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Research Article

A Note on Hybrid Numbers with Generalized Hybrid k-Pell Numbers as Coefficients

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Abstract
In this study, we define a new generalization of the hybrid k-Pell sequence consisting of hybrid numbers with generalized hybrid k-Pell numbers as coefficients. We establish some algebraic properties and also the Binet formula, generating function, and exponential generating function related to this new sequence. In addition, some identities are provided as sum identities, and Catalan, Cassini, and d'Ocagne's identities. The particular cases are studied, namely, the hybrid numbers with hybrid k-Pell numbers as coefficients, the hybrid numbers with hybrid k-Pell–Lucas numbers as coefficients, and the hybrid numbers with hybrid Modified k-Pell numbers as coefficients.

Keywords: Binet's formula, Generating function, Hybrid numbers, Hybrid k-Pell sequence, Identities
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1. Introduction

The classical Pell sequence is defined by the recurrence relation

$$P_n = 2P_{n-1} + P_{n-2},$$

with initial values $P_0 = 0$ and $P_1 = 1$. The first elements of the Pell sequence are 0, 1, 2, 5, 12, 29, 70, and so on. These numbers have interesting properties and connections to various mathematical concepts, such as continued fractions and square roots (see more about this sequence and others in [1]). Several researchers have studied generalizations of the Pell sequences and their properties (see, for instance, references [2], [3], and [4]).

In this work, we have, as a base, one of these sequences of integers - the k-Pell sequence - which is also related to two other examples - that of the k-Pell sequence. For any positive real number k, a generalization for Pell numbers, k-Pell numbers, was introduced by the second author in [5], denoted by $\{P_{k,n}\}_{n\geq 0}$ and defined by

$$P_{k,n} = 2P_{k,n-1} + kP_{k,n-2},$$

with $P_{k,0}=0$ and $P_{k,1}=1$. Others important generalizations are k-Pell-Lucas, $\{Q_{k,n}\}_{n\geq 0}$ and Modified k-Pell, $\{q_{k,n}\}_{n\geq 0}$, defined, respectively, by the recurrences relations

$$Q_{k,n}=2Q_{k,n-1}+kQ_{k,n-2},$$

$$q_{k,n}=2q_{k,n-1}+kq_{k,n-2},$$

with $Q_{k,0}=Q_{k,1}=2$ and $q_{k,0}=q_{k,1}=1$ as the respective initial values. The first k-Pell numbers are $0,\ 1,\ 2,\ 4+k,\ 8+4k,\ 16+12k+k^2,\ 32+32k+8k^2,\ 64+80k+28k^2+k^3$. The first k-Pell-Lucas numbers are $2,\ 2,\ 4+2k,\ 8+6k,\ 16+16k+2k^2,\ 32+40k+10k^2,\ 64+96k+36k^2+2k^3$. The first Modified k-Pell numbers are $1,\ 1,\ 2+2k,\ 4+5k,\ 8+12k+2k^2,\ 16+28k+9k^2,\ 32+64k+30k^2+2k^3$. These sequences are generalizations of the sequence of Pell numbers.

By considering the extensions of sequences in other sets of numbers, let us consider the set of numbers introduced in [6], where the author defined the set of hybrid numbers that is a generalization of complex, hyperbolic, and dual number systems, as

$$\mathbb{K}=\{a+bi+c\varepsilon+d\mathbf{h}:a,b,c,d\in\mathbb{R},i^2=-1,\varepsilon^2=0,\mathbf{h}^2=1,i\mathbf{h}=-\mathbf{h}i=\varepsilon+i\},$$

where $i,\varepsilon,\mathbf{h}$ are hybrid units (see, for instance, [6,7]). From the definition of hybrid numbers, the multiplication table of the hybrid units is given by Table 1.1.

$\bullet$	1	$\mathbf{i}$	ε	$\mathbf{h}$
1	1	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	-1	$1-\mathbf{h}$	$\varepsilon+i$
ε	ε	$\mathbf{h}+1$	0	-ε
$\mathbf{h}$	$\mathbf{h}$	$-(\varepsilon+i)$	ε	1

Table 1.1. The multiplication table for Hybrid units

Table 1.1 shows that the multiplication of hybrid numbers is not commutative. It is important to remark that, in this article, we chose to multiply imaginary units to the right of the coefficients. According [6], the set of hybrid numbers $(\mathbb{K},+)$ is an Abelian group.

Also, in Catarino [8], and in Catarino and Bilgici [9] were studied the hybrid k-Pell, k-Pell-Lucas and Modified k-Pell sequence, denoted respectively by $\{HP_{k,n}\}$,
$\{HQ_{k,n}\}_{n\geq 0}$ and $\{Hq_{k,n}\}_{n\geq 0}$, and defined by

$$HP_{k,n}=P_{k,n}+P_{k,n+1}\mathbf{i}+P_{k,n+2}\varepsilon+P_{k,n+3}\mathbf{h}\ ,$$

where $P_{k,n}$ are terms of k-Pell sequence,

$$HQ_{k,n}=Q_{k,n}+Q_{k,n+1}\mathbf{i}+Q_{k,n+2}\varepsilon+Q_{k,n+3}\mathbf{h}\ ,$$

where $Q_{k,n}$ are terms of k-Pell-Lucas sequence,

$$Hq_{k,n}=q_{k,n}+q_{k,n+1}\mathbf{i}+q_{k,n+2}\varepsilon+q_{k,n+3}\mathbf{h}\ ,$$

where $P_{k,n}$ are terms of Modified k-Pell sequence.

Consider a positive real number k. Let $\{P^*_{k,n}\}_{n\geq 0}$ be a sequence defined by the recurrence relation

$$P^*_{k,n}=2P^*_{k,n-1}+kP^*_{k,n-2},\tag{1.1}$$

with arbitrary initial values $P^*_{k,0}=\alpha_0$ and $P^*_{k,1}=\alpha_1$. The generalized hybrid k-Pell sequence, denoted by $\{HP^*_{k,n}\}_{n\geq n}$, are defined by the recurrence relation

$$HP^*_{k,n}=P^*_{k,n}+P^*_{k,n+1}\mathbf{i}+P^*_{k,n+2}\varepsilon+P^*_{k,n+3}\mathbf{h},$$

where $P^*_{k,n}$ are given by (1.1). Observe that, if we take Expression (1.1) with initial values $P^*_{k,0}=0$ and $P^*_{k,1}=1$, or $P^*_{k,0}=P^*_{k,1}=2$, or $P^*_{k,0}=P^*_{k,1}=1$ we obtain, the hybrid k-Pell numbers, hybrid k-Pell–Lucas numbers and hybrid Modified k-Pell numbers, respectively.

