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RESEARCH PAPER

Threshold properties of a stochastic epidemic model with a variable vaccination rate

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Abstract

This paper aims to improve the analysis of a stochastic SIVR epidemic model with an imperfect vaccination process, taking into consideration the fact that a fraction of vaccinated individuals becomes susceptible to infection. The uniqueness of the positive solution is shown. Further, we obtain the threshold of the stochastic SIVR model which determines whether the epidemic will persist or die out. In the extinction case, we prove that the solution converges almost surely toward the disease-free equilibrium of the deterministic SIVR model. Some numerical illustrations are given to confirm our theoretical results.

Keywords: Stochastic epidemic model; imperfect vaccination; disease persistence; exponential extinction; almost sure convergence

AMS 2020 Classification: 34A34; 58Z05; 60H10; 60H30; 92D25

1 Introduction

Mathematical treatment using differential equations is largely used to forecast the spread of infectious diseases. This approach consists of dividing the total population within which the diseases spread, into several compartments [1–8]. Most compartmental epidemic models descend from the works [1–3]. To control the spread of diseases, quarantine and vaccination are commonly used. Vaccination is considered to be the most effective strategy to mitigate or stop the spread of infectious diseases and reduce their associated morbidity and mortality rates [9–11].
In deterministic models, the indicator that measures the ability of an infectious disease to spread within a population is called the "basic reproduction number". It is considered as the average number of new infections caused by an infected individual in a fully susceptible population at the onset of contamination [12]. Let $R_0 = \frac{\beta}{\mu+\gamma}$ be the basic reproduction number of model (1) and $R_{a,b} = R_0 \frac{\mu+b+ac}{\mu+a+b}$ the basic reproduction number in a population such that a proportion had

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been vaccinated and by $R_{a,0}$ the value of $R_{a,b}$ when $b = 0$. If $R_{a,0} < 1$, the system (1) has a unique disease-free equilibrium $(w_0, x_0, y_0, z_0) = \left(\frac{\mu}{\mu+a}, 0, \frac{a}{\mu+a}, 0\right)$ which is globally asymptotically stable. If $R_{a,0} > 1$, model (1) presents a bifurcation leading to the existence of multiple endemic equilibria [13, 14].

It is known that epidemic models are inevitably affected by random environmental disturbance (see [15–17]). For this reason, Tornatore et al. [18] formulated and studied various aspects of stochastic stability related to the stochastic counterpart of system (1), obtained by substituting the contact rate β in (1) by $\beta + \sigma \frac{dB(t)}{dt}$, where $B(t)$ is a standard Brownian motion on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ with intensity $\sigma^2 > 0$.

In this paper, we consider the following deterministic epidemic model

$$\begin{cases} dw = [\mu - (\mu + a)w - \beta x w + by]\, dt, \\ dx = [\beta(w + cy)x - (\mu + \gamma)x]\, dt, \\ dy = [aw - c\beta xy - (\mu + b)y]\, dt, \\ dz = [\gamma x - \mu z]\, dt, \end{cases} \tag{1}$$

where w, x, y and z denote the number of susceptible, infected, vaccinated, and recovered individuals, respectively.

The other parameters involved in system (1) are defined below:

μ : The natural birth and death rate .

γ : The recovery rate of infected individuals.

a : The rate at which susceptible individuals become vaccinated.

b: A fraction of vaccinated individuals becoming susceptibles.

c: A positive parameter satisfying: $0 \leq c \leq 1$, with $c = 0$ means that the vaccine is perfectly effective, and $c = 1$ implies that the vaccination has no impact.

The concerned stochastic model is defined by the following stochastic differential system

$$\begin{cases} dw = [\mu - (\mu + a)w - \beta x w + by]\, dt - \sigma x w dB(t), \\ dx = [\beta(w + cy)x - (\mu + \gamma)x]\, dt + \sigma(w + cy)x dB(t), \\ dy = [aw - c\beta xy - (\mu + b)y]\, dt - \sigma cxy dB(t), \\ dz = [\gamma x - \mu z]\, dt. \end{cases} \tag{2}$$

In case $b = 0$, conditions guaranteeing the persistence and extinction of model (2) are established in relation with the threshold $R_0^S = \frac{1}{\mu+\gamma}[\beta \frac{\mu+ac}{\mu+a} - \frac{\sigma^2}{2}(\frac{\mu+ac}{\mu+a})^2]$ (see [19]).

In this paper, we are going to improve the analysis of (2) by taking into consideration the existence of the parameter b and we will determine conditions ensuring the extinction and persistence of the disease.

Remark 1 *Tornatore et al. [20] have shown the existence of a positive global solution and they have studied the stochastic stability of the disease-free equilibrium related to the following model*

$$\begin{cases} dw = [\mu - (\mu + a)w - \beta x w + by + \gamma x]\, dt - \sigma x w dB(t), \\ dx = [\beta(w + cy)x - (\mu + \gamma)x]\, dt + \sigma(w + cy)x dB(t), \\ dy = [aw - c\beta xy - (\mu + b)y]\, dt - \sigma cxy dB(t). \end{cases} \tag{3}$$

At first sight, it seems that the two models (2) and (3) are the same, but there exists a systematic difference.

