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A NEW APPLICATION OF ABSOLUTE MATRIX SUMMABILITY

HİKMET SEYHAN ÖZARSLAN AND MEHMET ÖNER ŞAKAR

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Abstract. In the present paper, a theorem dealing with the $|\bar{N}, p_n|_k$ summability factors of infinite series has been generalized to absolute matrix summability factors by using δ-quasi monotone sequences. This new theorem also includes several new results.

1. INTRODUCTION

Definition 1.1. A sequence (b_n) of positive numbers is said to be quasi monotone, if $n\Delta b_n \geq -\alpha b_n$ for some α and it is said to be δ-quasi monotone, if $b_n \to 0$, $b_n > 0$ ultimately and $\Delta b_n \geq -\delta_n$, where (δ_n) is a sequence of positive numbers (see [1]).

Definition 1.2. Let $\sum a_n$ be a given infinite series with partial sums (s_n). By u_n and t_n we denote the n-th $(C,1)$ means of the sequences (s_n) and (na_n), respectively. The series $\sum a_n$ is said to be summable $|C,1|_k$, $k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n - u_{n-1}|^k < \infty. \tag{1.1}$$

But since $t_n = n(u_n - u_{n-1})$ (see [6]), condition (1.1) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty. \tag{1.2}$$

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Definition 1.3. Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1)\,. \tag{1.3}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.4}$$

defines the sequence (σ_n) of the $\left(\bar{N}, p_n\right)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [5]). The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\sigma_n - \sigma_{n-1}|^k < \infty. \tag{1.5}$$

Definition 1.4. Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, \ldots \tag{1.6}$$

The series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [9])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\bar{\Delta} A_n(s)|^k < \infty, \tag{1.7}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

2. Known result

In [3], Bor has proved the following main theorem dealing with $\left|\bar{N}, p_n\right|_k$ summability factors of infinite series.

Theorem 2.1. *Let (X_n) be a positive non-decreasing sequence,$(\lambda_n) \to 0$ as $n \to \infty$ and (p_n) be a sequence of positive numbers such that*

$$P_n = O(np_n) \qquad as \quad n \to \infty. \tag{2.1}$$

Suppose that there exist a sequence of numbers (A_n) which is δ-quasi monotone with $\sum nX_n\delta_n < \infty$, $\sum A_n X_n$ is convergent and $|\Delta\lambda_n| \leq |A_n|$ for all n. If

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad as \qquad m \to \infty, \tag{2.2}$$

then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

3. The Main Result

The aim of this paper is to generalize Theorem 2.1 to $|A, p_n|_k$ summability. Before stating the main theorem we must first introduce some further notations. Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{3.1}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{3.2}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{3.3}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{3.4}$$

Now we shall prove the following theorem.

Theorem 3.1. *Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \quad n = 0, 1, ..., \tag{3.5}$$

$$a_{n-1,v} \geq a_{nv} \quad for \quad n \geq v + 1, \tag{3.6}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{3.7}$$

$$|\hat{a}_{n,v+1}| = O\left(v \left|\Delta_v \hat{a}_{nv}\right|\right). \tag{3.8}$$

If (X_n) is a non-decreasing sequence and the conditions of Theorem 2.1 are satisfied, then the series $\sum a_n \lambda_n$ is summable $|A, p_n|_k$, $k \geq 1$.

Remark 3.1. It should be noted that, if we take (X_n) as a positive non-decreasing sequence and $a_{nv} = \dfrac{p_v}{P_n}$ for all values of n in Theorem 3.1, then we get Theorem 2.1. Also, if we take $p_n = 1$, $a_{nv} = \dfrac{p_v}{P_n}$ and $X_n = \log n$ for all values of n in Theorem 3.1, then we get a result due to Mazhar [7].

Lemma 3.1. *(see [3]) Under the conditions of Theorem 3.1, we have that*

$$|\lambda_n| X_n = O(1) \quad as \quad n \to \infty. \tag{3.9}$$

Lemma 3.2. *(see [3]) If (A_n) is δ-quasi monotone with $\sum nX_n\delta_n < \infty$, $\sum A_nX_n$ is convergent, then*

$$mX_mA_m = O(1) \quad as \quad m \to \infty, \tag{3.10}$$

$$\sum_{n=1}^{\infty} nX_n|\Delta A_n| < \infty. \tag{3.11}$$

4. Proof of Theorem 3.1

Let (T_n) denotes A-transform of the series $\sum a_n\lambda_n$. Then, we have by (3.3) and (3.4)

$$T_n = \sum_{v=0}^{n} a_{nv} \sum_{i=0}^{v} a_i\lambda_i = \sum_{v=0}^{n} a_v\lambda_v\bar{a}_{nv}.$$

Hence we get

$$\bar{\Delta}T_n = \sum_{i=0}^{n} a_i\lambda_i\bar{a}_{ni} - \sum_{i=0}^{n-1} a_i\lambda_i\bar{a}_{n-1,i} = \sum_{i=0}^{n} \frac{\hat{a}_{ni}\lambda_i}{i} ia_i.$$

By Abel's transformation, we have

$$\begin{aligned}
\bar{\Delta}T_n &= \sum_{v=1}^{n-1} \Delta_v\Big(\frac{\hat{a}_{nv}\lambda_v}{v}\Big) \sum_{r=1}^{v} ra_r + \frac{\hat{a}_{nn}\lambda_n}{n} \sum_{r=1}^{n} ra_r \\
&= \sum_{v=1}^{n-1} \Delta_v\Big(\frac{\hat{a}_{nv}\lambda_v}{v}\Big)(v+1)t_v + \frac{\hat{a}_{nn}\lambda_n}{n}(n+1)t_n \\
&= \frac{n+1}{n}a_{nn}\lambda_nt_n + \sum_{v=1}^{n-1} \frac{v+1}{v}\Delta_v(\hat{a}_{nv})\lambda_vt_v \\
&+ \sum_{v=1}^{n-1} \frac{v+1}{v}\hat{a}_{n,v+1}\Delta\lambda_vt_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\lambda_{v+1}\frac{t_v}{v} \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \qquad say.
\end{aligned}$$