The study of hybrid numbers has been explored through several mathematical perspectives, including binomial transformations [10], algebraic structures [11], and ratio analysis [12], offering new insights into the connection between numerical sequences and hybrid algebraic systems. Motivated by the generalizations and results given in [13], in this article, we will introduce the hybrid numbers with generalized hybrid k-Pell numbers as coefficients. To do this, we consider the sequence of generalized hybrid k-Pell numbers defined by (1.1) with arbitrary initial values as coefficients.

This article is organized as follows. Section 1 introduces the hybrid numbers with generalized hybrid k-Pell numbers as coefficients, and two recurrence relations for this new sequence were provided. Section 2 is devoted to studying the Binet formula for the hybrid numbers with generalized hybrid k-Pell numbers as coefficients. Moreover, the generating function and exponential generating function are stated. In Section 3, we established the Catalan, Cassini, and d'Ocagne identities, each one of two types, related to the hybrid numbers with generalized hybrid k-Pell numbers as coefficients. Moreover, some identities are provided in this section. In addition, results for the hybrid numbers with hybrid k-Pell numbers as coefficients, the hybrid numbers with hybrid k-Pell–Lucas numbers as coefficients, and hybrid numbers with hybrid Modified k-Pell numbers as coefficients are established as particular cases of the general propositions and theorems.

2. Hybrid Numbers with Generalized Hybrid k-Pell Numbers as Coefficients

In this section, we define hybrid numbers with generalized hybrid k-Pell numbers as coefficients and provide some properties of this new sequence of numbers. We start considering the following definition,

Definition 2.1. *Consider a positive real number k. For integers $n \geq 0$, the generalized hybrid numbers with generalized hybrid k-Pell number coefficients are defined recursively by*

$$\mathbb{P}^*_{k,n} = HP^*_{k,n} + HP^*_{k,n+1}\mathbf{i} + HP^*_{k,n+2}\varepsilon + HP^*_{k,n+3}\mathbf{h},$$

where $HP^*_{k,n}$ is the n-th generalized hybrid k-Pell number, $\mathbf{i}, \varepsilon$ and $\mathbf{h}$ are hybrid units.

According [8], [9], the sequence $\{HP_{k,n}\}$ of hybrid k-Pell numbers satisfies the following second order recursive relation:

$$HP_{k,n+1} = 2HP_{k,n} + kHP_{k,n-1},$$

with initial values $HP_{k,0} = 0 + \mathbf{i} + 2\varepsilon + (4+k)\mathbf{h}$ and $HP_{k,1} = 1 + 2\mathbf{i} + (4+k)\varepsilon + (8+4k)\mathbf{h}$. We can show that the generalized hybrid k-Pell verifies this property sequence, i.e., it is valid the recurrence relation

$$HP^*_{k,n+1} = 2HP^*_{k,n} + kHP^*_{k,n-1}, \tag{2.1}$$

with initial values $HP^*_{k,0} = \alpha_0 + \alpha_1\mathbf{i} + P^*_{k,2}\varepsilon + P^*_{k,3}\mathbf{h}$ and $HP^*_{k,1} = \alpha_1 + P^*_{k,2}\mathbf{i} + P^*_{k,3}\varepsilon + P^*_{k,4}\mathbf{h}$. More generally, as a consequence of the recurrence relation given in equation (2.1), we have the next result.

Proposition 2.2. *Consider a positive real number k. The sequence of the hybrid numbers with generalized hybrid k-Pell numbers as coefficients satisfies the following second-order recursive relation:*

$$\mathbb{P}^*_{k,n+1} = 2\mathbb{P}^*_{k,n} + k\mathbb{P}^*_{k,n-1}, \tag{2.2}$$

with initial values $\mathbb{P}^*_{k,0} = HP^*_{k,0} + HP^*_{k,1}\mathbf{i} + HP^*_{k,2}\varepsilon + HP^*_{k,3}\mathbf{h}$ and $\mathbb{P}^*_{k,1} = HP^*_{k,1} + HP^*_{k,2}\mathbf{i} + HP^*_{k,3}\varepsilon + HP^*_{k,4}\mathbf{h}$, where $HP^*_{k,n}$ is the n-th generalized hybrid k-Pell number.

Now, we obtain another recurrence relation for the hybrid numbers with generalized hybrid k-Pell numbers as coefficients, given in the next result.

Proposition 2.3. *Consider a positive real number k. For $n \geq 2$ the hybrid sequence $\{\mathbb{P}^*_{k,n}\}_{n\geq 0}$ satisfies the recurrence relation*

$$\mathbb{P}^*_{k,n+1} = (k+4)\mathbb{P}^*_{k,n-1} + 2k\mathbb{P}^*_{k,n-2},$$

with initial values $\mathbb{P}^*_{k,0} = HP^*_{k,0} + HP^*_{k,1}\mathbf{i} + HP^*_{k,2}\varepsilon + HP^*_{k,3}\mathbf{h}$ and $\mathbb{P}^*_{k,1} = HP^*_{k,1} + HP^*_{k,2}\mathbf{i} + HP^*_{k,3}\varepsilon + HP^*_{k,4}\mathbf{h}$, where $HP^*_{k,n}$ is the n-th generalized hybrid k-Pell number.