In model (2), the parameter γ signifies a fraction of infected individuals who come back into the recovered class, whereas in model (3) it signifies a fraction of infected individuals whose goes back in the susceptible class.

For simplicity, we define

$$R_b^S = \frac{1}{\mu+\gamma}\left[\beta\frac{\mu+b+ac}{\mu+a+b} - \frac{\sigma^2}{2}\left(\frac{\mu+b+ac}{\mu+a+b}\right)^2\right],$$

$$\langle h\rangle = \frac{1}{t}\int_0^t h(s)\,ds,$$

and

$$\mathbb{R}_+^4 = \{(x_1,...,x_4)\in\mathbb{R}^4 : x_i \geqslant 0, i=1,...,4\}.$$

The rest of the paper is organized as follows. In Section 2, we prove the existence of a global positive solution for system (2). In Section 3, we present several lemmas that will be used in the following sections. Section 4 is devoted to establishing conditions under which the disease goes extinct exponentially. Moreover, the solution of system (2) converges almost surely (abbreviated as a.s.) to the disease-free equilibrium $(w_b, x_b, y_b, z_b) = \left(\frac{\mu+b}{\mu+a+b}, 0, \frac{a}{\mu+a+b}, 0\right)$ of the deterministic system (1), when the extinction conditions are fulfilled. If the condition $R_b^S > 1$ holds, we show that the disease will be persistent almost surely, in Section 5. In the penultimate section (Section 6), numerical simulations are given to confirm the theoretical results. The paper ends with a brief conclusion Section 7.

2 Well-posedness of model (2)

First of all, we define the subset

$$\Delta = \{(w,\ x,\ y,\ z)\ :\ w\geq 0,\ x\geq 0,\ y\geq 0,\ z\geq 0,\ w+x+y+z=1\}.$$

Since

$$d(w(t)+x(t)+y(t)+z(t)) = (\mu - \mu(w(t)+x(t)+y(t)+z(t)))dt,$$

then

$$w(t)+x(t)+y(t)+z(t) = 1 + (w(0)+x(0)+y(0)+z(0)-1)e^{-\mu t}.$$

Therefore, Δ is a positive invariant set for the stochastic model (2).

Theorem 1 *Let $(w(0), x(0), y(0), z(0)) \in \Delta$, then the system (2) admits a unique positive solution $(w(t), x(t), y(t), z(t))$ on $t \geq 0$.*

Proof Since the coefficients of system (2) satisfy the local Lipschitz property, then for any $(w(0), x(0), y(0), z(0)) \in \mathbb{R}_+^4$, there is unique local solution $(w(t), x(t), y(t), z(t)) \in \mathbb{R}_+^4$ which may blow up at time τ_e, such that $t \in [0, \tau_e)$ (see [21]). To show this solution is global, we only need to prove that $\tau_e = \infty$ a.s. Let $\epsilon_0 > 0$ such that $w(0), x(0), y(0), z(0) > \epsilon_0$. For each $\epsilon \geq \epsilon_0$, we define the stopping times

$$\tau_\epsilon = \inf\{t \in [0, \tau_\epsilon) : w(t) \leq \epsilon \text{ or } x(t) \leq \epsilon \text{ or } y(t) \leq \epsilon \text{ or } z(t) \leq \epsilon\},$$
$$\tau = \lim_{\epsilon \to 0} \tau_\epsilon = \inf\{t \in [0, \tau_\epsilon) : w(t) \leq \epsilon \text{ or } x(t) \leq \epsilon \text{ or } y(t) \leq \epsilon \text{ or } z(t) \leq \epsilon\}.$$

Let us consider the twice differentiable function π defined, for $X = (w, x, y, z) \in \Delta$, by

$$\pi(X) = -\ln w - \ln x - \ln y - \ln z.$$

Applying Itô formula to π, we obtain for all $t \geq 0,\ s \in [0, t \wedge \tau_\epsilon]$,

$$\begin{aligned} d\pi(X(s)) =& \left[-\frac{\mu}{w(s)} + \beta x(s) + c\beta x(s) - \beta(w(s) + cy(s)) - \frac{aw(s)}{y(s)} - \frac{\gamma x(s)}{z(s)}\right] ds \\ &+ \left[\frac{1}{2}\sigma^2 x^2(s) + \frac{1}{2}\sigma^2 (w(s) + cy(s))^2 + \frac{1}{2}\sigma^2 c^2 x^2(s) + 4\mu + a + \gamma\right] ds \\ &+ \sigma\Big[(1+c)x(s) - (w(s) + cy(s))\Big] dB(s) \\ \leq& \left[(1+c)\beta + \frac{3}{2}\sigma^2 + 4\mu + a + \gamma\right] ds + \sigma\Big[(1+c)x(s) - (w(s) + cy(s))\Big] dB(s) \\ =& D.ds + \sigma\Big[(1+c)x(s) - (w(s) + cy(s))\Big] dB(s). \end{aligned}$$

Integrating both sides of the above inequality from 0 to $t \wedge \tau_\epsilon$ followed by taking the expectation $\mathbb{E}$ on both sides, one obtains

$$\mathbb{E}(\pi(X(t \wedge \tau_\epsilon))) \leq \pi(X(0)) + D\mathbb{E}(t \wedge \tau_\epsilon) \leq \pi(X(0)) + Dt.$$

