left(z\right) = 0$ and $\mathcal{S}_c\left(z\right) = -\frac{2}{\Delta}$.*

2.3. **Geometrical considerations.** In this section, some geometrical approaches will be developed. For this purpose, we first introduce the following equations:

$$x = \mp\frac{A\alpha^t - B\alpha^{-t}}{\Delta} \text{ and } y = \frac{A\alpha^t + B\alpha^{-t}}{\Delta}, \tag{2.29}$$

where the parameter t is the hyperbolic angle. From this, we can write

$$y^2 - x^2 = \frac{4AB}{\Delta^2}$$

or

$$\frac{y^2}{\left(\frac{2\sqrt{AB}}{\Delta}\right)^2} - \frac{x^2}{\left(\frac{2\sqrt{AB}}{\Delta}\right)^2} = 1, \tag{2.30}$$

which is a rectangular hyperbola equation, with center at the origin, and with a horizontal real axis. We may call Eq. (2.30) the Stakhov hyperbola. Note that the exchange of preferences in Eq. (2.29) yields to obtain the equation of the conjugate hyperbola.

According to Eq. (2.30), the foci, vertices, and co-vertices of the hyperbola lie in $\left(\mp\frac{2\sqrt{2AB}}{\Delta}, 0\right)$, $\left(\mp\frac{2\sqrt{AB}}{\Delta}, 0\right)$, and $\left(0, \mp\frac{2\sqrt{AB}}{\Delta}\right)$, respectively. In addition, the equations of asymptotes and directrices are $y = \mp x$ and $x = \mp\frac{\sqrt{2AB}}{\Delta}$. In particular, the Modified Pell hyperbola is unit. This means that since the Pseudo Euclidean plane is represented by a unit hyperbola that also describes Minkowski space-time, the Modified Pell hyperbola can be used.

On the other hand, rotating the Stakhov hyperbola completely around the vertical axis generates a hyperboloid of one sheet. In this case, we have the equation

$$\frac{x^2}{\left(\frac{2\sqrt{AB}}{\Delta}\right)^2} + \frac{y^2}{\left(\frac{2\sqrt{AB}}{\Delta}\right)^2} - \frac{z^2}{\left(\frac{2\sqrt{AB}}{\Delta}\right)^2} = 1 \tag{2.31}$$

and its parametric representation is

$$\begin{cases} x = \frac{2\sqrt{AB}}{\Delta}\mathcal{S}_c(t)\cos\theta \\ y = \frac{2\sqrt{AB}}{\Delta}\mathcal{S}_c(t)\sin\theta \\ z = \frac{2\sqrt{AB}}{\Delta}\mathcal{S}_s(t) \end{cases} \tag{2.32}$$

where θ is azimuth angle and $t \in [0, \infty)$. By the way, this may be called the hyperbolic Stakhov hyperboloid. Further, for $z \in [0, \infty)$, the projection of the hyperbolic Stakhov hyperboloid on xy-plane is a planar spiral that looks like an Archimedean spiral.

If rotation is made along the horizantal axis, an hyperboloid of two sheets occurs. So, we have the hyperboloid equation as follows:

$$\frac{x^2}{\left(\frac{2\sqrt{AB}}{\Delta}\right)^2} + \frac{y^2}{\left(\frac{2\sqrt{AB}}{\Delta}\right)^2} - \frac{z^2}{\left(\frac{2\sqrt{AB}}{\Delta}\right)^2} = -1 \tag{2.33}$$

and its parametric representation is

$$\begin{cases} x = \frac{2\sqrt{AB}}{\Delta}S_s(t)\cos\theta \\ y = \frac{2\sqrt{AB}}{\Delta}S_s(t)\sin\theta \\ z = \frac{2\sqrt{AB}}{\Delta}S_c(t) \end{cases} \tag{2.34}$$

Similarly, the last surface may be called the elliptic Stakhov hyperboloid. Since the denominators are equal, there is a hyperboloid of revolution in both cases. It should be noted that there are no umbilics on a hyperboloid of one sheet, but two on each sheet of the two-sheeted variety.