Proof. We have that

$$\begin{aligned} \mathbb{P}^*_{k,n+1} &= 2\mathbb{P}^*_{k,n} + k\mathbb{P}^*_{k,n-1} \\ \mathbb{P}^*_{k,n} &= 2\mathbb{P}^*_{k,n-1} + k\mathbb{P}^*_{k,n-2}. \end{aligned}$$

Then,

$$2\mathbb{P}^*_{k,n+1} - k\mathbb{P}^*_{k,n} = (4\mathbb{P}^*_{k,n} + 2k\mathbb{P}^*_{k,n-1}) - (2k\mathbb{P}^*_{k,n-1} + k^2\mathbb{P}^*_{k,n-2})$$

$$= \quad 4\mathbb{P}^*_{k,n} - k^2\mathbb{P}^*_{k,n-2} \,.$$

Thus, we obtain

$$2\mathbb{P}^*_{k,n+1} = (4+k)\mathbb{P}^*_{k,n} - k^2\mathbb{P}^*_{k,n-2} \,. \tag{2.3}$$

By replacing the equation (2.2) in equation (2.3) we get

$$\begin{aligned} 2\mathbb{P}^*_{k,n+1} &= (4+k)(2\mathbb{P}^*_{k,n-1} + k\mathbb{P}^*_{k,n-2}) - k^2\mathbb{P}^*_{k,n-2} \\ &= (8+2k)\mathbb{P}^*_{k,n-1} + 4k\mathbb{P}^*_{k,n-2} \,. \end{aligned}$$

And we get the result. □

3. The Binet Formula and the Generating Functions

This section provides the Binet formula and the exponential and ordinary generating functions for the hybrid numbers with generalized hybrid k-Pell numbers as coefficients. The next subsection establishes the Binet formula.

3.1 The Binet formula

In this subsection, we establish the Binet formula for the hybrid numbers with generalized hybrid k-Pell numbers as coefficients.

The characteristic equation associated with the recurrence (2.2) is given by

$$r^2 - 2r - k = 0, \tag{3.1}$$

whose roots are $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$. Thus, we have the next result.

Theorem 3.1. *Consider a positive real number k. For positive integer n, the Binet formula for hybrid numbers with generalized hybrid k-Pell numbers coefficients is given by,*

$$\mathbb{P}^*_{k,n} = \frac{\underline{r_1}(r_1)^n + \underline{r_2}(r_2)^n}{r_1 - r_2} \,, \tag{3.2}$$

where $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$ are the roots of the characteristic equation (3.1), $\underline{r_1} = \mathbb{P}^*_{k,1} - \mathbb{P}^*_{k,0}r_2$ and $\underline{r_2} = -\mathbb{P}^*_{k,1} + \mathbb{P}^*_{k,0}r_1$, and initial values $\mathbb{P}^*_{k,0}$ and $\mathbb{P}^*_{k,1}$.

Proof. Since $r_1 = 1 + \sqrt{1+k}$ and $r_1 = 1 - \sqrt{1+k}$ are the roots of the characteristic equation $x^2 - 2x - k = 0$. The sequence

$$\mathbb{P}^*_{k,n} = c_1(r_1)^n + c_2(r_2)^n \tag{3.3}$$

is the solution of the equation (2.2). Giving to n the values $n = 0$ and $n = 1$ and solving this system of linear equations, we obtain a unique value for c_1 and c_2. So, we get the following distinct values,

$$c_1 = \frac{\mathbb{P}^*_{k,1} - \mathbb{P}^*_{k,0}r_2}{r_1 - r_2}$$

$$c_2 = \frac{-\mathbb{P}^*_{k,1} - \mathbb{P}^*_{k,0}r_1}{r_1 - r_2}.$$

Since $\underline{r_1} = \mathbb{P}^*_{k,1} - \mathbb{P}^*_{k,0}r_2$ and $\underline{r_2} = -\mathbb{P}^*_{k,1} + \mathbb{P}^*_{k,0}r_1$, we can express c_1 and c_2, respectively by $c_1 = \dfrac{\underline{r_1}}{r_1 - r_2}$ and $c_2 = \dfrac{\underline{r_2}}{r_1 - r_2}$. Now, using (3.3), we obtain (3.2) as required. □

In particular cases, considering the hybrid numbers with hybrid k-Pell numbers as coefficients, hybrid k-Pell–Lucas numbers as coefficients, and hybrid Modified k-Pell numbers as coefficients, we get the following corollaries.

Corollary 3.2. *Consider a positive real number k. For $n \geq 0$, the Binet formula for hybrid numbers with hybrid k-Pell numbers as coefficients is given by,*

$$\mathbb{P}^*_{k,n} = \frac{\underline{r_1}(r_1)^n + \underline{r_2}(r_2)^n}{r_1 - r_2} \,,$$

where $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$ are the roots of the characteristic equation (3.1), $\underline{r_1} = 1 + (2 - r_2)i + (4 + k - 2r_2)\varepsilon + (8 + 4k - 4r_2 - kr_2)h$ and $\underline{r_2} = -1 + (r_1 - 2)i + (-4 - k + 2r_1)\varepsilon + (-8 - 4k + 4r_1 + kr_1)h$.

Corollary 3.3. *Consider a positive real number k. For $n \geq 0$, the Binet formula for hybrid numbers with hybrid k-Pell–Lucas numbers as coefficients is given by,*

$$\mathbb{P}^*_{k,n} = \frac{\underline{r_1}(r_1)^n + \underline{r_2}(r_2)^n}{r_1 - r_2} ,$$

where $r_1 = 1+\sqrt{1+k}$ and $r_2 = 1-\sqrt{1+k}$ are the roots of the characteristic equation (3.1), $\underline{r_1} = (2-2r_2)+(4+2k-2r_2)i+(8+6k-(4+2k)r_2)\varepsilon+(16+16k+2k^2-(8+6k)r_2)h$ and $\underline{r_2} = (-2+2r_1)+(-4-2k+2r_1)i+(-8-6k+(4+2k)r_1)\varepsilon+(-16-16k-2k^2+(8+6k)r_1)h$.

Corollary 3.4. *Consider a positive real number k. For $n \geq 0$, the Binet formula for hybrid numbers with hybrid Modified k-Pell numbers as coefficients is given by,*

$$\mathbb{P}^*_{k,n} = \frac{\underline{r_1}(r_1)^n + \underline{r_2}(r_2)^n}{r_1 - r_2} ,$$

where $r_1 = 1+\sqrt{1+k}$ and $r_2 = 1-\sqrt{1+k}$ are the roots of the characteristic equation (3.1), $\underline{r_1} = (1-r_2)+(2+2k-r_2)i+(4+5k-(2+2k)r_2)\varepsilon+(8+12k+2k^2-(4+5k)r_2)h$ and $\underline{r_2} = (-1+r_1)+(-2-2k+r_1)i+(-4-5k+(2+2k)r_1)\varepsilon+(-8-12k-2k^2+(4+5k)r_1)h$.

3.2 Generating functions

Theorem 3.1 allows us to provide the exponential generating function for the generating function of the hybrid numbers with generalized hybrid k-Pell coefficients by considering equation (3.2).