Since $\pi(X(t \wedge \tau_\epsilon)) > 0$, then

$$\begin{aligned} \mathbb{E}\left[\pi(X(t \wedge \tau_\epsilon))\right] &= \mathbb{E}\left[\pi(X(t \wedge \tau_\epsilon))\mathbf{1}_{\{\tau_\epsilon \leq t\}}\right] + \mathbb{E}\left[\pi(X(t \wedge \tau_\epsilon))\mathbf{1}_{\{\tau_\epsilon > t\}}\right] \\ &\geq \mathbb{E}\left[\pi(X(t \wedge \tau_\epsilon))\mathbf{1}_{\{\tau_\epsilon \leq t\}}\right], \end{aligned}$$

where $\mathbf{1}_{\mathcal{A}}$ is the indicator function of set $\mathcal{A}$. Note that there are some components of $X(\tau_\epsilon)$ equal to ϵ. Therefore, $\pi(X(\tau_\epsilon)) \geq -\ln \epsilon$. Thus

$$\mathbb{E}\left[\pi(X(t \wedge \tau_\epsilon))\right] \geq \mathbb{E}\left[\pi(X(t \wedge \tau_\epsilon))\mathbf{1}_{\{\tau_\epsilon \leq t\}}\right] \geq -\mathbb{P}(\tau_\epsilon \leq t) \ln \epsilon. \quad (4)$$

Hence

$$\mathbb{P}(\tau_\epsilon \leq t) \leq -\frac{\pi(X(0)) + Dt}{\ln \epsilon}.$$

Letting ϵ to 0, we obtain for all $t > 0$, $\mathbb{P}(\tau \leq t) = 0$, which means that $\mathbb{P}(\tau = \infty) = 1$. As $\tau \leq \tau_e$, then $\tau_\infty = \tau_e = \infty$ a.s. ∎

In the following section, we present two lemmas that will be useful in the upcoming sections.

3 Toolbox

Lemma 1 *Let* $\big(w(t), x(t), y(t), z(t)\big)$ *be a solution of system* (2) *with* $\big(w(0), x(0), y(0), z(0)\big) \in \Delta$. *Then*

$$\langle w + cy \rangle = \frac{\mu + b + ac}{\mu + a + b} - \langle G_1 \rangle + \langle \varphi \rangle - \sigma (1-c) K_1(t),$$

where

$$\begin{aligned} G_1(t) =& c\, x + c\, \gamma \int_0^t x(s) e^{-\mu (t-s)} ds + b\,(1-c\,) \int_0^t x(s) e^{-(\mu + a + b)(t-s)} ds \\ & + \beta(1-c) \int_0^t w(s)x(s) e^{-(\mu + a+b)(t-s)} ds + b\,(1-c\,) \int_0^t z(s) e^{-(\mu + a+b)(t-s)} ds, \\ \varphi\,(t) =& - \left[c\, z(0) e^{-\mu\, t} + (1-c)\big(\frac{\mu + b}{\mu + a + b} - w(0)\,\big) e^{-(\mu + a+b)t} \right], \end{aligned}$$

and

$$K_1(t) = \frac{1}{t} \int_0^t \left[\int_0^v w(s)x(s) e^{-(\mu + a+b)(v-s)} dB(s) \right] dv.$$

Proof From Eq. (2), we get

$$\begin{aligned} w(t) =& \frac{\mu + b}{\mu + a + b} - \Big(\frac{\mu + b}{\mu + a + b} - w(0)\Big) e^{-(\mu + a+b)t} - c \int_0^t (x(s) + z(s)) e^{-(\mu + a+b)(t-s)} ds \\ & - \beta \int_0^t w(s)x(s) e^{-(\mu + a+b)(t-s)} ds - \sigma \int_0^t w(s)x(s) e^{-(\mu + a+b)(t-s)} dB(s), \\ z(t) =& z(0) e^{-\mu\, t} + \gamma \int_0^t x(s) e^{-\mu\,(t-s)} ds. \end{aligned}$$

Then

$$\begin{aligned} w(t) + c\, y(t) =& \; c\,(w(t) + y(t)) + (1-c) w(t) \\ =& \; \frac{\mu + b + ab}{\mu + a + b} - \left[c\, x + c\, \gamma \int_0^t x(s) e^{-\mu\,(t-s)} ds + b\,(1-c) \int_0^t x(s) e^{-(\mu + a+b)(t-s)} ds \right. \\ & \left. + \beta\,(1-c) \int_0^t w(s)x(s) e^{-(\mu + a+b)(t-s)} ds + b\,(1-c) \int_0^t z(s) e^{-(\mu + a+b)(t-s)} ds \right] \\ & - \left[c\, z(0) e^{-\mu\, t} + (1-c)\big(\frac{\mu + b}{\mu + a + b} - w(0)\,\big) e^{-(\mu + a+b)t} \right] \\ & - \sigma\,(1-c) \int_0^t w(s)x(s) e^{-(\mu + a+b)(t-s)} dB(s). \end{aligned}$$

This leads to the desired result. ∎

Lemma 2 *Let* $\big(w(t), x(t), y(t), z(t)\big)$ *be a solution of system* (2) *with* $\big(w(0), x(0), y(0), z(0)\big) \in \Delta$.

