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NEW PRESENTATIONS FOR REAL NUMBERS

NİHAL YILMAZ ÖZGÜR AND SÜMEYRA UÇAR

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Abstract. In this paper we prove that every real number can be uniquely represented as the sum of the squares of consecutive k- Fibonacci numbers. To do this, we give new presentation theorems for any real number $u \neq 0$ using k- Fibonacci series.

1. Introduction

Zeckendorf's theorem is a theorem about representation of integers as follows.

Theorem 1.1. (*Zeckendorf's theorem*) *Every positive integer can be uniquely represented as the sum of distinct Fibonacci numbers if no two consecutive Fibonacci numbers are used in any given sum.*

Let $k > 0$ be any positive real number. In [7], Hoggatt used the sequence defined by

$$F_{k,0} = 0,\ F_{k,1} = 1 \text{ and } F_{k,n+1} = kF_{k,n} + F_{k,n-1} \text{ for } n \geq 1, \tag{1.1}$$

to generalize Zeckendorf's theorem (see [5] for more details about this sequence). For $k = 1$ and $k = 2$ we obtain the well-known Fibonacci and Pell sequences, respectively. Recently it was given the relationship between this new sequence and the recursive application of two geometrical transformations used in well-known four-triangle longest-edge (4TLE) partition (see [3] and [4]). This new number sequence called k-Fibonacci. Here we prefer the notation $F_{k,n}$ used in [3] and [4].

The first seven k-Fibonacci numbers are showed in the following table :

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$$\begin{aligned} F_{k,1} &= 1 \\ F_{k,2} &= k \\ F_{k,3} &= k^2+1 \\ F_{k,4} &= k^3+2k \\ F_{k,5} &= k^4+3k^2+1 \\ F_{k,6} &= k^5+4k^3+3k \\ F_{k,7} &= k^6+5k^4+6k^2+1 \end{aligned}$$

In this paper we obtain new presentations of any real number $u \neq 0$ using k-Fibonacci series and hence we deduce that every real number can be uniquely represented as the sum of the squares of consecutive k-Fibonacci numbers. Especially we have found infinitely many presentations for any real number. As an application we present a new infinite family of complex series for $\frac{1}{\pi}$ (see [1] for more details).

2. Representation Theorems

We begin the following theorem.

Theorem 2.1. *Let $k > 0$ be any real number, $r \geq 0$ be an integer and $F_{k,n}$ be the k-th Fibonacci number. Then we have*

$$\sum_{n=1}^{\infty} \frac{1}{F_{k,n+r}F_{k,n+r+2}} = \frac{1}{kF_{k,r+1}F_{k,r+2}}. \tag{2.1}$$

Proof. Using (1.1) we obtain

$$\frac{1}{F_{k,n+r}F_{k,n+r+2}} = \frac{1}{kF_{k,n+r}F_{k,n+r+1}} - \frac{1}{kF_{k,n+r+1}F_{k,n+r+2}}$$

and then we find

$$\begin{aligned} s_n &= \sum_{l=1}^{n} \frac{1}{F_{k,l+r}F_{k,l+r+2}} \\ &= \sum_{l=1}^{n} \left(\frac{1}{kF_{k,l+r}F_{k,l+r+1}} - \frac{1}{kF_{k.l+r+1}F_{k,l+r+2}}\right) \\ &= \frac{1}{kF_{k,r+1}F_{k,r+2}} - \frac{1}{kF_{k.r+n+1}F_{k,r+n+2}}. \end{aligned}$$

So we have

$$\sum_{n=1}^{\infty} \frac{1}{F_{k,n+r}F_{k,n+r+2}} = \frac{1}{kF_{k.r+1}F_{k,r+2}}.$$

□

Theorem 2.2. *For any real number $u \neq 0$ and any positive integer r, there exists unique positive real number k satisfying the following equation*

$$\sum_{n=1}^{\infty} \frac{1}{F_{k,n+r}F_{k,n+r+2}} = \frac{1}{u}. \tag{2.2}$$

Proof. Let $u > 0$ be any real number. From the Theorem 2.2 we know the following equation:

$$\sum_{n=1}^{\infty} \frac{1}{F_{k,n+r}F_{k,n+r+2}} = \frac{1}{kF_{k.r+1}F_{k,r+2}}.$$

Then we get

$$\frac{1}{kF_{k.r+1}F_{k,r+2}} = \frac{1}{u}$$

and so we get the following equation

$$kF_{k,r+1}F_{k,r+2} - u = 0. \tag{2.3}$$

Using $F_{k,r+2} = kF_{k,r+1} + F_{k,r}$, we obtain

$$kF_{k,r+1}F_{k,r+2} = k^2(F_{k,1}^2 + F_{k,2}^2 + F_{k,3}^2 + ... + F_{k,r}^2 + F_{k,r+1}^2).$$