Proposition 3.5. *Consider a positive real number k. For $n \geq 0$, the exponential generating function for hybrid numbers with generalized hybrid k-Pell numbers as coefficients is given by,*

$$\sum_{n=0}^{\infty} \mathbb{P}^*_{k,n} \frac{t^n}{n!} = \frac{\underline{r_1}e^{r_1 t} + \underline{r_2}e^{r_2 t}}{r_1 - r_2},$$

where $r_1 = 1+\sqrt{1+k}$ and $r_1 = 1-\sqrt{1+k}$ are the roots of the characteristic equation (3.1), $\underline{r_1} = \mathbb{P}^*_{k,1} - \mathbb{P}^*_{k,0} r_2$ and $\underline{r_2} = -\mathbb{P}^*_{k,1} + \mathbb{P}^*_{k,0} r_1$, and initial values $\mathbb{P}^*_{k,0}$ and $\mathbb{P}^*_{k,1}$.

Proof. Let $\sum_{n=0}^{\infty} \mathbb{P}^*_{k,n} \frac{t^n}{n!}$ be the exponential generating function of the hybrid numbers with hybrid k-Pell number coefficients. By using Expression (3.2) we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathbb{P}^*_{k,n} \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \left(\frac{\underline{r_1}(r_1)^n + \underline{r_2}(r_2)^n}{r_1 - r_2} \right) \frac{t^n}{n!} \\
&= \left(\frac{\underline{r_1}}{r_1 - r_2} \right) \sum_{n=0}^{\infty} \frac{(r_1 t)^n}{n!} + \left(\frac{\underline{r_2}}{r_1 - r_2} \right) \sum_{n=0}^{\infty} \frac{(r_2 t)^n}{n!} \\
&= \frac{\underline{r_1}e^{r_1 t} + \underline{r_2}e^{r_2 t}}{r_1 - r_2}.
\end{aligned}$$

□

Next, we establish the ordinary generating function of the hybrid numbers with generalized hybrid k-Pell numbers as coefficients. For reach our goal, let $\sum_{n=0}^{\infty} \mathbb{P}^*_{k,n} t^n$ be the generating function of $\mathbb{P}^*_{k,n}$.

Given the recurrence relation $\mathbb{P}^*_{k,n+1} = 2\mathbb{P}^*_{k,n-1} + k\mathbb{P}^*_{k,n-2}$, we obtain

$$\begin{aligned}
\sum_{n=2}^{\infty} \mathbb{P}^*_{k,n} t^n &= 2t \sum_{n=2}^{\infty} \mathbb{P}^*_{k,n-1} t^{n-1} + kt^2 \sum_{n=2}^{\infty} \mathbb{P}^*_{k,n-2} t^{n-2} \\
&= 2t \Big(\sum_{n=1}^{\infty} \mathbb{P}^*_{k,n-1} t^{n-1} - \mathbb{P}^*_{k,0} \Big) + kt^2 \sum_{n=2}^{\infty} \mathbb{P}^*_{k,n-2} t^{n-2}.
\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{P}^{*}_{k,n}t^{n} &= \mathbb{P}^{*}_{k,0}+t\mathbb{P}^{*}_{k,1}+\sum_{n=2}^{\infty}\mathbb{P}^{*}_{k,n}t^{n}\\ &= \mathbb{P}^{*}_{k,0}+t\mathbb{P}^{*}_{k,1}+t\sum_{n=2}^{\infty}\mathbb{P}^{*}_{k,n-1}t^{n-1}+kt^{2}\sum_{n=2}^{\infty}\mathbb{P}^{*}_{k,n-2}t^{n-2}\\ &= \mathbb{P}^{*}_{k,0}+t\mathbb{P}^{*}_{k,1}+2t(\sum_{n=1}^{\infty}\mathbb{P}^{*}_{k,n-1}t^{n-1}-\mathbb{P}^{*}_{k,0})+kt^{2}\sum_{n=2}^{\infty}\mathbb{P}^{*}_{k,n-2}t^{n-2}\\ &= (1-2t)\mathbb{P}^{*}_{k,0}+t\mathbb{P}^{*}_{k,1}+2t\sum_{n=1}^{\infty}\mathbb{P}^{*}_{k,n-1}t^{n-1}+kt^{2}\sum_{n=2}^{\infty}\mathbb{P}^{*}_{k,n-2}t^{n-2}.\end{aligned}$$

Then, we obtain

$$(1-2t-kt^{2})\sum_{n=0}^{\infty}\mathbb{P}^{*}_{k,n}t^{n}=(1-2t)\mathbb{P}^{*}_{k,0}+t\mathbb{P}^{*}_{k,1}$$

Therefore, we get the following result.

Theorem 3.6. *Consider a positive real number k. For $n\geq 0$, the generating function for hybrid numbers with generalized hybrid k-Pell numbers as coefficients is given by,*

$$\sum_{n=0}^{\infty}\mathbb{P}^{*}_{k,n}t^{n}=\frac{(1-2t)\mathbb{P}^{*}_{k,0}+t\mathbb{P}^{*}_{k,1}}{(1-2t-kt^{2})}$$

where $\mathbb{P}^{*}_{k,0}=HP^{*}_{k,0}+HP^{*}_{k,1}\mathbf{i}+HP^{*}_{k,2}\varepsilon+HP^{*}_{k,3}\mathbf{h}$ and $\mathbb{P}^{*}_{k,1}=HP^{*}_{k,1}+HP^{*}_{k,2}\mathbf{i}+HP^{*}_{k,3}\varepsilon+HP^{*}_{k,4}\mathbf{h}$, and $HP^{*}_{k,n}$ is the n-th generalized hybrid k-Pell number.

Theorem 3.6 allows us to obtain the generating function for the hybrid numbers with the hybrid *k*-Pell numbers as coefficients, the hybrid *k*-Pell–Lucas numbers as coefficients, and the hybrid Modified *k*-Pell numbers as coefficients, given in the next corollaries.