Then

$$\langle (w+cy)^2\rangle = \Big(\frac{\mu+b+ac}{\mu+a+b}\Big)^2 + \langle G_2\rangle + \psi(t) + K_2(t),$$

where

$$\begin{aligned}
G_2(t) = &-\frac{1}{\mu}(w+cy)(\beta w+c^2\beta y)x + \frac{\sigma^2}{2\mu}x^2(w+c^2y)^2 \\
&+\frac{c(1-c)a}{\mu}w(x+z) - (1-c)\frac{b^2a+\mu ab+abc}{\mu(\mu+a+b)^2} - \frac{\beta c^2}{\mu}xy \\
&-\frac{b(1-c)}{\mu}y(w+cy) - \frac{\beta}{\mu}xw \\
&+\frac{ac(1-c)}{\mu}\Big[\frac{b}{\mu+a+b}(x+z) + \frac{\beta}{\mu+a+b}xw\Big] \\
&+\Big[\frac{a(1-c)}{\mu} + \frac{a(1-c)^2(\mu+b)}{\mu(\mu+a+b)}\Big]\Big[\frac{b}{\mu+a+b}(x+z) + \frac{\beta}{\mu+a+b}xw\Big] \\
&-\frac{b(1-c)}{\mu}\Big[\frac{a}{\mu+a+b}x + \frac{a}{\mu+a+b}z + \frac{\beta c}{\mu+a+b}xy\Big] \\
&+\frac{a(1-c)^2}{\mu}\Big[\frac{b}{\mu+a+b}w(x+z) + \frac{\beta}{\mu+a+b}w^2x - \frac{\sigma^2}{2(\mu+a+b)}w^2x^2\Big],
\end{aligned}$$

$$\begin{aligned}
\psi(t) = &-\frac{(w(t)+cy(t))^2-(w(0)+cy(0))^2}{2\mu t} - \frac{(w(t)+cy(t))-(w(0)+cy(0))}{\mu t} \\
&+\Big[\frac{a(1-c^2)}{\mu} + \frac{a(1-c)^2(\mu+b)}{\mu(\mu+a+b)}\Big]\frac{w(t)-w(0)}{(\mu+a+b)t} \\
&-\frac{b(1-c)}{\mu}\frac{y(t)-y(0)}{(\mu+a+b)t} + \frac{a(1-c)^2}{\mu}\frac{w^2(t)-w^2(0)}{2(\mu+a+b)t},
\end{aligned}$$

and

$$\begin{aligned}
K_2(t) = &\frac{\sigma}{\mu+a+b}\Big(\frac{a(1-c)}{\mu} + \frac{a(1-c)^2(\mu+b)}{\mu(\mu+a+b)}\Big)\frac{1}{t}\int_0^t x(s)w(s)\,dB(s) \\
&-\frac{\sigma}{\mu t}\int_0^t (w(s)+c^2y(s))x(s)\,dB(s) - \frac{\sigma bc(1-c)}{\mu(\mu+a+b)}\frac{1}{t}\int_0^t x(s)y(s)\,dB(s) \\
&+\frac{\sigma a(1-c)^2}{\mu(\mu+a+b)t}\int_0^t w^2(s)x(s)\,dB(s) + \frac{\sigma ac(1-c)}{\mu(\mu+a+b)t}\int_0^t x(s)w(s)\,dB(s) \\
&-\frac{\sigma}{\mu t}\int_0^t (w(s)+cy(s))(w(s)+c^2y(s))x(s)\,dB(s).
\end{aligned}$$

Proof Applying Itô formula to system (2), we obtain

$$\begin{aligned}
\langle w\rangle &= -\frac{w(t)-w(0)}{(\mu+a+b,)t}+\frac{\mu+b}{\mu+a+b}-\frac{b}{\mu+a+b}\langle x+z\rangle-\frac{\beta}{\mu+a+b}\langle xw\rangle\\
&\quad -\frac{\sigma}{(\mu+a+b)t}\int_0^t x(s)w(s)\,dB(s),\\
\langle y\rangle &=-\frac{y(t)-y(0)}{(\mu+a+b)t}+\frac{a}{\mu+a+b}-\frac{a}{\mu+a+b}\langle x+z\rangle-\frac{\beta c}{\mu+a+b}\langle xy\rangle\\
&\quad -\frac{\sigma c}{(\mu+a+b)t}\int_0^t x(s)y(s)\,dB(s),\\
\langle w^2\rangle &= -\frac{w^2(t)-w^2(0)}{2(\mu+a+b)t}+\frac{\mu+b}{\mu+a+b}\langle w\rangle-\frac{b}{\mu+a+b}\langle w(x+z)\rangle\\
&\quad -\frac{\beta}{\mu+a+b}\langle w^2x\rangle+\frac{\sigma^2}{2(\mu+a+b)}\langle x^2w^2\rangle-\frac{\sigma}{(\mu+a+b)t}\int_0^t x^2(s)w(s)\,dB(s),\\
\langle w+cy\rangle &= -\frac{(w(t)+cy(t))-(w(0)-cy(0))}{\mu t}+1-\frac{a(1-c)}{\mu}\langle w\rangle-\frac{\beta}{\mu}\langle xw\rangle-\frac{\beta c^2}{\mu}\langle xy\rangle\\
&\quad +\frac{b(1-c)}{\mu}\langle y\rangle-\frac{\sigma}{\mu t}\int_0^t (w(s)+c^2y(s))x(s)\,dB(s),
\end{aligned}$$