Fig. 2 displays special forms of the hyperbolic and elliptic Stakhov hyperboloids for cases where the hyperbolic and elliptic Fibonacci (Fig. 2. a & b), the hyperbolic

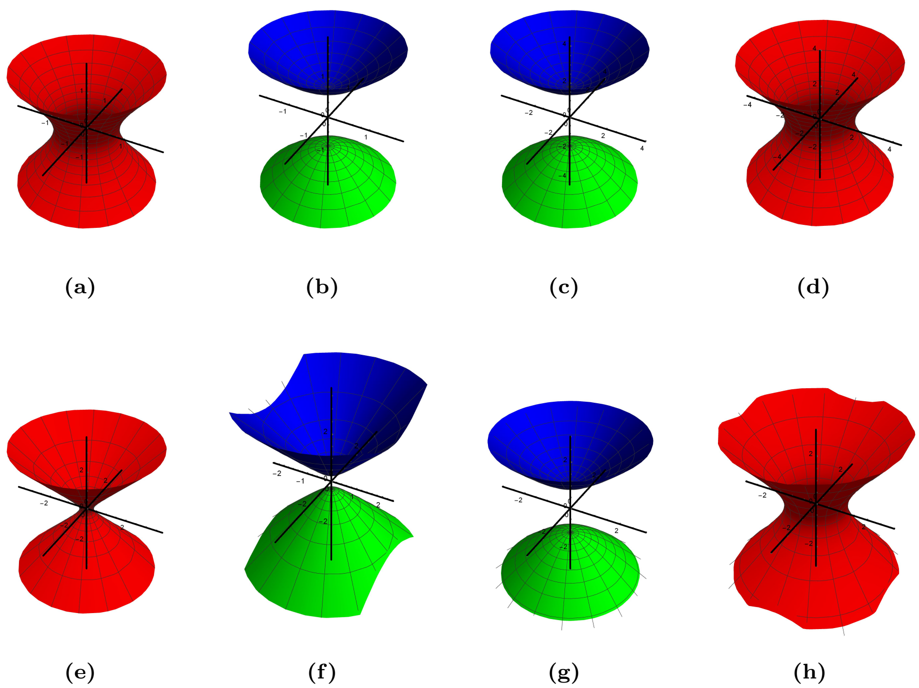


Figure 2. Some special surfaces: (a) Hyperbolic Fibonacci, (b) Elliptic Fibonacci, (c) Hyperbolic Lucas, (d) Elliptic Lucas, (e) Hyperbolic Pell, (f) Elliptic Pell, (g) Hyperbolic Modified Pell, and (h) Elliptic Modified Pell

and elliptic Lucas (Fig. 2. c & d), the hyperbolic and elliptic Pell (Fig. 2. e & f), and the hyperbolic and elliptic Modified Pell (Fig. 2. g & h) hyperboloids according to Table 1. Fig. 2 reveals that oscillating characters of the Fibonacci and Pell surface symmetrically exchange with the ones of Lucas and Modified Pell.

Theorem 2.6. *The Gaussian and mean curvatures of the hyperbolic Stakhov hyperboloid are given by*

$$K_h = -\frac{1}{\left[\{S_s(t)\}^2 + \{S_c(t)\}^2\right]^2} \; and \, H_h = \frac{\Delta [S_s(t)]^2}{2\sqrt{AB}\left[\{S_s(t)\}^2 + \{S_c(t)\}^2\right]^{\frac{3}{2}}}. \quad (2.35)$$

Proof. It is clear that the hyperbolic Stakhov hyperboloid is a regular surface with a differentiable field of unit normal vectors N. Let us compute the coefficients of the first and second fundamental forms of hyperboloid so that the Gaussian and mean curvatures can be obtained in terms of whose coefficients. To do this, we shall express dN as a matrix in terms of the natural basis X_u, X_v, where $X(t,\theta) = \frac{2\sqrt{AB}}{\Delta}(S_c(t)\cos\theta, S_c(t)\sin\theta, S_s(t))$. In this case, the coefficients of the first fundamental form are given by

$$E = \langle X_t, X_t \rangle = \left\{ \tfrac{2\sqrt{AB}\ln\alpha}{\Delta} \right\}^2 \left[\{S_s(t)\}^2 + \{S_c(t)\}^2 \right],$$
$$F = \langle X_t, X_\theta \rangle = 0, \text{ and } G = \langle X_\theta, X_\theta \rangle = \left\{ \tfrac{2\sqrt{AB}}{\Delta} \right\}^2 \{S_c(t)\}^2.$$