Corollary 3.7. *Consider a positive real number k. For $n\geq 0$, the generating function for hybrid numbers with hybrid k-Pell numbers as coefficients is given by,*

$$\sum_{n=0}^{\infty}\mathbb{P}^{*}_{k,n}t^{n}=\frac{(1-2t)\mathbb{P}^{*}_{k,0}+t\mathbb{P}^{*}_{k,1}}{(1-2t-kt^{2})}$$

where $\mathbb{P}^{*}_{k,0}=HP_{k,0}+HP_{k,1}\mathbf{i}+HP_{k,2}\varepsilon+HP_{k,3}\mathbf{h}$ and $\mathbb{P}^{*}_{k,1}=HP_{k,1}+HP_{k,2}\mathbf{i}+HP_{k,3}\varepsilon+HP_{k,4}\mathbf{h}$, and $HP^{*}_{k,n}$ is the n-th hybrid k-Pell number.

Corollary 3.8. *Consider a positive real number k. For $n\geq 0$, the generating function for hybrid numbers with hybrid $k-$Pell–Lucas numbers as coefficients is given by,*

$$\sum_{n=0}^{\infty}\mathbb{P}^{*}_{k,n}t^{n}=\frac{(1-2t)\mathbb{P}^{*}_{k,0}+t\mathbb{P}^{*}_{k,1}}{(1-2t-kt^{2})}$$

where $\mathbb{P}^{*}_{k,0}=HQ_{k,0}+HQ_{k,1}\mathbf{i}+HQ_{k,2}\varepsilon+HQ_{k,3}\mathbf{h}$ and $\mathbb{P}^{*}_{k,1}=HQ_{k,1}+HQ_{k,2}\mathbf{i}+HQ_{k,3}\varepsilon+HQ_{k,4}\mathbf{h}$, and $HQ_{k,n}$ is the n-th hybrid $k-$Pell–Lucas number.

Corollary 3.9. *Consider a positive real number k. For $n\geq 0$, the generating function for hybrid numbers with hybrid Modified k-Pell numbers as coefficients is given by,*

$$\sum_{n=0}^{\infty}\mathbb{P}^{*}_{k,n}t^{n}=\frac{(1-2t)\mathbb{P}^{*}_{k,0}+t\mathbb{P}^{*}_{k,1}}{(1-2t-kt^{2})}$$

where $\mathbb{P}^{*}_{k,0}=Hq_{k,0}+Hq_{k,1}\mathbf{i}+Hq_{k,2}\varepsilon+Hq_{k,3}\mathbf{h}$ and $\mathbb{P}^{*}_{k,1}=Hq_{k,1}+Hq_{k,2}\mathbf{i}+Hq_{k,3}\varepsilon+Hq_{k,4}\mathbf{h}$, and $Hq_{k,n}$ is the n-th hybrid Modified k-Pell number.

4. Some Identities and Sum Formulas

This section provides the Catalan, Cassini, and d'Ocagne identities related to the hybrid numbers with generalized hybrid k-Pell coefficients. In addition, sum formulas are provided.

4.1 Identities

The Catalan, Cassini, and d'Ocagne identities related to a sequence of numbers are important because they describe an elegant relationship between the elements of the sequence. In this subsection, we establish the Catalan, Cassini, and d'Ocagne identities related to the hybrid numbers with generalized hybrid k-Pell numbers as coefficients. Observe that we have two versions of each identity since the multiplication of hybrid numbers is not commutative.

Theorem 4.1 (First Catalan's identity)**.** *Consider a positive real number k. For positive integers n and r, with $n \geq r$, then the following identity is verified*

$$(\mathbb{P}^*_{k,n})^2 - \mathbb{P}^*_{k,n-r}\mathbb{P}^*_{k,n+r} = \frac{(-k)^n \left(\underline{r_1}\underline{r_2}(1-r_1^{-r}r_2^r) + \underline{r_2}\underline{r_1}(1-r_1^r r_2^{-r})\right)}{4+4k}. \tag{4.1}$$

where $r_1 = 1+\sqrt{1+k}$ and $r_2 = 1-\sqrt{1+k}$ are the roots of the characteristic equation (3.1), $\underline{r_1} = \mathbb{P}^*_{k,1} - \mathbb{P}^*_{k,0}r_2$ and $\underline{r_2} = -\mathbb{P}^*_{k,1} + \mathbb{P}^*_{k,0}r_1$, and initial values $\mathbb{P}^*_{k,0}$ and $\mathbb{P}^*_{k,1}$.

Proof. From the Binet formula given in equation (3.2), we obtain

$$\begin{aligned}
(\mathbb{P}^*_{k,n})^2 - \mathbb{P}^*_{k,n-r}\mathbb{P}^*_{k,n+r} &= \left(\frac{\underline{r_1}(r_1)^n + \underline{r_2}(r_2)^n}{r_1 - r_2}\right)^2 - \left(\frac{\underline{r_1}(r_1)^{n-r} + \underline{r_2}(r_2)^{n-r}}{r_1 - r_2}\right)\left(\frac{\underline{r_1}(r_1)^{n+r} + \underline{r_2}(r_2)^{n+r}}{r_1 - r_2}\right) \\
&= \frac{\left(\underline{r_1}^2(r_1)^{2n} + \underline{r_1}(r_1)^n\underline{r_2}(r_2)^n + \underline{r_2}(r_2)^n\underline{r_1}(r_1)^n + \underline{r_2}^2(r_2)^{2n}\right)}{(r_1-r_2)^2} \\
&\quad - \frac{\left(\underline{r_1}^2(r_1)^{2n} - \underline{r_1}(r_1)^{n-r}\underline{r_2}(r_2)^{n+r} - \underline{r_2}(r_2)^{n-r}\underline{r_1}(r_1)^{n+r} - \underline{r_2}^2(r_2)^{2n}\right)}{(r_1-r_2)^2} \\
&= \frac{\underline{r_1}\underline{r_2}(r_1r_2)^n\left(1-r_1^{-r}r_2^r\right) + \underline{r_2}\underline{r_1}(r_1r_2)^n\left(1-r_1^r r_2^{-r}\right)}{(r_1-r_2)^2}.
\end{aligned}$$

Since $r_1r_2 = -k$ and $r_1 - r_2 = 2\sqrt{1+k}$, then we obtain the result . □

As a consequence, according to Theorem 4.1, we have the Cassini identity in the next result.