and

$$\begin{aligned}
\langle (w+cy)^2\rangle &= -\frac{(w(t)+cy(t))^2-(w(0)+cy(0))^2}{2\mu t}+\langle w+cy\rangle-\frac{a(1-c)^2}{\mu}\langle w^2\rangle\\
&\quad -\frac{1}{\mu}\langle (w+cy)(\beta w+c^2\beta y)x\rangle+\frac{\sigma^2}{2\mu}\langle x^2(w+c^2y)^2\rangle\\
&\quad -\frac{ac(1-c)}{\mu}\langle w\rangle+\frac{ac(1-c)}{\mu}\langle w(x+z)\rangle-\frac{b(1-c)}{\mu}\langle y(w+cy)\rangle\\
&\quad -\frac{\sigma}{\mu t}\int_0^t (w(s)+cy(s))(w(s)+c^2y(s))x(s)\,dB(s).
\end{aligned}$$

Injecting the expressions of $\langle w\rangle$, $\langle y\rangle$, $\langle w^2\rangle$ and $\langle w+cy\rangle$ into the expression of $\langle (w+cy)^2\rangle$, we get

$$\langle (w+cy)^2\rangle = \left(\frac{\mu+b+ac}{\mu+a+b}\right)^2+\langle G_2\rangle+\psi(t)+K_2(t).$$

4 Exponential extinction of the disease

Theorem 2 *Let $(w(t),x(t),y(t),z(t))$ be a solution of system (2) with $(w(0),x(0),y(0),z(0))\in\Delta$. We consider the two following assumptions*

(A) $\dfrac{\beta^2}{2\sigma^2}<\mu+\gamma$;

(B) $\sigma^2\dfrac{\mu+b+ac}{\mu+a+b}-\beta\le 0$ *and* $R_b^S<1$.

Then

$$\limsup_{t\to\infty}\frac{1}{t}\ln x(t)\leq\frac{\beta^2}{2\sigma^2}-(\mu+\gamma)<0\quad \text{a.s., if } \textbf{(A)} \text{ holds;}$$
$$\limsup_{t\to\infty}\frac{1}{t}\ln x(t)\leq(\mu+\gamma)(R_b^S-1)+(\sigma^2\frac{\mu+b+ac}{\mu+a+b}-\beta)\langle H_1\rangle<0\quad \text{a.s., if } \textbf{(B)} \text{ holds.}$$

Proof Applying Itô formula, gives

$$\begin{aligned}\frac{1}{t}\ln\frac{x(t)}{x(0)}&\leq\beta\langle w+cy\rangle-(\mu+\gamma)-\frac{\sigma^2}{2}\langle w+cy\rangle^2+\sigma M(t)\\&\leq\frac{\beta^2}{2\sigma^2}-(\mu+\gamma)-\frac{\sigma^2}{2}\left[\langle w+cy\rangle-\frac{\beta}{\sigma^2}\right]^2+\sigma M(t)\\&\leq\frac{\beta^2}{2\sigma^2}-(\mu+\gamma)+\sigma M(t),\end{aligned}\tag{5}$$

where

$$M(t)=\frac{\sigma}{t}\int_0^t(w(r)+cy(r))dB(r).$$

If the assumption **(A)** holds, we get

$$\limsup_{t\to\infty}\frac{1}{t}\ln x(t)<0\quad \text{a.s.}$$

Returning to Eq. (5) and applying Lemma 1, one has

$$\begin{aligned}\frac{1}{t}\ln\frac{x(t)}{x(0)}\leq&\beta\frac{\mu+b+ac}{\mu+a+b}-\beta\langle G_1\rangle+\beta\langle\varphi\rangle-\sigma\beta(1-c)K_1(t)\\&-(\mu+\gamma)+\sigma M(t)-\frac{\sigma^2}{2}\Bigg[\Big(\frac{\mu+b+ac,}{\mu+a+b}-\langle G_1\rangle\Big)^2+\Big(\langle\varphi\rangle-\sigma(1-c)K_1(t)\Big)^2\\&+2\Big(\frac{\mu+b+ac}{\mu+a+b}-\langle G_1\rangle\Big)\Big(\langle\varphi\rangle-\sigma(1-c)K_1(t)\Big)\Bigg].\end{aligned}$$

We define

$$\Psi(t)=\Big(\langle\varphi\rangle-\sigma(1-c)K_1(t)\Big)^2+2\Big(\frac{\mu+b+ac}{\mu+a+b}-\langle G_1\rangle\Big)\Big(\langle\varphi\rangle-\sigma(1-c)K_1(t)\Big).$$

Since $(w(0),x(0),y(0),z(0))\in\Delta$, then

$$\lim_{t\to\infty}\Psi(t)=0\quad \text{a.s.,}$$

$$\limsup_{t\to\infty}\frac{1}{t}\ln x(t)\leq\beta\frac{\mu+b+ac}{\mu+a+b}-(\mu+\gamma)-\frac{\sigma^2}{2}\Big(\frac{\mu+b+ac}{\mu+a+b}\Big)^2+\Big(\sigma^2\frac{\mu+b+ac}{\mu+a+b}-\beta\Big)\langle G_1\rangle\quad \text{a.s.}$$