Since

$$|T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}|^k \le 4^k(|T_{n,1}|^k + |T_{n,2}|^k + |T_{n,3}|^k + |T_{n,4}|^k)$$

to complete the proof of Theorem 3.1, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,r}|^k < \infty \quad for \quad r = 1, 2, 3, 4.$$

Firstly, by using (3.7) condition, we have that

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,1}|^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^k|\lambda_n|^k|t_n|^k \\
&= O(1)\sum_{n=1}^{m}\frac{p_n}{P_n}|\lambda_n||\lambda_n|^{k-1}|t_n|^k \\
&= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{r=1}^{n}\left(\frac{p_r}{P_r}\right)|t_r|^k+O(1)|\lambda_m|\sum_{r=1}^{m}\left(\frac{p_r}{P_r}\right)|t_r|^k \\
&= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n+O(1)|\lambda_m|X_m \\
&= O(1)\sum_{n=1}^{m-1}|A_n|X_n+O(1)|\lambda_m|X_m \\
&= O(1)\sum_{n=1}^{m-1}A_nX_n+O(1)|\lambda_m|X_m=O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by the hypotheses of Theorem 3.1 and Lemma 3.1. Now, applying Hölder's inequality with indices k and k' where $1/k+1/k'=1$, as in $T_{n,1}$ we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v||t_v|\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\right) \\
&\times \left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\right)\times a_{nn}^{k-1} \\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^k|t_v|^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})| \\
&= O(1)\sum_{v=1}^{m}|\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})| \\
&= O(1)\sum_{v=1}^{m}|\lambda_v||t_v|^k a_{vv} \\
&= O(1)\sum_{v=1}^{m}|\lambda_v||t_v|^k\frac{p_v}{P_v} \\
&= O(1) \qquad as \quad m\to\infty.
\end{aligned}$$

Now, since $vA_v = O\Big(\frac{1}{X_v}\Big) = O(1)$, by (3.8), we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{k-1}|T_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{k-1}\Big(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\Big)^k \\
&= O(1)\sum_{n=2}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{k-1}\Big(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||A_v||t_v|\Big)^k \\
&= O(1)\sum_{n=2}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||A_v||t_v|^k \\
&\times \Big(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||A_v|\Big)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{k-1}\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{n,v})||A_v||t_v|^k \\
&\times \Big(\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{n,v})||A_v|\Big)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{k-1}\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{n,v})||A_v||t_v|^k \\
&\times \Big(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{n,v})|\Big)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{n,v})||A_v||t_v|^k \\
&= O(1)\sum_{v=1}^{m}v|A_v||t_v|^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})| \\
&= O(1)\sum_{v=1}^{m}v|A_v||t_v|^k\frac{p_v}{P_v} \\
&= O(1)\sum_{v=1}^{m-1}\Delta(v|A_v|)\sum_{r=1}^{v}\frac{p_r}{P_r}|t_r|^k + m|A_m|\sum_{r=1}^{m}\frac{p_r}{P_r}|t_r|^k \\
&= O(1)\sum_{v=1}^{m-1}v\Delta(|A_v|)X_v + O(1)\sum_{v=1}^{m-1}|A_{v+1}|X_v + O(1)m|A_m|X_m \\
&= O(1)\sum_{v=1}^{m-1}v|\Delta A_v|X_v + O(1)\sum_{v=1}^{m-1}A_{v+1}X_v + O(1)mA_mX_m \\
&= O(1) \qquad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypothesises of Theorem 3.1 and by Lemma 3.2.

Finally, as in $T_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{k-1}|T_{n,4}|^k &\leq \sum_{n=2}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{k-1}\Big(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|\frac{|t_v|}{v}\Big)^k \\
&= O(1)\sum_{n=2}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{k-1}\Big(\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{nv})||\lambda_{v+1}|\frac{|t_v|}{v}\Big)^k \\
&= O(1)\sum_{n=2}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{k-1}\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{n,v})||\lambda_{v+1}||t_v|^k \\
&\times \Big(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{n,v})|\Big)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_{v+1}|^k|t_v|^k \\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}||t_v|^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})| \\
&= O(1)\sum_{v=1}^{m}\frac{p_v}{P_v}|\lambda_{v+1}||t_v|^k \\
&= O(1) \quad as \quad m\to\infty.
\end{aligned}$$

Therefore, we obtain that

$$\sum_{n=1}^{m}(\frac{P_n}{p_n})^{k-1}|T_{n,r}|^k = O(1) \quad as \quad m\to\infty, \quad for \quad r=1,2,3,4.$$

This completes the proof of the Theorem 3.1.

It should be noted that, if we take $a_{nv}=\frac{p_v}{P_n}$ for all values of n, then we get Theorem 2.1. Also, if we take $p_n=1$ for all values of n, then we get $|A|_k$ summability (see [8]). Furthermore, if we take $p_n=1$ and $a_{nv}=\frac{p_v}{P_n}$ for all values of n, then we get $|C,1|_k$ summability. Finally, if we take $p_n=1$, $a_{nv}=\frac{p_v}{P_n}$ and $X_n=\log n$ for all values of n, then we get a result due to Mazhar [7].

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Kayseri-TURKEY
E-mail address: seyhan@erciyes.edu.tr and mehmethaydaroner@hotmail.com