Proposition 4.2 (First Cassini's identity)**.** *Let k be a positive real number. For a positive integer n, the following identity is verified*

$$(\mathbb{P}^*_{k,n})^2 - \mathbb{P}^*_{k,n-1}\mathbb{P}^*_{k,n+1} = \frac{(-k)^{n-1}\left(\underline{r_1}\underline{r_2}(k-r_2^2) + \underline{r_2}\underline{r_1}(k-r_1^2)\right)}{4+4k}.$$

where $r_1 = 1+\sqrt{1+k}$ and $r_2 = 1-\sqrt{1+k}$ are the roots of the characteristic equation (3.1), $\underline{r_1} = \mathbb{P}^*_{k,1} - \mathbb{P}^*_{k,0}r_2$ and $\underline{r_2} = -\mathbb{P}^*_{k,1} + \mathbb{P}^*_{k,0}r_1$, and initial values $\mathbb{P}^*_{k,0}$ and $\mathbb{P}^*_{k,1}$.

Proof. Making $r = 1$ in equation (4.1) we obtain

$$(\mathbb{P}^*_{k,n})^2 - \mathbb{P}^*_{k,n-1}\mathbb{P}^*_{k,n+1} = \frac{(-k)^n\left(\underline{r_1}\underline{r_2}(1-r_1^{-1}r_2) + \underline{r_2}\underline{r_1}(1-r_1r_2^{-1})\right)}{4+4k}.$$

Since $\frac{r_1}{r_2} = \frac{-(r_1)^2}{k}$ and $\frac{r_2}{r_1} = \frac{-(r_2)^2}{k}$, then we get

$$(\mathbb{P}^*_{k,n})^2 - \mathbb{P}^*_{k,n-1}\mathbb{P}^*_{k,n+1} = \frac{(-k)^{n-1}\left(\underline{r_1}\underline{r_2}(k-r_2^2) + \underline{r_2}\underline{r_1}(k-r_1^2)\right)}{4+4k}.$$

□

In a similar way to Theorem 4.1, we obtain

Theorem 4.3 (First d'Ocagne's identity)**.** *Let k be a positive real number. For positive integers n,m and t with $n=m+t$, then the following identity is verified*

$$\mathbb{P}^*_{k,n}\mathbb{P}^*_{k,m+1}-\mathbb{P}^*_{k,n+1}\mathbb{P}^*_{k,n}=\frac{\underline{r_1}\underline{r_2}\left((r_1)^n(r_2)^{m+1}-(r_1)^{m+1}(r_2)^n\right)}{(r_1-r_2)^2}+\frac{\underline{r_2}\underline{r_1}(((r_2)^n(r_1)^{m+1}-(r_2)^{m+1}(r_1)^n))}{(r_1-r_2)^2},$$

where $r_1=1+\sqrt{1+k}$ and $r_2=1-\sqrt{1+k}$ are the roots of the characteristic equation (3.1), $\underline{r_1}=\mathbb{P}^*_{k,1}-\mathbb{P}^*_{k,0}r_2$ and $\underline{r_2}=-\mathbb{P}^*_{k,1}+\mathbb{P}^*_{k,0}r_1$, and initial values $\mathbb{P}^*_{k,0}$ and $\mathbb{P}^*_{k,1}$.

Proof. From the Binet formula given by equation (3.2), we obtain

$$\mathbb{P}^*_{k,n}\mathbb{P}^*_{k,m+1}-\mathbb{P}^*_{k,n+1}\mathbb{P}^*_{k,n}=\left(\frac{\underline{r_1}(r_1)^n+\underline{r_2}(r_2)^n}{r_1-r_2}\right)\left(\frac{\underline{r_1}(r_1)^{m+1}+\underline{r_2}(r_2)^{m+1}}{r_1-r_2}\right).$$

Thus,

$$\mathbb{P}^*_{k,n}\mathbb{P}^*_{k,m+1}-\mathbb{P}^*_{k,n+1}\mathbb{P}^*_{k,n}=\frac{\underline{r_1}\underline{r_2}\left((r_1)^n(r_2)^{m+1}-(r_1)^{m+1}(r_2)^n\right)}{(r_1-r_2)^2}\frac{+\underline{r_2}\underline{r_1}(((r_2)^n(r_1)^{m+1}-(r_2)^{m+1}(r_1)^n))}{(r_1-r_2)^2}.$$

□

Since multiplication in $\mathbb{K}$ is not commutative, we also have the following results.

Theorem 4.4 (Second Catalan's identity)**.** *Let k be a positive real number. For positive integers n and r with $n\geq r$, then the following identity is verified*

$$(\mathbb{P}^*_{k,n})^2-\mathbb{P}^*_{k,n+r}\mathbb{P}^*_{k,n-r}=\frac{(-k)^n\left(\underline{r_1}\underline{r_2}(1-(r_1)^r(r_2)^{-r})+\underline{r_2}\underline{r_1}(1-(r_1)^{-r}(r_2)^r)\right)}{4+4k},$$

where $r_1=1+\sqrt{1+k}$ and $r_2=1-\sqrt{1+k}$ are the roots of the characteristic equation (3.1), $\underline{r_1}=\mathbb{P}^*_{k,1}-\mathbb{P}^*_{k,0}r_2$ and $\underline{r_2}=-\mathbb{P}^*_{k,1}+\mathbb{P}^*_{k,0}r_1$, and initial values $\mathbb{P}^*_{k,0}$ and $\mathbb{P}^*_{k,1}$.

Proposition 4.5 (Second Cassini's identity)**.** *Consider a positive real number k. For positive integer n, then the following identity is verified*

$$(\mathbb{P}^*_{k,n})^2-\mathbb{P}^*_{k,n+1}\mathbb{P}^*_{k,n-1}=\frac{(-k)^{n-1}\left(\underline{r_1}\underline{r_2}(k-r_1^2)+\underline{r_2}\underline{r_1}(k-r_2^2)\right)}{4+4k},$$

where $r_1=1+\sqrt{1+k}$ and $r_2=1-\sqrt{1+k}$ are the roots of the characteristic equation (3.1), $\underline{r_1}=\mathbb{P}^*_{k,1}-\mathbb{P}^*_{k,0}r_2$ and $\underline{r_2}=-\mathbb{P}^*_{k,1}+\mathbb{P}^*_{k,0}r_1$, and initial values $\mathbb{P}^*_{k,0}$ and $\mathbb{P}^*_{k,1}$.