If assumption **(B)** is taking into consideration, we obtain

$$\limsup_{t\to\infty} \frac{1}{t} \ln x(t) \; < \; 0 \quad \text{a.s.}$$

■

Theorem 3 *If one of the two assumptions in Theorem 2 holds, then*

$$(w(t), x(t), y(t), z(t)) \to (w_b, x_b, y_b, z_b) \qquad a.s. \qquad as \;\; t \to \infty.$$

Proof Applying Itô formula to the first equation in system (2) leads to

$$\begin{aligned} d\Big(\frac{\mu+b}{\mu+a+b} - w\Big)^2 = & \Big[2\beta\, xw(\frac{\mu+b}{\mu+a+b} - w) - 2(\mu+a+b)(\frac{\mu+b}{\mu+a+b} - w)^2 \\ & + 2b\,(\frac{\mu+b}{\mu+a+b} - w)(x+z) + \sigma^2 x^2 w^2\Big] dt \\ & + 2\sigma\, xw\Big(\frac{\mu+b}{\mu+a+b} - w\Big) dB(t). \end{aligned}$$

It follows that

$$\begin{aligned} \int_0^t \Big(\frac{\mu+b}{\mu+a+b} - w(s)\Big)^2 ds \leq & \frac{1}{2(\mu+a+b)} \Big[\, 1 + 2(\beta+b)w_0 \int_0^t x(s)ds + \sigma^2 \int_0^t x^2(s)ds \\ & + 2b\, w_0 \int_0^t z(s)ds \,\Big] + \widehat{M}(t), \end{aligned}$$

where

$$\widehat{M}(t) = \frac{\sigma}{\mu+a+b} \int_0^t x(s)w(s)dB(s)$$

is a continuous local martingale. For the rest of the proof, we use the idea exposed in [19]. ■

5 Persistence in mean

Theorem 4 *Let $(w(t), x(t), y(t), z(t))$ be a solution of system (2) with $(w(0), x(0), y(0), z(0)) \in \Delta$. If $R_b^S > 1$, then the disease will be persistent in mean, that is,*

$$\liminf_{t\to\infty} \langle\, x\,\rangle \;\geqslant\; \frac{\mu+\gamma}{\beta D_1 + \frac{\sigma^2}{2} D_2} (R_b^S - 1) \quad a.s.,$$

where

$$\begin{aligned} D_1 &= c + \frac{\gamma\,(b + c\,\mu + ac\,)}{\mu\,(\mu + a + b\,)} + \frac{(1-c\,)(b+\beta\,)}{\mu + a + b}, \\ D_2 &= \frac{\sigma^2}{2\mu} + \frac{c\,(1-c\,)a}{\mu}\Big(1 + \frac{\gamma}{\mu}\Big) \\ &\quad + \Bigg(\frac{a\,(1-c\,)}{\mu} + \frac{a\,(1-c\,)^2(\mu + b\,)}{\mu\,(\mu + a + b\,)} + \frac{a\,(1-c\,)^2}{\mu} + \frac{ac\,(1-c\,)}{\mu}\Bigg)\Bigg(\frac{\mu\,\beta + \mu\, b + \gamma\, b}{\mu\,(\mu + a + b\,)}\Bigg). \end{aligned}$$

Furthermore,

$$\liminf_{t\to\infty}\langle w\rangle \geqslant \frac{\mu}{\beta+\mu+a+b} \quad a.s.$$

$$\liminf_{t\to\infty}\langle y\rangle \geqslant \frac{\mu a}{(\beta c+\mu+b)(\beta+\mu+a+b)} \quad a.s.$$

and

$$\liminf_{t\to\infty}\langle z\rangle = \frac{\gamma}{\mu}\liminf_{t\to\infty}\langle x\rangle \quad a.s.$$

Proof From system (2), one has

$$\frac{1}{t}\ln\frac{x(t)}{x(0)} = \beta\langle w+cy\rangle - (\mu+\gamma) - \frac{\sigma^2}{2}\langle (w+cy)^2\rangle + \sigma M(t),$$

where

$$M(t) = \frac{1}{t}\int_0^t (w(s)+cy(s))\,dB(s)$$

is a martingale. According to Lemma 1 and Lemma 2, we obtain

$$\begin{aligned}\frac{1}{t}\ln\frac{x(t)}{x(0)} =& \beta\frac{\mu+b+ac}{\mu+b+\beta a} - \beta\langle G_1\rangle + \beta\langle\varphi\rangle - \beta\sigma(1-c)K_1(t) - (\mu+\gamma)\\ &-\frac{\sigma^2}{2}\left(\frac{\mu+b+ac}{\mu+b+\beta a}\right)^2 - \frac{\sigma^2}{2}\langle G_2\rangle - \frac{\sigma^2}{2}\psi(t) - \frac{\sigma^2}{2}K_2(t) + \sigma M(t).\end{aligned}$$

One can see that

$$K_1(t) = \frac{1}{\mu+a+b}\frac{1}{t}\int_0^t (1-e^{-(\mu+a+b)(t-s)})x(s)w(s)\,dB(s).$$