Besides, we can write down

$$N = \frac{X_t \times X_\theta}{\|X_t \times X_\theta\|} = \frac{-iS_c(t)\cos\theta - jS_c(t)\sin\theta + kS_s(t)}{\left\{ \{S_c(t)\}^2 + \{S_s(t)\}^2 \right\}^{\frac{1}{2}}},$$

and from this, the coefficients of the second fundamental form can be computed as

$$e = \langle N, X_{tt} \rangle = -\tfrac{8AB\sqrt{AB}(\ln\alpha)^2}{\Delta^3\left\{ \{S_c(t)\}^2 + \{S_s(t)\}^2 \right\}^{\frac{1}{2}}}, \; f = \langle N, X_{t\theta} \rangle = 0,$$
$$\text{and } g = \langle N, X_{\theta\theta} \rangle = \tfrac{2\sqrt{AB}\{S_c(t)\}^2}{\Delta\left\{ \{S_c(t)\}^2 + \{S_s(t)\}^2 \right\}^{\frac{1}{2}}}.$$

Considering

$$K = \frac{eg - f^2}{EG - F^2} \text{ and } H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2},$$

the results follows after some mathematical operations. □

Since $K < 0$, the principal curvatures κ_1 and κ_2 are of opposite sign at any point P. So the surface near P is a hyperboloid. We can call P a hyperbolic point of the surface.

Theorem 2.7. *The Gaussian and mean curvatures of the elliptic Stakhov hyperboloid are given by*

$$K_e = \frac{1}{\left[\{S_s(t)\}^2 + \{S_c(t)\}^2 \right]^2} \text{ and } H_e = \frac{\Delta [S_c(t)]^2}{2\sqrt{AB}\left[\{S_s(t)\}^2 + \{S_c(t)\}^2 \right]^{\frac{3}{2}}}. \quad (2.36)$$

Proof. Repeating the same procedure in the previous theorem, the proof can easily be done. □

Since $K > 0$, the sings of the principal curvatures κ_1 and κ_2 are the same. The normal curvature κ in any tangent direction t is equal to $\kappa = \kappa_1 \cos\theta + \kappa_2 \sin\theta$, where θ is the angle between t and the principal vector corresponding to κ_1. So the sign of κ is the same as that of κ_1 and κ_2. The surface is bending away from its tangent plane in all tangent directions at any point P. The quadratic approximation of the surface near P is the paraboloid $z^2 = \kappa_1 x^2 + \kappa_2 y^2$. In addition, the Gaussian curvatures K of two surfaces are invariant by local isometries.

Theorem 2.8. *Both hyperbolic and elliptic Stakhov hyperboloids are geodesic. To be clear, we have*

$$\kappa_h = 0 \text{ and } \kappa_e = 0. \quad (2.37)$$

Proof. We only present proof for the hyperbolic Stakhov hyperboloid here. Other can be proved similarly. Let χ be cut out of the hyperbolic Stakhov hyperboloid by the form $z = c$. In this case, $\frac{2\sqrt{AB}}{\Delta}\mathcal{S}_s(t)$ is constant and so t is also constant. Then a unit-speed parameterization of χ can be defined as

$$\chi(s) = \frac{2\sqrt{AB}}{\Delta}\left(S_c(\tilde{s})\cos t_0, S_c(\tilde{s})\sin t_0, S_s(\tilde{s}) \right), \; \tilde{s} = \frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}.$$

Then we can compute the Frenet frame as follow:

$$\mathbf{t}(s)=\frac{\left(S_s\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\cos t_0, S_s\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\sin t_0, S_c\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\right)}{\sqrt{\left\{S_c\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\right\}^2+\left\{S_s\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\right\}^2}}$$

$$\mathbf{n}(s)=\frac{\left(S_c\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\cos t_0, S_c\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\sin t_0, S_s\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\right)}{\sqrt{\left\{S_c\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\right\}^2+\left\{S_s\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\right\}^2}}$$

$$\mathbf{b}(s)=\frac{\frac{4AB}{\Delta^2}\left(-\sin t_0, \cos t_0, 0\right)}{\left\{S_c\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\right\}^2+\left\{S_s\left(\frac{s}{\frac{2\sqrt{AB}}{\Delta}S_s(t_0)}\right)\right\}^2}$$

From $\kappa_h=\langle\chi''(s), \mathbf{b}(s)\rangle$, the result follows. □

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