Theorem 4.6 (Second d'Ocagne's identity)**.** *Consider a positive real number k. For positive integers n,m and t with $n=m+t$, then the following identity is verified*

$$\mathbb{P}^*_{k,n+1}\mathbb{P}^*_{k,m}-\mathbb{P}^*_{k,n}\mathbb{P}^*_{k,m+1}=\frac{\underline{r_1}\underline{r_2}\left((r_1)^{m+1}(r_2)^n-(r_1)^n(r_2)^{m+1}\right)}{(r_1-r_2)^2}\frac{+\underline{r_2}\underline{r_1}(((r_2)^{m+1}(r_1)^n-(r_2)^n(r_1)^{m+1}))}{(r_1-r_2)^2},$$

where $r_1=1+\sqrt{1+k}$ and $r_2=1-\sqrt{1+k}$ are the roots of the characteristic equation (3.1), $\underline{r_1}=\mathbb{P}^*_{k,1}-\mathbb{P}^*_{k,0}r_2$ and $\underline{r_2}=-\mathbb{P}^*_{k,1}+\mathbb{P}^*_{k,0}r_1$, and initial values $\mathbb{P}^*_{k,0}$ and $\mathbb{P}^*_{k,1}$.

In particular, considering the hybrid numbers with hybrid k-Pell numbers as coefficients, hybrid k-Pell–Lucas numbers as coefficients, and hybrid Modified k-Pell numbers as coefficients, we get interesting identities. Next, we consider the hybrid numbers with hybrid k-Pell numbers as coefficients, but the analogous result can be provided for hybrid k-Pell–Lucas numbers and hybrid Modified k-Pell numbers as coefficients.

Corollary 4.7. *For $n,m\geq 0$, the hybrid numbers with hybrid k-Pell numbers as coefficients verify the following identities*

1.

$$(\mathbb{P}^*_{k,n})^2-\mathbb{P}^*_{k,n-r}\mathbb{P}^*_{k,n+r}=\frac{(-k)^n\left(\underline{r_1}\underline{r_2}(1-r_1^{-r}r_2^r)+\underline{r_2}\underline{r_1}(1-r_1^rr_2^{-r})\right)}{4+4k},$$

2.

$$(\mathbb{P}^*_{k,n})^2 - \mathbb{P}^*_{k,n-1}\mathbb{P}^*_{k,n+1} = \frac{(-k)^{n-1}\left(\underline{r_1}\underline{r_2}(k-r_2^2) + \underline{r_2}\underline{r_1}(k-r_1^2)\right)}{4+4k},$$

3.

$$\mathbb{P}^*_{k,n}\mathbb{P}^*_{k,m+1} - \mathbb{P}^*_{k,n+1}\mathbb{P}^*_{k,m} = \frac{\underline{r_1}\underline{r_2}\left((r_1)^n(r_2)^{m+1} - (r_1)^{m+1}(r_2)^n\right)}{(r_1-r_2)^2} + \frac{\underline{r_2}\underline{r_1}(((r_2)^n(r_1)^{m+1} - (r_2)^{m+1}(r_1)^n))}{(r_1-r_2)^2},$$

$$(\mathbb{P}^*_{k,n})^2 - \mathbb{P}^*_{k,n+r}\mathbb{P}^*_{k,n-r} = \frac{(-k)^n\left(\underline{r_1}\underline{r_2}(1-r_1^r r_2^{-r}) + \underline{r_2}\underline{r_1}(1-r_1^{-r}r_2^r)\right)}{4+4k},$$

4.

$$(\mathbb{P}^*_{k,n})^2 - \mathbb{P}^*_{k,n+1}\mathbb{P}^*_{k,n-1} = \frac{(-k)^{n-1}\left(\underline{r_1}\underline{r_2}(k-r_1^2) + \underline{r_2}\underline{r_1}(k-r_2^2)\right)}{4+4k},$$

5.

$$\mathbb{P}^*_{k,n+1}\mathbb{P}^*_{k,m} - \mathbb{P}^*_{k,n}\mathbb{P}^*_{k,m+1} = \frac{\underline{r_1}\underline{r_2}\left((r_1)^{m+1}(r_2)^n - (r_1)^n(r_2)^{m+1}\right)}{(r_1-r_2)^2} + \frac{\underline{r_2}\underline{r_1}(((r_2)^{m+1}(r_1)^n - (r_2)^n(r_1)^{m+1}))}{(r_1-r_2)^2}$$

where $r_1 = 1+\sqrt{1+k}$ *and* $r_2 = 1-\sqrt{1+k}$ *are the roots of the characteristic equation (3.1),* $\underline{r_1} = 1+(2-r_2)i+(4+k-2r_2)\varepsilon+(8+4k-4r_2-kr_2)h$ *and* $\underline{r_2} = -1+(r_1-2)i+(-4-k+2r_1)\varepsilon+(-8-4k+4r_1+kr_1)h$.

4.2 Some sum formulas

Next, we present results concerning the recurrence of sums of terms of hybrid numbers with the generalized hybrid k-Pell numbers as coefficients by using some results of generalized hybrid k-Pell sequence.

Here S_n represents the partial sum of n terms of $\mathbb{P}_{k,n}$, that is,

$$S_n = \mathbb{P}^*_{k,0} + \mathbb{P}^*_{k,1} + \cdots + \mathbb{P}^*_{k,n} = \sum_{i=0}^{n} \mathbb{P}^*_{k,i}.$$

Since

$$\begin{aligned}
&\mathbb{P}^*_{k,0}\\
&\mathbb{P}^*_{k,1}\\
&\mathbb{P}^*_{k,2} = 2\mathbb{P}^*_{k,1} + k\mathbb{P}^*_{k,0}\\
&\mathbb{P}^*_{k,3} = 2\mathbb{P}^*_{k,2} + k\mathbb{P}^*_{k,1}\\
&\quad\vdots = \vdots\\
&\mathbb{P}^*_{k,n} = 2\mathbb{P}^*_{k,n-1} + k\mathbb{P}^*_{k,n-2},
\end{aligned}$$

then

$$\begin{aligned}
S_n &= 2(\mathbb{P}^*_{k,1} + \mathbb{P}^*_{k,2} + \cdots + \mathbb{P}^*_{k,n-1}) + k(\mathbb{P}^*_{k,0} + \mathbb{P}^*_{k,1} + \cdots + \mathbb{P}^*_{k,n-2})\\
&= 2(\mathbb{P}^*_{k,0} + \mathbb{P}^*_{k,1} + \cdots + \mathbb{P}^*_{k,n-1} + \mathbb{P}^*_{k,n} + k(\mathbb{P}^*_{k,0} + \cdots + \mathbb{P}^*_{k,n-2} + \mathbb{P}^*_{k,n-1} + \mathbb{P}^*_{k,n})2(\mathbb{P}^*_{k,0} + \mathbb{P}^*_{k,n}) - k(\mathbb{P}^*_{k,n-1} + \mathbb{P}^*_{k,n})\\
&= 2S_n + kS_n - 2(\mathbb{P}^*_{k,0} + \mathbb{P}^*_{k,n}) - k(\mathbb{P}^*_{k,n-1} + \mathbb{P}^*_{k,n}).
\end{aligned}$$

Therefore, we have

$$kS_n + S_n = 2(\mathbb{P}^*_{k,0} + \mathbb{P}^*_{k,n}) + k(\mathbb{P}^*_{k,n-1} + \mathbb{P}^*_{k,n}).$$

Indeed, we will show that S_n can also be given by a recurrence, and we get the following result.