Since $(w(0), x(0), y(0), z(0)) \in \Delta$, then the strong law of large numbers for martingales implies

$$\lim_{t\to\infty} K_1(t) = \lim_{t\to\infty} K_2(t) = \lim_{t\to\infty} M(t) = 0 \quad \text{a.s.}$$

Notice that

$$\langle G_1\rangle \leqslant D_1\langle x\rangle + \frac{b(1-c)}{\mu+a+b}z(0)\langle e^{-\mu t}\rangle,$$

and

$$\langle G_2\rangle \leqslant \left[\frac{a(1-c)(c+b+bc)}{\mu} + \frac{ab(1-c)^2}{\mu}\right]z(0)\langle e^{-\mu t}\rangle + D_2\langle x\rangle.$$

Thus

$$\liminf_{t\to\infty}\langle x\rangle \geqslant \frac{\mu+\gamma}{\beta D_1+\frac{\sigma^2}{2}D_2}(R_b^S-1) > 0 \quad \text{a.s.}$$

Next, to show that system (2) is persistent in mean, it remains to show that w, y and z are persistent. From system (2), we have

$$\left(1+\frac{\beta}{\mu+a+b}\right)\langle w\rangle \geq -\frac{w(t)-w(0)}{(\mu+a+b)t}+\frac{\mu}{\mu+a+b}-\frac{\sigma}{(\mu+a+b)t}\int_0^t x(s)w(s)\,dB(s),$$

$$\frac{y(t)-y(0)}{t}+\frac{\sigma c}{t}\int_0^t x(s)y(s)\,dB(s) \geq a\langle w\rangle-(\beta c+\mu+b)\langle y\rangle,$$

and

$$\frac{z(t)-z(0)}{t}=\gamma\langle x\rangle-\mu\langle z\rangle.$$

By using the strong law of large numbers for martingales, we obtain the desired results. ■

6 Numerical simulations

In this section, we report three experiments to verify the theoretical results shown in the previous sections. We denote $w(t)=S(t), x(t)=I(t), y(t)=V(t)$ and $z(t)=R(t),$ for all $t\geq 0$.

Example 1 *We choose the parameters as follows:*

$$\sigma=0.8, \mu=0.1, \beta=0.6, a=0.8, c=0.5, \gamma=0.3 \text{ and } b=0.2.$$

The initial value is $(w(0),x(0),y(0),z(0))=(0.6,0.2,0.1,0.1)$.
Then

$$\frac{\beta^2}{2\sigma^2}-(\mu+\gamma)=-0.1188,$$

which confirms Theorem 2, condition (A) (see Figure 1).

Example 2 *Let the noise intensity* $\sigma=0.602$, *and all the other parameters be the same as in the previous example. In this case, we have*

$$R_b^S=0.9248 \quad \text{and} \quad \sigma^2\frac{\mu+b+ac}{\mu+a+b}-\beta=-0.2920.$$

Then

$$\limsup_{t\to\infty}\frac{1}{t}\ln x(t) \leq (\mu+\gamma)(R_b^S-1)+(\sigma^2\frac{\mu+b+ac}{\mu+a+b}-\beta)\langle H_1\rangle < 0.$$

That is, $x(t)$ *tends exponentially to zero, according to Theorem 2, condition (B) (see Figure 2).*

Example 3 *In model (2), we choose the parameters as follows:*

$$\sigma = 0.2, \mu = 0.2, \beta = 0.6, a = 0.2, c = 0.4, \gamma = 0.2099 \text{ and } b = 0.4.$$

The initial value is $(w(0), x(0), y(0), z(0)) = (0.6, 0.2, 0.1, 0.1)$.
We compute that

$$R_b^S = \frac{1}{\mu + \gamma}\left[\beta \frac{\mu + b + ac}{\mu + a + b} - \frac{\sigma^2}{2}\left(\frac{\mu + b + ac}{\mu + a + b}\right)^2\right] = 1.2090 > 1.$$

According to Theorem 4, the solution of model (2) is persistent in mean (see Figure 3).

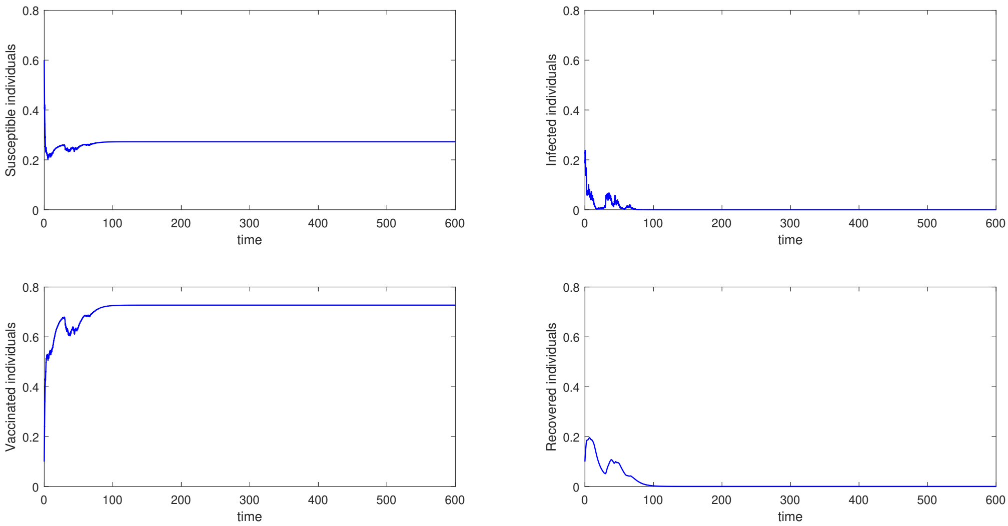

Figure 1. Numerical simulations of the populations in model (2) for the given values in Example 1.