Let us consider the following polynomial

$$f_r(k) = k^2(F_{k,1}^2 + F_{k,2}^2 + F_{k,3}^2 + ... + F_{k,r}^2 + F_{k,r+1}^2) - u. \tag{2.4}$$

Applying the Descartes rule of signs, $f_r(k)$ has unique positive real zero, say k_0, and then we have equation (2.2) for k_0. If $u < 0$, then for the real number $a = |u|$ we get

$$\sum_{n=1}^{\infty} \frac{1}{F_{k,n+r}F_{k,n+r+2}} = \frac{1}{a}.$$

and hence

$$-\sum_{n=1}^{\infty} \frac{1}{F_{k,n+r}F_{k,n+r+2}} = \frac{1}{u}.$$

Thus, for all real numbers different zero, there exists unique k such that (2.2) holds. □

Corollary 2.1. *For any positive integer r, there exists unique positive real number k satisfying the following equation*

$$\sum_{n=1}^{\infty} \frac{1}{F_{k,n+r}F_{k,n+r+2}} = \frac{1}{\pi}. \tag{2.5}$$

Thus, we have an infinite family of real series converging to $\frac{1}{\pi}$.

Theorem 2.3. *Let $u \neq 0$ be any real number. For any positive integer r, there exists unique positive real number k such that we have the following polynomial representation for u :*

$$u = k^2(F_{k,1}^2 + F_{k,2}^2 + F_{k,3}^2 + ... + F_{k,r}^2 + F_{k,r+1}^2). \tag{2.6}$$

Proof. The proof follows easily from the proof of Theorem 2.2. □

Thus, we have an infinite family of polynomials representing u.

Example 2.1. Let $u = \frac{1}{97}$ and $r = 3$. From (2.3) we have $kF_{k,4}F_{k,5} - \frac{1}{97} = 0$. Hence we have the equation $k(k^3 + 2k)(k^4 + 3k^2 + 1) - \frac{1}{97} = 0$. The unique positive root of this equation is $0,07116$. Thus, we have $\sum_{n=1}^{\infty} \frac{1}{F_{0,07116,n+3}F_{0,07116,n+5}} = 97$.

Now we restrict our attention to polynomial representation and the equation (2.6), so let u be any real number. By Theorem 2.3, for any positive integer r, there exists unique positive real number k such that we have the following polynomial representation for u

$$u = k^2(F_{k,1}^2 + F_{k,2}^2 + F_{k,3}^2 + ... + F_{k,r}^2 + F_{k,r+1}^2). \tag{2.7}$$

Let

$$u_r(k) = k^2(F_{k,1}^2 + F_{k,2}^2 + F_{k,3}^2 + ... + F_{k,r}^2 + F_{k,r+1}^2).$$

Example 2.2. For the prime number $u = 41$, we have the following table.

u	r	$u_r(k)$	k
41	1	$k^2(1 + k^2)$	2, 43364
41	2	$k^2(1 + k^2 + (1 + k^2)^2)$	1, 59526
41	3	$k^2(1 + k^2 + (1 + k^2)^2 + (k^3 + 2k)^2)$	1, 22655
41	4	$k^2(1 + k^2 + (1 + k^2)^2 + (k^3 + 2k)^2 + (k^4 + 3k^2 + 1)^2)$	1, 00449

So we have seen that one of the presentations representing any prime looks like very simple, $u_1(k) = k^2(1+k^2)$. Now we focus on the case $r = 1$. At first we consider the following example.

Example 2.3. Let $r = 1$ be fixed. Then we have $u_1(k) = k^2(1 + k^2)$. In the following table we can see the values of k such that the corresponding polynomial $u_1(k) = k^2(1 + k^2)$ represents the first ten primes.

p	k
2	1
3	1, 14139
5	1, 33839
7	1, 48074
11	1, 68941
13	1, 77202
17	1, 91136
19	1, 97167
23	2, 0789
29	2, 21547

Notice that k can not be an integer for $p \geq 3$. We have only integer value of $k = 1$ in the case $p = 2$.

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Balikesir University, Department of Mathematics, 10145 Balikesir, TURKEY
E-mail address: nihal@balikesir.edu.tr

Balikesir University, Department of Mathematics, 10145 Balikesir, TURKEY
E-mail address: sumeyraucar@balikesir.edu.tr