Theorem 4.8. *Consider $S_o = \mathbb{P}^*_{k,0}$, $S_1 = \mathbb{P}^*_{k,0} + \mathbb{P}^*_{k,1}$ and for all $n \geq 2$, then the following sum identity is verifies*

$$\sum_{i=0}^{n} \mathbb{P}^*_{k,i} = \frac{\mathbb{P}^*_{k,n+1} + (2+k)\mathbb{P}^*_{k,n}}{k+1} \,. \tag{4.2}$$

As a consequence of Theorem 4.8, we have the following proposition.

Proposition 4.9. *For $n \geq 0$,*

1.

$$\sum_{i=0}^{n} \mathbb{P}^*_{k,2i} = \frac{\mathbb{P}^*_{k,n+2} + (4+k)\mathbb{P}^*_{k,n} - k(k+1)\mathbb{P}^*_{k,2n}}{(3-k)(k+1)} \,. \tag{4.3}$$

2.

$$\sum_{i=0}^{n} \mathbb{P}^*_{k,2i+1} = \frac{\mathbb{P}^*_{k,n+2} + (4+k)\mathbb{P}^*_{k,n} - k(k+1)(4-k)\mathbb{P}^*_{k,2n} - 2(3-k)(k+1)\mathbb{P}_{k,0}}{(1-k)(3-k)(k+1)} \,. \tag{4.4}$$

Proof. First, we prove the identity (4.3). Note that

$$\mathbb{P}^*_{k,0} + \mathbb{P}^*_{k,2} + \cdots + \mathbb{P}^*_{k,2n} = 2(\mathbb{P}^*_{k,1} + \mathbb{P}^*_{k,3} + \cdots + \mathbb{P}^*_{k,2n-1}) + k(\mathbb{P}^*_{k,0} + \mathbb{P}^*_{k,2} + \cdots + \mathbb{P}^*_{k,2(n-1)}) \,.$$

Thus,

$$3(\mathbb{P}^*_{k,0} + \mathbb{P}^*_{k,2} + \cdots + \mathbb{P}^*_{k,2n}) = 2(\mathbb{P}^*_{k,0} + \mathbb{P}^*_{k,1} + \cdots + \mathbb{P}^*_{k,2n-1} + \mathbb{P}^*_{k,2n}) + k(\mathbb{P}^*_{k,0} + \cdots + \mathbb{P}^*_{k,2(n-1)} + \mathbb{P}^*_{k,2n}) - k\mathbb{P}^*_{k,2n} \,,$$

or equivalently

$$(3-k)S_{2n} = 2S_n - k\mathbb{P}^*_{k,2n}.$$

Therefore by equation (4.2), we obtain

$$(3-k)S_{2n} = 2\left(\frac{\mathbb{P}^*_{k,n+1} + (2+k)\mathbb{P}^*_{k,n}}{k+1}\right) - k\mathbb{P}^*_{k,2n} \,,$$

and the first result is verified.

Now, consider the following identity,

$$\begin{aligned}\mathbb{P}^*_{k,1} + \mathbb{P}^*_{k,3} + \cdots + \mathbb{P}^*_{k,2n+1} &= 2(\mathbb{P}^*_{k,2} + \mathbb{P}^*_{k,4} + \cdots + \mathbb{P}^*_{k,2n}) + k(\mathbb{P}^*_{k,1} + \mathbb{P}^*_{k,3} + \cdots + \mathbb{P}^*_{k,2n-1}) \\ &= 2(\mathbb{P}^*_{k,0} + \mathbb{P}^*_{k,2} + \cdots + \mathbb{P}^*_{k,2n}) + k(\mathbb{P}^*_{k,1} + \mathbb{P}^*_{k,3} + \cdots + \mathbb{P}^*_{k,2n-1} + \mathbb{P}^*_{k,2n+1}) - 2\mathbb{P}_{k,0} - k\mathbb{P}^*_{k,2n+1} \,.\end{aligned}$$

Then

$$S_{2n+1} = 2S_{2n} + kS_{2n+1} - 2\mathbb{P}_{k,0} - k\mathbb{P}^*_{k,2n+1} \,,$$

or equivalently

$$(1-k)S_{2n+1} = 2S_{2n} - 2\mathbb{P}_{k,0} - k\mathbb{P}^*_{k,2n+1} \,.$$

Thus, by equation (4.3)

$$(1-k)S_{2n+1} = 2\left(\frac{\mathbb{P}^*_{k,n+2} + (4+k)\mathbb{P}^*_{k,n} - k(k+1)\mathbb{P}^*_{k,2n}}{(3-k)(k+1)}\right) - 2\mathbb{P}_{k,0} - k\mathbb{P}^*_{k,2n+1} \,,$$

and equation (4.4) is valid. □

5. Conclusion

In this study, we introduced a new generalization of the hybrid k-Pell sequence, the hybrid numbers with generalized hybrid k-Pell numbers as coefficients. We provide two recurrence relations of this new sequence and also the Binet formula, generating function, and exponential generating function. In addition, we established several identities, namely, the identities of the First and Second Catalan, the identities of the First and Second Cassini, and the identities of the First and Second d'Ocagne. As a special case, we derived all results for the hybrid numbers with hybrid k-Pell numbers as coefficients, the hybrid numbers with hybrid k-Pell–Lucas numbers as coefficients, and the hybrid numbers with hybrid Modified k-Pell numbers as coefficients. It seems that all the results given here are new in the literature.

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