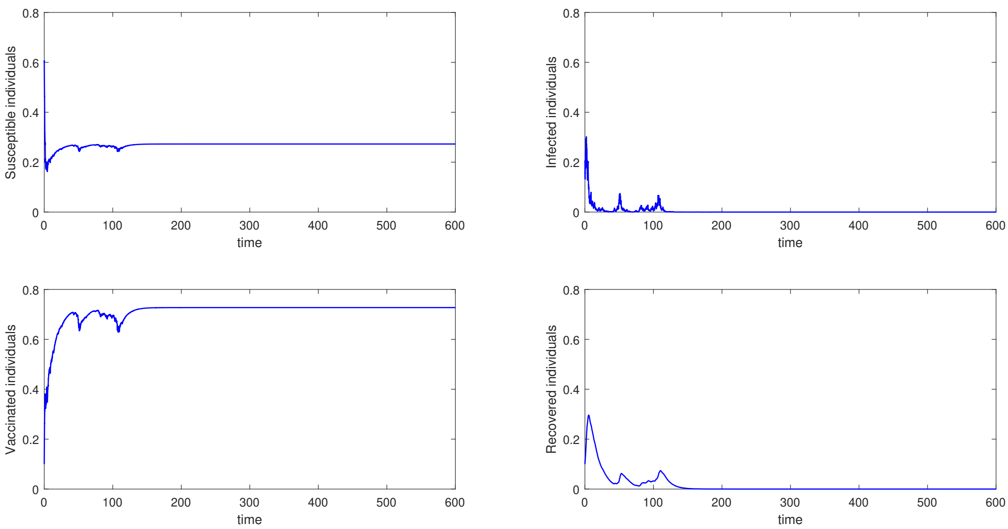

Figure 2. Numerical simulations of the populations in model (2) for the given values in Example 2.

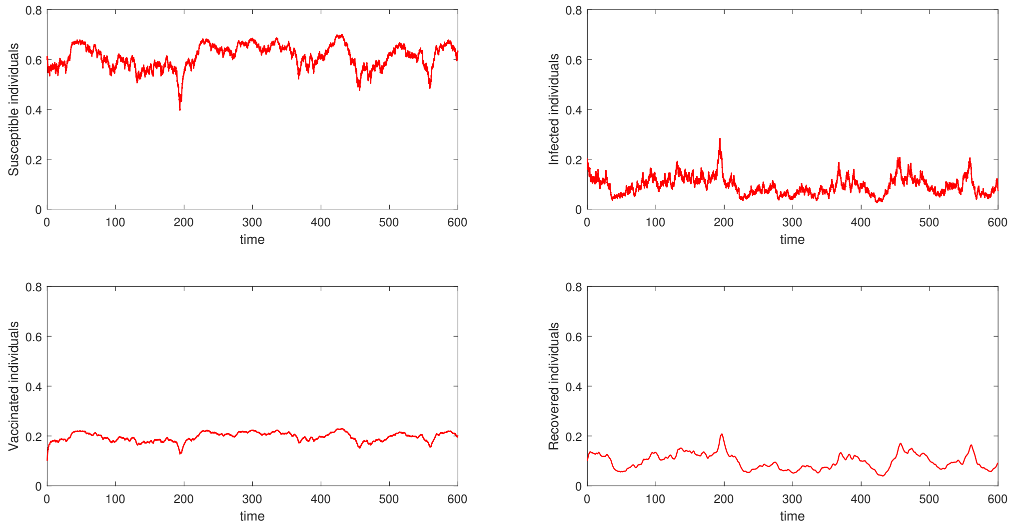

Figure 3. Numerical simulations of the populations in model (2) for the given values in Example 3.

7 Conclusion

In this paper, we considered a stochastic epidemic model with a variable vaccination rate. First, we showed the uniqueness of a positive solution for system (2). Second, we gave two conditions for the disease extinction. Under these two conditions, we proved that the solution of system (2) converges almost surely toward (w_b, x_b, y_b, z_b). Further, we showed that the disease is permanent in mean when $R_b^S > 1$. We also notice that the large noise suppresses the disease. If the noise is small enough, the condition $R_\theta^S > 1$ is sufficient for guaranteeing the prevalence of the disease.

Declarations

Ethical approval

The authors state that this research complies with ethical standards. This research does not involve either human participants or animals.

Consent for publication

Not applicable

Conflicts of interest

The author declares that he has no known competing interests regarding the work reported in this article.

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Author's contributions

The author has made substantial contributions to the conception, and design of the work, the acquisition, analysis, interpretation of data, and the creation of new software used in the work.

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